
Chapter VI

CHAPTER VI

Fuzzy Soft Topological Spaces

Section 6.1

Fuzzy Soft Sets

Notation: 6.1.1

Let U be an initial universe set and E be the set of parameters and $\tilde{P}(U)$, the set of all fuzzy subset of U and also by (U, E) , we mean the universal set U and parameter set E .

Definition: 6.1.2

A pair (F, A) is called a **fuzzy soft set** over U where $F: A \rightarrow \tilde{P}(U)$ is a mapping from A into $\tilde{P}(U)$.

The fuzzy soft set (F, A) is also denoted as F_A .

Definition: 6.1.3

For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class (U, E) we say that (F, A) is a **fuzzy soft subset** of (G, B) , if

i. $A \subseteq B$

ii. For all $\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$ and is written as $(F, A) \subseteq (G, B)$.

Definition: 6.1.4

Union of two fuzzy soft sets (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H, C) where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A \setminus B \\ G(\varepsilon) & \text{if } \varepsilon \in B \setminus A \\ F(\varepsilon) \cup G(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

and is denoted as $(F, A) \cup (G, B) = (H, C)$.

Definition: 6.1.5

Let (F, A) and (G, B) be two fuzzy soft sets in a soft class (U, E) with $A \cap B = \emptyset$. Then **intersection** of two fuzzy soft sets (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H, C) where $C = A \cap B$ and $\forall \varepsilon \in C$, $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. We write $(F, A) \cap (G, B) = (H, C)$.

Definition: 6.1.6

The **complement** of a fuzzy soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ where $F^c: A \rightarrow \tilde{P}(U)$ is a mapping given by

$$F^c(\alpha) = [F(\alpha)]^c, \forall \alpha \in A.$$

The set of all fuzzy soft set over (U, E) is denoted by $FS(U, E)$.

Definition: 6.1.7

The fuzzy soft set $F_\emptyset \in FS(U, E)$ is called **null fuzzy soft set** and is denoted by $\tilde{\emptyset}$, if $F_\emptyset(e) = \bar{0}$ where $\bar{0}$ is the constant fuzzy set zero, for every $e \in E$.

Definition: 6.1.8

The fuzzy soft set $F_E \in FS(U, E)$ is called **absolute fuzzy soft set** and is denoted by \tilde{E} , if $F_E(e) = \bar{1}$ where $\bar{1}$ is the constant fuzzy set one, for every $e \in E$.

Theorem: 6.1.9

Let $F_A, G_B, H_C \in FS(U, E)$. It can be verified that the following hold according to our notion of fuzzy soft sets.

- i. $\tilde{\emptyset} \subseteq F_A \subseteq \tilde{E}$
- ii. $F_A \cup G_B = G_B \cup F_A, F_A \cap G_B = G_B \cap F_A$
- iii. $F_A \cup (G_B \cap H_C) = (F_A \cup G_B) \cap H_C, F_A \cap (G_B \cap H_C) = (F_A \cap G_B) \cap H_C$
- iv. $F_A \cup (G_B \cap H_C) = (F_A \cup G_B) \cap (F_A \cup H_C)$
 $F_A \cap (G_B \cup H_C) = (F_A \cap G_B) \cup (F_A \cap H_C)$
- v. $F_A, G_B \subseteq F_A \cup G_B, F_A, G_B \subseteq F_A \cap G_B$
- vi. $F_A \subseteq G_B \Rightarrow G_B^c \subseteq F_A^c$
- vii. $\tilde{\emptyset}^c = \tilde{E}, \tilde{E}^c = \tilde{\emptyset}$
- viii. $(F_A^c)^c = F_A$
- ix. $(\tilde{\cap}_\alpha F_{A_\alpha}^\alpha)^c = \tilde{\cup}_\alpha (F_{A_\alpha}^{\alpha c})$
- x. $(\tilde{\cup}_\alpha F_{A_\alpha}^\alpha)^c = \tilde{\cap}_\alpha (F_{A_\alpha}^{\alpha c})$

Section 6.2

Definitions and Basic Properties of Fuzzy Soft Topological Spaces

Definition: 6.2.1

A **fuzzy soft topology** τ on (U, E) is a family of fuzzy soft sets over (U, E) satisfying the following properties

- i. $\tilde{\emptyset}, \tilde{E} \in \tau$
- ii. if $F_A, G_B \in \tau$ then $F_A \cap G_B \in \tau$
- iii. If $F_{A_\alpha}^\alpha \in \tau$ for all $\alpha \in \Delta$, an indexed set, then $\tilde{\cup}_{\alpha \in \Delta} F_{A_\alpha}^\alpha \in \tau$.

