COUNTABLE CONNECTED SPACE

BY

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INTRODUCTION

INTRODUCTION

> "There are no rules of architecture for
> a castle in the clods"
>
> G.K.CHESTERTON

The main aim of this dissertation is to study a few countable connected spaces with different interesting characteristics. Here we discuss the following articles.

1. V.Kannan and M. Rajagopalan, Regularity and Dispersion in countable spaces (6)
2. Prabir Roy, A countable connected Urysohn space with a dispersion point (10)
3. R.W. Bagley, Another way of constructing countable connected Hausdorff spaces (1)
4. Ivan Baggs, A connected Hausdorff space which is not contained in a maximal connected space(3)

The concept of a dispersion point was first introduced by Knaster and Kuratowski (8).

A point x of a connected Hausdorff space X is called a dispersion point if $X | \{x\}$ is totally disconnected.

In 1966, J. Martin (4) gave the first example of a countable Hausdorff space with a dispersion point. In the same year Prabir Roy (10) gave the first example of a countable Urysohn space with a dispersion point. In all these spaces there is atmost one point of regularity. Prabir Roy (10) posed a question whether such spaces can be regular at a dense set of points. In 1972, V.Kannan and M. Rajagopalan (6) answered Prabir Roy's question affirmatively in their article "Regularity and Dispersion in countable spaces" (6).

The interesting feature of the results of R.W.Bagley is that a countable connected Hausdorff space can be constructed having any given countable Hausdorff space as a subspace.

Ivan Baggs in his paper studies a modification (X, σ) of the example constructed by Prabir Roy. The topology σ is such that if γ is finer than σ and if (X,γ) is connected then there exists a topology γ' on X such that γ' is finer than γ and (X, γ') is connected. Thus the space (X,σ) cannot be embedded in any maximal connected space.

First chapter deals with the preliminary definitions and notations.

In chapter II, examples obtained by Knaster and Kurtowski (8), J. Martin (4), Prabir Roy (10), V.Kannan and M. Rajagopalan (6) are analysed.

In chapter III the example of a countable connected Hausdorff space constructed by R.W. Bagley (1) is studied in detail.

A detailed description of the example of Ivan Baggs is presented in the fourth chapter.

CHAPTER I

CHAPTER I

PRELIMINARY DEFINITIONS AND NOTATIONS

Definition 1.1

A topological space (X, τ) is said to be T_1- space or Frechet space iff given any pair of distinct points x and y of X there exist open sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

Definition 1.2

A topological space (X, τ) is said to bea Hausdorff space (T_2 - Space) iff for every pair of distinct points of x,y of X there exist disjoint neighbourhoods of x and y, that is, there exist neighbourhoods N of x and N of y such that $N \cap M = \phi$.

Remark 1.3

Every discrete space is Hausdorff.

Definition 1.4

A set is said to be countable if it is either finite or countably infinite.

Definition 1.5

A topological space (S, τ) is said to be connected if it is not disconnected. (ie) we cannot express $S = A \cup B$ where A and B are two non-empty disjoint open sets of S.

Definition 1.6

A topological space (S, τ) is said to be disconnected if we can express S as $S = A \cup B$ where A and B are two non-empty disjoint sets of S.

Definition 1.7

A space (S, τ) is said to be totally disconnected if its only connected subsets are one-point sets.

Definition 1.8

Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

Result 1.9

A subspace of a regular space is regular.

Result 1.10

Any regular space is Hausdorff.

Definition 1.11

Suppose that one-point sets are closed in X. The space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B respectively.

Definition 1.12.

A topological space (S, τ) is said to be Urysohn space iff $x, y \in S$ with $x \neq y$ implies that there exist open sets $U, V \in \tau$ with $x \in U$ and $y \in V$ and $\overline{U} \cap \overline{V} = \phi$.

Definition 1.13.

Let (S, τ) be a topological space. Let $A, B \subset 2^S$. We say A is dense in B if $B \subset \overline{A}$ ($\overline{A}$, the closure of A).

Definition 1.14

Let (S, τ) be a topological space. Let $A \subset S$. we say A is nowhere dense in S if $\bar{A}$ does not contain any non-empty open set of (S, τ).