Definition: 6.2.2

If τ is a fuzzy soft topology on (U, E) , the triple (U, E, τ) is said to be a **fuzzy soft topological space**. Also each member of τ is called a **fuzzy soft open set** in (U, E, τ) .

Example: 6.2.3

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_4\} \subseteq E$, $B = \{e_1, e_2, e_3, e_4\} \subseteq E$.

$$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0)\}$$

$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_4) = \{(a, 0.2), (b, 0.6), (c, 0.3)\}$$

$$G_B = \{G(e_1) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G(e_2) = \{(a, 0.5), (b, 0.2), (c, 0.1)\}$$

$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G(e_4) = \{(a, 0.1), (b, 0.3), (c, 0.2)\}$$

we consider a collection τ of fuzzy soft sets over (U, E) as $\tau = \{\tilde{\emptyset}, \tilde{E}, F_A, G_B\}$.

We see that

i. $\tilde{\emptyset}, \tilde{E} \in \tau$

ii. $\tilde{\emptyset} \tilde{\cup} \tilde{E} = \tilde{\emptyset}$, $\tilde{\emptyset} \tilde{\cup} F_A = \tilde{\emptyset}$, $\tilde{\emptyset} \tilde{\cup} G_B = \tilde{\emptyset}$

$$\tilde{E} \tilde{\cup} F_A = F_A, \tilde{E} \tilde{\cup} G_B = G_B, F_A \tilde{\cup} G_B = G_B$$

iii. $\tilde{\emptyset} \tilde{\cup} \tilde{E} = \tilde{E}$, $\tilde{\emptyset} \tilde{\cup} F_A = F_A$, $\tilde{\emptyset} \tilde{\cup} G_B = G_B$

$$\tilde{E} \tilde{\cup} F_A = \tilde{E}, \tilde{E} \tilde{\cup} G_B = \tilde{E}, F_A \tilde{\cup} G_B = F_A,$$

$$\tilde{\emptyset} \tilde{\cup} \tilde{E} \tilde{\cup} F_A = \tilde{E}, \tilde{\emptyset} \tilde{\cup} \tilde{E} \tilde{\cup} G_B = \tilde{E}$$

$$\tilde{E} \tilde{\cup} F_A \tilde{\cup} G_B = \tilde{E} \text{ and } \tilde{\emptyset} \tilde{\cup} \tilde{E} \tilde{\cup} F_A \tilde{\cup} G_B = \tilde{E}.$$

Thus τ is a fuzzy soft topology on (U, E) and the triple (U, E, τ) is a fuzzy soft topological space. The open sets in (U, E, τ) are $\tilde{\emptyset}, \tilde{E}, F_A, G_B$.

Definition: 6.2.4

A fuzzy soft set F_A over (U, E) is called a **fuzzy soft closed** set in (U, E, τ) if and only if its complement F_A^c is a fuzzy soft open set in (U, E, τ) .

Definition: 6.2.5

A fuzzy soft topological space (U, E, τ) is called **indiscrete** if it contains only $\tilde{\emptyset}$ and \tilde{E} while the **discrete** fuzzy soft topology consists of all fuzzy soft sets over (U, E) .

Theorem: 6.2.6

If $\{\tau_\lambda : \lambda \in I\}$ is a family of fuzzy soft topologies on (U, E) , then $\bigcap_\lambda \{\tau_\lambda : \lambda \in I\}$ is also a fuzzy soft topology on (U, E) .