Definition 1.15

We define a set $A \subset (S, \tau)$ is perfect if A is closed and $A \subset A'$. (A' - the set of limit of points of A)

Definition 1.16

Let (X, τ) be a connected topological space. A point p in (X, τ) is called a dispersion point if $X - \{p\}$ is totally disconnected.

Definition 1.17

A relation $<$ on a set A is called a linear order relation (or a total order or a simple order) if it has the following properties:

(i) Comparability : For every X and Y in A for which $X \neq Y$ either $X < Y$ or $Y < X$

(ii) Non-reflexivity: For no X in A does the relation $X < X$ hold.

(iii) Transitivity: If $X < Y$ and $Y < Z$ then $X < Z$

A set on which a linear order is defined is called a linearly ordered set or a chain.

Definition 1.18

A point $x \varepsilon A$ is isolated point of A if there exist a neighbourhood of x which does not contain any other point of A other than x.

Definition 1.19

Let $I = [0,1]$ and let $A_1 = I - (\frac{1}{3}, \frac{2}{3})$ be that subset of I obtained by removing those points which lie in the open middle thirds of I, that is $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let A_2 be the subset of A_1 obtained by removing the open middle third of $[0, \frac{1}{3}]$ and of $[\frac{2}{3}, 1]$. Continue this process and define A_3, A_4, ... The set $C = \bigcap_{n=1}^{\infty} A$ is called the cantor set.

Definition 1.20

A topological space (S, τ) is said to be compact if every open covering of S has a finite subcovering.

Example 1.21

A compact space which is not Hausdorff. Consider the topology $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ on $X = \{a, b, c\}$. Since X is finite, it is compact. But X is not Hausdroff since a,b are distinct points, having no disjoint neighbourhoods.

Definition 1.22

The collection $\mathcal{F}$ of non-empty subsets of S satisfying the following two conditions is a filter:

(i) If $A, B \in \mathcal{F} \Rightarrow A \quad B \in \mathcal{F}$

(ii) $A \in \mathcal{F}$ & $A \subset B \Rightarrow B \in \mathcal{F}$

Definition 1.23

A maximal filter in $(S, \mathcal{U})$ is a filter $\mathcal{F} \in (S, \mathcal{U})$ ie, the maximal element in the collection of all filters in $(S, \mathcal{U})$ partially ordered by inclusion ie, a filter that is not properly contain in any other filter in $(S, \mathcal{U})$.

CHAPTER II

CHAPTER II

SOME INTERESTING EXAMPLES OF COUNTABLE CONNECTED HAUSDORFF SPACES

The main aim of this chapter is to answer the question raised by Prabir Roy (10) concerning the points of regularity of a countable space with a dispersion point. The concept of a dispersion point was first introduced by Knaster and Kurotowski (8).

Definition 2.1

Let (X,τ) be a connected topological space. A point p in (X, τ) is called a dispersion point if $X-\{p\}$ is totally disconnected.

Example 2.2

Example of a space with a dispersion point

Let x be any infinite set. Let $p\in X$. Define a topology τ on X by specifying a subset $U\in\tau \Leftrightarrow p\in U$. or $U = \phi$. Clearly (X, τ) is connected topological space and p is a dispersion point. Here $X - \{p\}$ is a discrete space.

It is interesting to note the following result:

Preposition 2.3

A connected T_2 - space cannot have two dispersion points.

Knaster and Kurotowski in 1921 gave the first example of a space with dispersion point. We shall discuss below this example. The space constructed is referred to as "Cantor's Teepee".

Construction of "Cantor's Teepee".

Consider the unit interval $[0,1]$ and the cantor set $C\subset[0,1]$

This set with the relative topology is compact and totally disconnected. In R^2 let $p = (\frac{1}{2},\frac{1}{2})$. Let $L(x)$ denote the line segment joining the point $x \in C$ to the point p. Let X be the union of all lines $L(x)$ with $x \in C$. Let E denote the subset of C consisting of the endpoints of the deleted intervals in the construction of the cantor set C. Let F=C-E. Let $X_E = U\{L(x)/x \in E\}$ and $X_F = U\{L(x)/x \in F\}$. Further Let $Y_E = \{(x,y) \in X_E \ /y \in Q\}$ and $Y_F = \{(x,y) \in X_F/y \in Q\}$.

Finally let $Y = Y_E \cup Y_F$. Y is connected set and $Y-\{p\}$ is totally disconnected. The space Y is called cantor Teepee.

It Y is disconnected there exists a seperation $Y = A \cup B$. We can show that there exists a dense subset S of C such that A contains all the points of Y which lie in the come over S except those which lie on the X-axis. This will imply $\overline{A}=Y$ and therefore $B=\emptyset$ which is a contradiction. Therefore Y is connected.

To prove $Y^* =Y- \{p\}$ is totally disconnected. Consider a connected subset A of Y^* . Then $A \subset L\ (X)$ for some X. But $L(x) \cap Y^*$ is totally disconnected. This implies A can contain only one point.

The first example of countable Hausdorff space with a dispersion point was given by Joseph Martin (4).

Consider a countable collection of C_0, C_1 ,C_2 ,----- of countable dense subsets of the real numbers satisfying the following requirements.

i.$C_i \cap C_j = \emptyset$ if $i \neq j$

ii.if $Z \in \bigcup_{i=0}^{\infty} C_i$ and n is an integer $(Z+n\pi)$ does not belong to $\bigcup_{i=0}^{\infty} C_i$ and $n \notin \bigcup_{i=0}^{\infty} C_i$

iii. Define $X=\{(X,Y)\in E^2/X\in C_y$ and Y is an integer$\}\cup\omega$ Where ω is an ideal point.

Let $n>0$ and let $P=(Z,n)\in X$. Let K be a positive integer and let δ be a positive number less than $\frac{\pi}{4}$. Let $V_{k\delta}$ (p) be the set $X\cap\{(x,y)/y=n,\ Z+n\pi-\delta<x<Z+n\pi+\delta\}$.

Let $U_\delta(p) = (\bigcup_{k=1}^{\infty} V_{k\delta}(p)) \cup \{P\}$.

Suppose $p=(Z,0)$ with $Z>0$. Suppose $n<Z<n+1$.

In this case we define

$$V_{k\delta}(p) = \{(x,y)\ /0\leq y\leq -n,\ Z+k\pi-\delta<x<Z+K\pi+\delta\}$$

$$\text{and } U_\delta(p) = (\bigcup_{k=1}^{\infty} V_{k\delta}(p))\cup\{p\}\ .$$

Finally let $p=\omega$ We define

$U_\delta(p) = X\cap\{(x,y)/y>\frac{1}{\delta}\}$. The family $\{U_\delta(p)/p\in X, \delta<\frac{\pi}{4}\}$ defines a basis for a topology τ on X.

Theorem 2.4

(X,τ) is a countable Hausdorff space.

Proof :

Since each C_l is countable, X is countable. Let p and q be two distinct points of (x,τ). Assume $p=\omega$. Let $q=(Z,n)$. Choose $\delta<\frac{\pi}{4}$ such that $\frac{1}{\delta}>n$. then $(U_\delta(p))\cap(U_\delta(q))=\phi$

Let $p=(Z_1, n_1)$ $q=(Z_2, n_2)$ $n_1>0,\ n_2>0$

If $n_1=n_2$ then $p\neq q \Rightarrow Z_1\neq Z_2$

If $n_1\neq n_2$ then $C_{n_1}\cap C_{n_2}=\phi$ and this implies $Z_1\neq Z_2$. In any case $p\neq q \Rightarrow Z_1\neq Z_2$. Assume $Z_1<Z_2$.

Let $\delta=\min\{d(Z_2, Z_1+K\pi)/k$ is an integer$\}$

$Z_2\notin(Z_1+k\pi-\frac{\delta}{4},\ Z_1+k\pi+\frac{\delta}{4})$ and

$Z_1\notin(Z_2+k\pi-\frac{\delta}{4},\ Z_2+k\pi+\frac{\delta}{4})$

First follows from the definition of δ

$Z_2 \in (Z_1 + k\pi - \delta/4,\ Z_1 + k\pi + \delta/4)$

$\Rightarrow d(Z_2,\ Z_1 + k\pi) < \delta/2$. This contradicts the definition of δ.

To prove the second

$Z_1 \in (Z_2 + k\pi - \delta/4,\ Z_2 + k\pi + \delta/4)$

$\Rightarrow d(Z_1,\ Z_2 + k\pi) < \delta/2$

$\Rightarrow d(Z_2,\ Z_1 - k\pi) < \delta/2$

This again contradicts the definition of δ. Similar argument shows that $(U_{\delta/4}(p)) \cap (U_{\delta/4}(q)) = \phi$

Hence (X, τ) is Hausdorff.