Proposition: 6.2.7

Let (U, E, τ) be a fuzzy soft topological space and Γ denote the collection of all closed fuzzy soft sets in (U, E, τ) . Then

- i. $\tilde{\emptyset}, \tilde{E} \in \Gamma$
- ii. If $F_{A_1}, F_{A_2} \in \Gamma$ then $F_{A_1} \tilde{\cup} F_{A_2} \in \Gamma$
- iii. If $F_{A_\alpha}^\alpha \in \tau$ for all $\alpha \in \Delta$, an indexed set, then $\tilde{\cap}_{\alpha \in \Delta} F_{A_\alpha}^\alpha \in \Gamma$

Definition: 6.2.8

Let τ_1 and τ_2 be two fuzzy soft topologies on (U, E) . We say that τ_1 is **coarser** than τ_2 or that τ_2 is **finer** than τ_1 if and only if $\tau_1 \subseteq \tau_2$ i.e. every τ_1 fuzzy soft open set is τ_2 fuzzy soft open set. If either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, we say that the topologies τ_1 and τ_2 are comparable.

If $\tau_1 \not\subseteq \tau_2$ and $\tau_2 \not\subseteq \tau_1$, we say that the topologies τ_1 and τ_2 are not comparable.

Example: 6.2.9

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0)\}$$

$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G_A = \{G(e_1) = \{(a, 0.6), (b, 0.1), (c, 0)\}$$

$$G(e_2) = \{(a, 0.7), (b, 0.9), (c, 0.5)\}$$

$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G(e_4) = \{(a, 0.5), (b, 0.3), (c, 0.9)\}$$

We consider a fuzzy soft topologies τ_1 and τ_2 on (U, E) as $\tau_1 = \{\tilde{\emptyset}, \tilde{E}, F_A\}$ and $\tau_2 = \{\tilde{\emptyset}, \tilde{E}, F_A, G_A\}$. We see that $\tau_1 \subseteq \tau_2$ and hence τ_1 is coarser than τ_2 .

Definition: 6.2.10

The fuzzy soft set F_A over (U, E) is called a **fuzzy soft point** in (U, E) denoted by $e(F_A)$, if for the element $e \in A$, $F(e) = \bar{0}$ and $F(e^1)$ for all $e^1 \in A - \{e\}$.

Example: 6.2.11

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$.

Consider a fuzzy soft set F_A over (U, E) as

$$F_A = \{F(e_1) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_2) = \{(a, 0.1), (b, 0.7), (c, 0.7)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}$$

Here $e_2 \in A$, $F(e_2) = \bar{0}$ and for $e^1 \in A - \{e_2\}$, $F(e^1) = \bar{0}$. Thus F_A is a fuzzy soft point in (U, E) denoted by $e_2(F_A)$.

Definition: 6.2.12

The **fuzzy soft point** $e(F_A)$ is said to be in the fuzzy soft set G_B if $A \subseteq B$ and for the element $e \in A$, $F(e) \subseteq G(e)$. Symbolically we write $e(F_A) \tilde{\in} G_B$.

Example: 6.2.13

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

Clearly $A \subseteq B$. We consider the fuzzy soft point $e_2(F_A)$ given in Example 6.2.11 and a fuzzy soft set G_B over (U, E) as

$$G_A = \{G(e_1) = \{(a, 0.1), (b, 0.7), (c, 0.9)\}$$

$$G(e_2) = \{(a, 0.1), (b, 0.3), (c, 0.8)\}$$

$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G(e_4) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$

Here $e_2 \in A$, and $F(e_2) \subseteq G(e_2)$. Thus by our definition $e(F_A) \tilde{\in} G_B$.

Definition: 6.2.14

A fuzzy soft set H_C in a fuzzy soft topological space (U, E, τ) is called a **fuzzy soft neighbourhood** of the fuzzy soft point $e(F_A) \tilde{\in} (U, E)$ if there is a fuzzy soft open set G_B such that $e(F_A) \tilde{\in} G_B \tilde{\subseteq} H_C$.

Example: 6.2.15

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

Clearly $A \subseteq B$. We consider the fuzzy soft point $e_2(F_A)$ given in Example 6.2.11 and a fuzzy soft set G_B over (U, E) given in Example 6.2.13 above.

Consider the fuzzy soft topology $\tau = \{\bar{\emptyset}, \bar{E}, G_B\}$.