Theorem 2.5

(x, τ) is a connected space.

Proof:

Suppose $X = A \cup B$ is a disconnection of X with $\omega \in A$ and $q \in B$. Since B is open there exists a neighbourhood $U_\delta(q) \subseteq B$. Let $q = (Z_1, n_1)$. choose a number $Z \in C_0$, Such that $-(n2) < Z < -(n2)+1$. Where $n_2 > n_1$ and Such that every neighbourhood of $(Z,0)$ meets $U_\delta(q)$ then $(Z,0) \in \overline{U_\delta(q)} \subseteq B$.

Given a positive integer N, there exists $Z \in C_N$ Such that $-N < Z < (-N)+1$, Since B is open and for every positive integer M there is a point $P_M = (Z_0, 0)$, Such that $Z_0 < -M$, $(Z,N) \in B$. This follows from the definition of neighbourhood system at $(Z_0, 0)$. Now every neighbourhood $U_\delta(\omega)$ contains all $(Z, n) \in X$ fpr which $N > \frac{1}{\delta}$. Therefore ω is a limit point of B. Since B is closed, $\omega \in B$. This isnot possible. Hence there is no disconnection for X.

Theorem 2.6

ω is a dispersion point

Proof :

Assume $p=(Z_1, n_1)$ and $q=(Z_2, n_2)$ are two distinct points of $X-\omega$. As in the proof of theorem 2.4 We see $Z_1 \neq Z_2$. Assume $Z_1 < Z_2$ Let $\delta_1 = \min\{d(Z_2, Z_1+k\pi), k$ is an integer$\}$. Choose such that

1. $\delta<\delta\frac{1}{2}$ and

2. If n is an integer then $(Z_1+n\pi+\delta)$ and $(Z_1+n\pi-\delta)$ do not belong to $\bigcup_{i=0}^{\infty} C_i$

Since $\bigcup_{i=0}^{\infty} C_i$ is countable, it is possible to select a δ satisfying (1) & (2). For each n,

Let $A_n = X\cap\{(x,y)/0\leq Y, (Z_1+n\pi-\delta)<x<(Z_1+n\pi+\delta)\}$

Let $A = \bigcup_{n=-\infty}^{\infty} A_n$. Then $p\in A$ and $q\notin A$. From the very definition, A is open. A is also closed since $Z_1+n\pi-\delta$ and $Z_1+n\pi+\delta$ do not belong to $\bigcup_{i=1}^{\infty} C_i$.

Roy (10) gave the first example of a countable Urysohn space with a dispersion point.

The construction of such a space is a slight modification of the Martin space.

Example 2.7

Consider $\{C_i\}_{i=-\infty}^{\infty}$ a countable collection of subsets of rational numbers Such that $C_i\cap C_j = \phi$ for $i \neq j$ and each C_i is dense in R - the space of reals. Let ω denote an ideal point. Our space E consists of the point ω and the set

$\{(x,y)/y$ is an integer, $x\in C_y\}$

for a point $p=(x,y)\in$ E define for each integer n, $D_n(p)$, a neighbourhood of p in E as follows:

If y is even

$D_n(p) = \{(Z,\omega)/\omega=y, Z\in C_\omega$ and $|Z-x|<1/n\}$

If y is odd

$D_n(p) = \{(Z,\omega)/\omega=y,\ Y+1 \text{ or } Y-1,\ Z \in C_\omega \text{ and } |Z-x| < 1/n\}$.

When $p = \omega$,

$Dn\ (p) = \{\omega\} \cup \{(Z,\omega)/\omega \text{ is an integer, } Z \in C_\omega \text{ and } |\omega| \geq 2n\}$

It can be easily checked that the collection of all $D_n(p)$ is the basis of a topology τ on E such that (E, τ) is connected and W is a dispersion point of E.