Now let us consider the fuzzy soft set H_B over (U, E) as

$$H_B = \{H(e_1) = \{(a, 0.1), (b, 0.8), (c, 0.9)\}$$

$$H(e_2) = \{(a, 0.2), (b, 0.3), (c, 0.8)\}$$

$$H(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$H(e_4) = \{(a, 0.7), (b, 0.6), (c, 0.2)\}$$

We see that $e(F_A) \tilde{\in} G_B \tilde{\subseteq} H_C$. Thus by our definition the fuzzy soft set H_B is a fuzzy soft neighbourhood of the fuzzy soft point $e_2(F_A)$.

Definition: 6.2.16

The neighbourhood system of a fuzzy soft point $e(F_A) \tilde{\in} (U, E)$ is the family of all neighbourhoods of $e(F_A)$ and is written as $N_\tau(e(F_A))$.

Definition: 6.2.17

A fuzzy soft set H_C in a fuzzy soft topological space (U, E, τ) is called a **fuzzy soft neighbourhood** of the fuzzy soft open G_B such that $F_A \tilde{\subseteq} G_B \tilde{\subseteq} H_C$.

Example: 6.2.18

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

Clearly $A \subseteq B$. We consider the fuzzy soft set G_B over (U, E) given in example 2.2.13 and the fuzzy soft set H_B over (U, E) given in example 2.2.15 above.

Consider the fuzzy soft topology $\tau = \{\tilde{\Phi}, \tilde{E}, G_B\}$.

Let us consider the fuzzy soft set I_A over (U, E) as

$$I_A = \{I(e_1) = \{(a, 0.1), (b, 0.6), (c, 0.7)\}$$

$$I(e_2) = \{(a, 0), (b, 0.1), (c, 0.8)\}$$

$$I(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$I(e_4) = \{(a, 0), (b, 0), (c, 0)\}$$

We see that $I_A \tilde{\subseteq} G_B \tilde{\subseteq} H_B$. It follows that the fuzzy soft set H_B is a fuzzy soft neighbourhood of the fuzzy soft set I_A .

Theorem: 6.2.19

In a fuzzy soft topological space (U, E, τ) the following hold.

i. $\tilde{E} \in N_\tau(e(F_A)) \forall e(F_A)$ and

$$G_B \in N_\tau(e(F_A)) \Rightarrow e(F_A) \tilde{\in} G_B$$

ii. If $G_B \in N_\tau(e(F_A))$ and $G_B \tilde{\subseteq} H_C$ then $H_C \in N_\tau(e(F_A))$

iii. If $G_B, H_C \in N_\tau(e(F_A))$ then $G_B \tilde{\cap} H_C \in N_\tau(e(F_A))$

iv. If $G_B \in N_\tau(e(F_A))$ then there is a $H_C \in N_\tau(e(F_A))$ such that $G_B \in N_\tau(e^1(M_D))$ for each $e^1(M_D) \tilde{\in} H_C$.

Proof

i. Let $\tilde{E} = H_E$

$$\text{Then } \forall \varepsilon \in E, H(\varepsilon) = \bar{1}$$

$$\text{Now } A \subseteq E \text{ and } F(e) = H(e)$$

$$\text{It follows that } e(F_A) \tilde{\in} H_E \tilde{\subseteq} H_E$$

$$\text{i.e. } e(F_A) \tilde{\in} \tilde{E} \tilde{\subseteq} \tilde{E}$$

Thus $\tilde{E} \in N_\tau(e(F_A)) \forall e(F_A)$

For the second part, we have $G_B \in N_\tau(e(F_A))$

So there is a fuzzy soft set H_C such that $e(F_A) \tilde{\subseteq} H_C \tilde{\subseteq} G_B$. It follows that $e(F_A) \tilde{\subseteq} G_B$

ii. We have $G_B \in N_\tau(e(F_A))$

So there is a fuzzy soft open sets I_D such that $e(F_A) \tilde{\subseteq} I_D \tilde{\subseteq} G_B$

Also $G_B \tilde{\subseteq} H_C$

Thus $e(F_A) \tilde{\subseteq} I_D \tilde{\subseteq} G_B \tilde{\subseteq} H_C$ and hence