Further the space E is Urysohn space. To prove this consider two distinct points $P = (x,y)$ and $q = (Z,\omega)$ Choose a positive integer n such that

$1/n < \frac{|x-Z|}{2}$ Then $\overline{D_n(p)} \cap \overline{D_n(q)} = \emptyset$

When $p = \omega$ and $q = (x,y)$. Let n be a positive integer such that $n \geq |y+2|$. Then $\overline{D_n(p)} \cap \overline{D_n(q)} = \emptyset$.

A simpler example of a countable connected Urysohn space with a dispersion point is contructed by V.Kannan (5)

Let R be the real line with the usual topology and let R^* be its one point compactification. Let $R^* = R \cup \{\infty\}$. Let Q be a copy of the set of rationals which is disjoint from R^* .

For each $q \in Q$. Let q' denote the corresponding rational number in R^* . Let $Y = R^* \cup Q$. We deine a topology on Y as follows:

The set R^* belongs to τ and every point in R^* has the usual neighborhood system as a point of R^* .

For a point $q \in Q$ given an $\epsilon > 0$ We define the ϵ-neighbourhood $U_\epsilon(q)$ to consists of

1. $\{q\}$
2. the usual ϵ- neighbourhood of $q'-\sqrt{Q}$ in R^*
3. the usual ϵ-neighbourhood of $q'+\sqrt{Q}$ in R^*

It can be checked that τ is a topology on τ .

Let X be the space got by delating all irrational points of R^*. This space X is a countable Urysohn space with ∞ as a dispersion point.

In all these examples there is atmost one point regularity. Roy (10) asked whether such spaces can be regular at a dense set of points. An affirmative answer was given by V.Kannan and M. Rajagopalan (6).

Definition 2.8

A totally disconnected Hausdorff space X is said to admit a regular dispersion point if there exists aspace Y with dispersion point y such that

(a) Y is regular at y and Y is Hausdorff and

(b) X is homeomorphic to $Y \setminus \{y\}$.

Remark 2.9

It can be easily seen that not every totally disconnected Hausdorff space can admit a regular dispersion point. For example, a zero dimensional space cannot.

Theorem 2.10

A countable totally disconnected Hausdorff space X admits a regular dispersion point if and only if X has a countable open cover $\{\mathcal{U}_n / n=1,2,\ldots\}$ such that

(i) $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ for every n=1,2,3....

and (ii) for every n it is true that no nonvoid subset of is clopen in X.

Proof:

Suppose X admits a regular dispersion point. ∴ there exist a space Y with a dispersion point ∞ such that Y is regular at ∞ and $Y\setminus\{\infty\}$ is homeomorphic to X.

∴ we can take Y as $X \cup \{\infty\}$. Write the elements of X as a sequence $x_1, x_2 \ldots\ldots x_n \ldots\ldots$. Since Y is Hausdorff we can choose a neighbourhood $\mathcal{U}_1$ of x_1 such that $\infty \notin \overline{\mathcal{U}}_1$. Consider the closed set $\overline{\mathcal{U}}_1 \cup \{x_2\}$ which does not contain ∞ . The regularity at ∞ implies that there exist an open set such that $\infty \notin \overline{\mathcal{U}}_2 \supset \mathcal{U}_2 \supset \overline{\mathcal{U}}_1 \cup \{x_2\}$. If we have already choosen $\mathcal{U}_n$ then choose $\mathcal{U}_{n+1}$ such that $\infty \notin \overline{\mathcal{U}}_{n+1} \supset \mathcal{U}_{n+1} \supset \overline{\mathcal{U}}_n \cup \{x\}$. This is possible since $X \cup \{\infty\}$ is Hausdorff and regularity at ∞ . Consider $\{\mathcal{U}_n / n=1,2,3\ldots\}$. This collection forms a cover for X since $x_n \in \mathcal{U}_{n+1}$ for every n=1,2,3... .

∴ condition (i) of the theorem is true we also have for every positive integer n, $\infty \varepsilon \overline{\mathcal{U}}$ but $\infty \varepsilon \overline{F}$ for every clopen subsets F of X. Hence condition (ii) of the theorem is true.

Conversely suppose X has a countable subcover $\{\mathcal{U}_n\}$ satisfying given condition (i) and (ii). Add an extra point ∞ to the space X. Consider its basic neighbourhood to the $X\cup\{\infty\}\setminus \overline{\mathcal{U}}_n$ for n=1,2,3.... . Declare X to be open in $X \cup \{\infty\}$ where X retains its topology. From the condition (i) we get $X\cup\{\infty\}$ is Hausdorff and $X\cup\{\infty\}$ is regular at ∞ , condition (ii) implies that X is connected.

Theorem 2.11

There exist a countable connected Hausdorff space with a dispersion point which is regular at a dense set of points.

Proof :

To prove this theorem, we need the following lemmas and a construction.

Lemma 2.12

There exist a countable totally disconnected Hausdorff space which is regular at a point but is not zero-dimensional at that point.

Proof :

Consider a countable space which has a regular dispersion point. Also assume that X contains a dense subspace homeomorphic to the space Q of all rational numbers (for example X may be the space obtained from Roy's example (10) after deleting the dispersion point).

for each positive integer n, let X_n be a homeomorphic copy of X and let $\sum X_n$ be the disjoint sum. Let x be a point not in $\sum X_n$. Let Y be the set $(\sum X_n) \cup \{x\}$. Define a topology on Y as follows: Declare that each X_n in Y to be open and closed, where X_n retains its topology. Next to define the neighbourhoods of x. By theorem 2.10 there exist open sets $\{ \mathcal{U}_{ij} \;/\; i,j \text{ are positive integer} \}$ such that for each i

(1) $\overline{\mathcal{U}_{ij}} \subset \mathcal{U}_{i,j+1}$, j=1,2,3...

(2) $\bigcup_{i=1}^{\infty} \mathcal{U}_{ij} = X_i$

(3) $\mathcal{U}_{ij}$ contains no clopen subset of X_i . For every positive integer n, put

$$B_n = \left[\bigcup\{\mathcal{U}_{ij} / i \geq n,\ j \geq n\} \cup \{x\}\right] - \left[\bigcup_{i \geq n} \mathcal{U}_{i,\,n-1}\right]$$

geometrically, each X_i , can be viewed to be made up of a countable number of levels by regarding $\mathcal{U}_{ij} \setminus \mathcal{U}_{ij}$ as the j^{th} level of X_i. Then B_n is obtained by omitting from Y the first n-1 copies of X and from each other copy of the first n-1 levels.

$\therefore$ $B_n \supset B_{n+1}$ $\forall$ n = 1,2,.... and $\bigcap_{n=1}^{\infty} B_n = \{x\}$

$\therefore$ $\{B_n / n = 1,2,\ldots..\}$ is a neighbourhood base at x.

$\therefore$ we get a topology on Y. It can be proved that Y is a Hausdorff space and $B_n \supset \overline{B_{n+1}}$, n=1,2,...

$\therefore$ the space is regular at x. It can also be proved that the only clopen sets in Y containing x are those contained in the complements of $\bigcup_{i \in I} X_i$ where I is a finite subset of positive integers.

$\therefore$ the space is not zero dimensional at x. Hence the lemma.

Lemma 2.13

There exists a countable totally disconnected Hausdorff space which is regular at a dense set of points and of which not every point of regularity is a point of zero-dimensionally. (That is, there is a space having all the properties stated in lemma 2.12 with the additional property that it is regular at a dense set of points)

Proof:

Consider the space Y constructed in the above lemma 2.12 change the topology of Y at some points in order to get the required extra property.

We first observe that Y/ {x} contains a dense subset D whose complement is also dense in Y. (for example, D may be taken as the union of all copies of dyadic rationals, one copy in each X_i, contained densely in X_i).

Declare D to be a discrete open subset of Y. ∴ the topology of Y is increased.

Let Y_1 be the new space. It can be seen that the special point x of Y_1 is still a point of regularity but not a point of zero-dimensionality because the clopen sets containing x are contained in the complements of $\bigcup_{i\in I} X_i$, where I is a finite subset of positive integers.

Y_1 is totally disconnected since its topology is finer than a totally disconnected topology. It is regular at a dense set of points, namely, at the points of D (which are isolated). Hence the lemma.

Construction 2.13

Consider the space Y_1 constructed in the lemma 2.13. Y_1 has a dense set of isolated points. For each isolated point of Y_1 take a disjoint copy of Y_1 and identify its "special" point with that isolated point. When this is done for each isolated point of Y_1, we get a bigger space Y_2 containing Y_1 as a closed subspace. Also Y_2 has a dense set of isolated points. Attaching a copy of Y_1 to each isolated point of Y_2 at the special point of Y_1, we get a still bigger space Y_3 containing Y_2 as a closed subspace.