$H_C \in N_\tau(e(F_A))$

iii. We have $G_B, H_C \in N_\tau(e(F_A))$

So there is a fuzzy soft open sets I_D, J_E such that

$e(F_A) \tilde{\subseteq} I_D \tilde{\subseteq} G_B$ and $e(F_A) \tilde{\subseteq} J_E \tilde{\subseteq} H_C$

Thus $e(F_A) \tilde{\subseteq} I_D \tilde{\cap} J_E \tilde{\subseteq} G_B \tilde{\cap} H_C$ and

Since $I_D \tilde{\cap} J_E \in \tau$, it follows that

$G_B \tilde{\cap} H_C \in N_\tau(e(F_A))$

iv. We have $G_B \in N_\tau(e(F_A))$

So there is an $I_P \in \tau$ such that $e(F_A) \tilde{\subseteq} I_P \tilde{\subseteq} G_B$.

We take $H_C = I_P$. Then for each $e^1(M_D) \tilde{\subseteq} H_C, e^1(M_D) \tilde{\subseteq} H_C \tilde{\subseteq} G_B$.

It follows that $G_B \in N_\tau(e^1(M_D))$

Definition: 6.2.20

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E) . The **fuzzy soft closure** of F_A is defined as the intersection of all fuzzy soft closed sets which contain F_A and is denoted by $\overline{F_A}$. We write

$$\overline{F_A} = \tilde{\cap} \{ G_B : G_B \text{ is a fuzzy soft closed and } F_A \tilde{\subseteq} G_B \}$$

It is obvious that

i. $\overline{F_A}$ is a fuzzy soft closed and

ii. $F_A \tilde{\subseteq} \overline{F_A}$.

Example: 6.2.21

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{ F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0)\} \}$$

$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$

$$F(e_3) = \{(a,0), (b,0), (c,0)\}$$

$$F(e_4) = \{(a,0), (b,0), (c,0)\}$$

$$G_B = \{G(e_1) = \{(a,0.6), (b,0.1), (c,0)\}\}$$

$$G(e_2) = \{(a,0.7), (b,0.9), (c,0.5)\}$$

$$G(e_3) = \{(a,0), (b,0), (c,0)\}$$

$$G(e_4) = \{(a,0.5), (b,0.3), (c,0.9)\}$$

We consider a fuzzy soft topology τ on (U, E) as $\tau = \{\tilde{\Phi}, \tilde{E}, F_A, G_B\}$. Then fuzzy soft closed sets are

$$F_A^c = \{F^c(e_1) = \{(a,0.6), (b, 0.9), (c,1)\}\}$$

$$F^c(e_2) = \{(a,0.4), (b, 0.5), (c,0.8)\}$$

$$F^c(e_3) = \{(a,1), (b, 1), (c,1)\}$$

$$F^c(e_4) = \{(a,1), (b, 1), (c,1)\}$$

$$G_B^c = \{G^c(e_1) = \{(a,0.4), (b, 0.9), (c,1)\}\}$$

$$G^c(e_2) = \{(a,0.3), (b, 0.1), (c,0.5)\}$$

$$G^c(e_3) = \{(a,1), (b, 1), (c,1)\}$$

$$G^c(e_4) = \{(a,0.5), (b, 0.7), (c,0.1)\}$$

We consider a fuzzy soft set H_C over (U, E) as

$$H_C = \{H(e_1) = \{(a,0.5), (b, 0.7), (c,0.8)\}\}$$

$$H(e_2) = \{(a,0.5), (b, 0.7), (c,0.8)\}$$

$$H(e_3) = \{(a,0), (b, 0), (c,0)\}$$

$$H(e_4) = \{(a,0), (b, 0), (c,0)\}$$

Then

$$\overline{H_C} = \text{Intersection of all fuzzy soft closed sets containing } H_C$$

$$= F_A^c \tilde{\cap} \tilde{E} = \{F_A^c = F^c(e_1) = \{(a,0.6), (b, 0.9), (c,1)\}\}$$

$$F^c(e_2) = \{(a,0.4), (b, 0.5), (c,0.8)\}$$

$$F^c(e_3) = \{(a,1), (b, 1), (c,1)\}$$

$$F^c(e_4) = \{(a,1), (b, 1), (c,1)\}$$

Definition: 6.2.22

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E) .