Repeating this process we construct by induction a direct limit system $\{Y_n, \alpha_{mn}\}$ of spaces $Y_1, Y_2 \ldots\ldots$ and homeomorphic embeddings $\alpha_{mn} : Y_m \to Y_n$ where $m \leqslant n$. Let L denote the direct limit system.

Lemma 2.14

The space L constructed above satisfies the covering condition of Theorem 2.10.

Proof:

We want to show that L has a countable open cover $\{\mathcal{U}_n / \ n = 1, 2, \ldots\}$ Such that

(i) $\overline{\mathcal{U}_n} \subset \mathcal{U}_{n+1}$ for every $n = 1, 2, \ldots$ and

(ii) there is no n such that $\mathcal{U}_n$ contains a clopen subset of L.

Note that the direct limit L is the set-union of Y_i's $i = 1, 2, \ldots$ with weak topology.

In Y_1 the special point y has the property that Y_1 is regular at y but not zero-dimensional at y. Consequently, there is a neighbourhood $\mathcal{U}_1$ of Y in Y_1 which contains no clopen subset of Y_1. For each positive integer n we choose an open neighbourhood V_r of y in Y_1 such that $V_r \subset \bar{V}_{r+1}$ for each r = 1, 2, ... It is possible to select such a sequence V_1, V_2, ... of open neighbourhoods of y since Y_1 is regular at y. We define $V_n = Y_1$ for each integer $n \leqslant o$.

If $x \in L$, then there is a unique integer m such that $x \in Y^m \setminus Y^{m-1}$. Also, no point of Y^m is isolated in Y^{m+1}. Therefore for each $x \in L$ there is at most one integer m such that x is isolated in Y^m. For those x for which such an integer exists let us denote by Y_x the copy of Y_1 attached at x (in the contruction of Y_{m+1}) Let f_x ; $Y_1 \to Y_x$ be a fixed homeomorphism.

Let r be any positive integer. Define $\mathcal{U}_r$ to be the smallest subset of L containing Y_1 such that the following holds. If $x \in \mathcal{U}_r$ and if x is isolated in Y^m, then it is true that $f_x(V_{m-r})$ is contained in $\mathcal{U}_r$. We claim that the family $\{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ has the required properties.

It can be checked in a straight forward way that each $\mathcal{U}_r$ is open. Since $f_n(V_{m-r}) = Y_x$ whenever $m \leqslant r$, it follows that $\mathcal{U}_r \supset Y_r$ for each positive integer r. An immediate consequence is that the above family covers X.

To prove that $\bar{\mathcal{U}}_r \subset \mathcal{U}_{r+1}$ it suffices to prove that $\bar{\mathcal{U}}_r$

$\cap Y_x \subset \mathcal{U}_{r+1} \cap Y_x$ for each $x \in L$ isolated in some Y_m. When $m \leqslant r+1$, then $\mathcal{U}_{r+1} \supset Y_x$, and so there is nothing to prove. If $m > r+1$, then $\overline{\mathcal{U}}_r \cap Y_x$ can be prove to be $f_x\ (\overline{V}_{m-r})$. This is certainly in $\mathcal{U}_{r+1} \cap Y_x = f_x\ (V_{m-(r-1)})$ since f_x is a homeomorphism and $\overline{V}_{m-r} \subset V_{m-r-1}$. (If $x \in \mathcal{U}_n$ is isolated in some Y_m, then $\mathcal{U}_n$ contains the colpy of V_{m-1} in that copy aof Y_1 which is attached at x and c ontains nothing more in that copy).

We have seen that the following assertions are true.

(i) Each $\mathcal{U}_n$ is open

(ii) $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ for each n = 1, 2, ...

(iii) $L = \bigcup_{n=1}^{\infty} \mathcal{U}_n$

we further claim the following

(iv) No $\mathcal{U}_n$ contains a clopen subset of L.

We have observeed that L is set - union of Y_1, Y_2, . where each Y_2, Y_3, ... is a set - union of a countable number of copies of Y_1. Also L has the weak topology given by these copies of Y_1. Therefore a set $F \subset L$ is open (closed) in L if and only if its intersection with each such copy of Y_1 is open (closed) in that copy.