The fuzzy soft interior of F_A is defined as the union of all fuzzy open sets contained in F_A and is denoted by F_A^o . We write

$$F_A^0 = \bigcup \{ G_B : G_B \text{ is a fuzzy soft open and } G_B \subseteq F_A \}.$$

It is obvious that

- i. F_A^0 is fuzzy soft open
- ii. F_A^0 is the largest fuzzy soft open set contained in F_A .

Example: 6.2.23

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{ F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0.1)\}$$

$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G_B = \{ G(e_1) = \{(a, 0.6), (b, 0.1), (c, 0.1)\}$$

$$G(e_2) = \{(a, 0.7), (b, 0.9), (c, 0.5)\}$$

$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$G(e_4) = \{(a, 0.5), (b, 0.3), (c, 0.9)\}$$

We consider a fuzzy soft set topology τ on (U, E) as $\tau = \{ \tilde{\Phi}, \tilde{E}, F_A, G_B \}$.

We consider a fuzzy soft set H_C over (U, E) as

$$H_C = \{ H(e_1) = \{(a, 0.6), (b, 0.3), (c, 0.2)\}$$

$$H(e_2) = \{(a, 0.7), (b, 0.3), (c, 0.2)\}$$

$$H(e_3) = \{(a, 0.2), (b, 0.8), (c, 0.6)\}$$

$$H(e_4) = \{(a, 0), (b, 0), (c, 0)\}$$

Then

$$H_C^0 = \text{Union of all fuzzy soft open sets contained in } H_C$$

$$= F_A \tilde{\cup} \tilde{\Phi} = F_A = \{ F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0.1)\}$$

$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}$$

Theorem: 6.2.24

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E) .

Then F_A is a fuzzy soft closed set if and only if $\overline{F_A} = F_A$.

Proof:

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E) such that $\overline{F_A} = F_A$. To prove that F_A is a fuzzy soft closed set. We have

$$\overline{F_A} = \tilde{\cap} \{ G_B : G_B \text{ is a fuzzy soft closed and } F_A \subseteq G_B \}$$

$\overline{F_A}$ is fuzzy soft closed, being an arbitrary intersection of fuzzy closed sets. Also $\overline{F_A}$ is fuzzy soft closed and $\overline{F_A} = F_A \Rightarrow F_A$ is fuzzy soft closed.

Conversely, suppose that F_A is fuzzy soft closed in (U, E, τ) . To prove that $\overline{F_A} = F_A$. It is clear from definition that any fuzzy soft closed set G_B , $F_A \subseteq G_B \Rightarrow \overline{F_A} \subseteq G_B$.

Since $F_A \subseteq F_A \Rightarrow \overline{F_A} \subseteq F_A$ and $F_A \subseteq \overline{F_A}$, it follows that $\overline{F_A} = F_A$.

Theorem: 6.2.25

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E) .

Then F_A is a fuzzy soft open set if and only if $F_A^o = F_A$.

Proof:

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E) such that $F_A^o = F_A$. To prove that F_A is fuzzy soft open. We have

$$F_A^o = \tilde{\cup} \{ G_B : G_B \text{ is a fuzzy soft open and } G_B \subseteq F_A \}$$

F_A^o is fuzzy soft open, being an arbitrary union of fuzzy soft open sets. Also F_A^o is fuzzy soft open and $F_A^o = F_A \Rightarrow F_A$ is fuzzy soft open.

Conversely, suppose that F_A is fuzzy soft open in (U, E, τ) . To prove that $F_A^o = F_A$. It is clear from definition that any fuzzy soft open set $G_B \subseteq F_A \Rightarrow G_B \subseteq F_A^o$.

Since $F_A \subseteq F_A \Rightarrow F_A \subseteq F_A^o$ and $F_A^o \subseteq F_A$, it follows that $F_A^o = F_A$.

Theorem: 6.2.26

Let (U, E, τ) be a fuzzy soft topological space. Let F_A, G_B be a fuzzy soft set over (U, E) . Then

- i. $\overline{\tilde{\Phi}} = \tilde{\Phi}$
- ii. $F_A \subseteq \overline{F_A}$
- iii. $F_A \subseteq G_B \Rightarrow \overline{F_A} \subseteq \overline{G_B}$
- iv. $\overline{F_A \tilde{\cup} G_B} = \overline{F_A} \tilde{\cup} \overline{G_B}$
- v. $\overline{F_A \tilde{\cap} G_B} = \overline{F_A} \tilde{\cap} \overline{G_B}$
- vi. $\overline{\overline{F_A}} = F_A$

Theorem: 6.2.27

Let (U,E,τ) be a fuzzy soft topological space. Let F_A, G_B be a fuzzy soft set over (U,E) . Then

- i. $\tilde{\Phi}^O = \tilde{\Phi}$
- ii. $\tilde{E}^O = \tilde{E}$
- iii. $F_A \subseteq G_B \implies F_A^O \subseteq G_B^O$
- iv. $(F_A^O)^O = F_A^O$

Definition: 6.2.28

Let (U,E,τ) be a fuzzy soft topological space. Let Y be an ordinary subset of U and H_E be a fuzzy soft set over (U,E) such that

$$\forall \varepsilon \in E, \mu^\varepsilon H_E(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

Let $T_Y = \{H_E \tilde{\cap} G_B : G_B \in \tau\}$.

It can be verified that T_Y is a fuzzy soft topology on (Y,E) . We would call T_Y the **fuzzy soft subspace** topology for (Y,E) .

Example: 6.2.29

Let $U = \{a,b,c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{F(e_1) = \{(a,0.4), (b,0.1), (c,0)\}\}$$

$$F(e_2) = \{(a,0.6), (b,0.5), (c,0.2)\}$$

$$F(e_3) = \{(a,0), (b,0), (c,0)\}$$

$$F(e_4) = \{(a,0), (b,0), (c,0)\}$$

$$G_B = \{G(e_1) = \{(a,0.6), (b,0.1), (c,0)\}\}$$

$$G(e_2) = \{(a,0.7), (b,0.9), (c,0.5)\}$$

$$G(e_3) = \{(a,0), (b,0), (c,0)\}$$

$$G(e_4) = \{(a,0.5), (b,0.3), (c,0.9)\}$$

We consider a fuzzy soft set topology τ on (U,E) as $\tau = \{\tilde{\Phi}, \tilde{E}, F_A, G_B\}$.

Let $Y = \{a,b\} \subseteq U$.

We consider a fuzzy soft set H_E over (Y,E) as

$$H_E = \{H(e_1) = \{(a,1), (b,1), (c,0)\}\}$$

$$H(e_2) = \{(a,1), (b,1), (c,0)\}$$

$$H(e_3) = \{(a,1), (b,1), (c,0)\}$$

$$H(e_4) = \{(a,1), (b,1), (c,0)\}$$

Then

i. $\tilde{\Phi} \tilde{\cap} H_E = \tilde{\Phi}$

ii. $\tilde{E} \tilde{\cap} H_E = \tilde{E}$

iii. $F_A \tilde{\cap} H_E = F_A = \{ F(e_1) = \{(a,0.4),(b,0.1),(c,0)\}$

$$F(e_2) = \{(a,0.6),(b,0.5),(c,0.2)\}$$

$$F(e_3) = \{(a,0),(b,0),(c,0)\}$$

$$F(e_4) = \{(a,0),(b,0),(c,0)\}$$

iv. $G_B \tilde{\cap} H_E = G_B = \{ G(e_1) = \{(a,0.6),(b,0.1),(c,0)\}$

$$G(e_2) = \{(a,0.7),(b,0.9),(c,0.5)\}$$

$$G(e_3) = \{(a,0),(b,0),(c,0)\}$$

$$G(e_4) = \{(a,0.5),(b,0.3),(c,0.9)\}$$

Thus the collection $T_Y = \{ H_E \tilde{\cap} G_B : G_B \in \tau \}$ is a fuzzy soft topology on (Y,E) .