Fix some positive integer n and consider $\mathcal{U}_n$. Let $x \in \mathcal{U}_n$ be an isolated point of Y_{n+1}. Let $F \subset L$ be any clopen set. Let z be the copy of Y_1 attached to the point x. Then by the above observation $F \cap Z$ is clopen in Z. If $\mathcal{U}_n \supset F$, then $\mathcal{U}_n \cap Z \supset F \cap Z$. But $\mathcal{U}_n \cap Z$ is the copy of V_1 in Z by the definition of V_r. This implies that in the original copy Y_1, V_1 contains a clopen subset of Y_1, which is not true.

This proves that $\mathcal{U}_n$ cannot contain any clopen subset of L..

Proof of the theorem 2.11 :

Consider the space L of construction 2.13. By lemma 2.14, L satisfies the covering condition of theorem 2.10. It can be easily verified that L is a totally disconnected Urysohn space. Therefore by theorem 2.10 L admits a regular dispersion point.

If we set $S = L \cup \{\infty\}$, declare L to be open in S, where S is retaining its topology, and declare the basic neighbourhood of the extra point infinity to be $S \setminus \mathcal{U}_n$ for $n = 1, 2, \ldots,$ then S is a countable connected Hausdorff space with a dispersion point.

Let $A = \{x \in L / x \text{ is isolated in some } Y_n\}$. Then it can be proved that A is dense in L, and therefore, in S. Using the regularity of the spoecial point y in the space Y_1, we can prove that S is regular at every point of A.

Let x be any point of A and let V be any open neighbourhood of x in S such that $\infty \notin V$. Then for each point $y \in A \cap V$ there is a neighbourhood V_y of y in A_y such that $V_y \subset V$, where A_y denotes the copy of Y_1, attached at y. Since A_y is regular at y, there exists an open neighbourhood W_y of y in A_y such that $\bar{W}_y \subset V_y$. Let W be the smallest subset of L containing x with the following property. Whenever $y \in W \cap A$, it is true that $W_y \subset W$. Then it can be proved that W is open in L and that $\bar{W} \subset V$.

Thus the space S satisfies all the conditions required in the theorem.

CHAPTER III

CHAPTER III

A NEW METHOD OF CONSTRUCTING COUNTABLE CONNECTED HAUSDORFF SPACES

Starting with an arbitrary countable Hausdorff space we construct a countable connected Hausdorff space by repeatedly attaching a countable Hausdorff space which is not Uryshon.

A simple example of a countale connected Hausdorff space can be constructed as follows:

Construction of a countable connected Hausdorff space

Let $X = A \cup B \cup C$. Let $A = B =$ the set of rationals in the interval $[0,1]$ excluding reciprocals of integers. Let $C =$ the set of reciprocals of all positive integers. On A and B the topology is the relative topology of reals.

Define the nieghbourhoods of $\frac{1}{n}$ in C as follows:

$$\{\tfrac{1}{n}\} \cup [(\tfrac{1}{n}-\varepsilon, \tfrac{1}{n}+\varepsilon) \cap (A \cup B)]$$

Here X is a countable Hausdorff space such that the points $a = 0 \in A$, $b = 0 \in B$ cannot be separated by open sets with disjoint closures.

Let Y, Z be topological spaces. Let C be a closed subset of Z. Let f be a continuous function from C into Y. Let $Z \cup_f Y$ be the adjunction space defined as follows:

The points of $Z \cup_f Y$ are those of $Z - C$ and Y and the open sets are those whose intersection with $Z - C$ and Y are open in Z and Y respectively. Let Y_1 be any countable Hausdorff space and choose $C \in Y_1$. Consider a family of functions F_1 from $\{a,b\}$

into Y_1 such that for every $y \in Y_1$ there is $f \in F_1$ such that $f(a) = c$ and $f(b) = y$. For each $f \in F_1$ we attach X to Y_1 by f and denote the resulting space by $X \cup_f Y_1$. Let $Y_2 = \bigcup_{f \in F_1} X \cup_f Y_1$. In this union the points of each $X \cup_f Y_1$ which are in $X - \{a,b\}$ are considered distinct from the same points of each other set. Thus, Y_2 contains only one copy of Y_1 but a copy of $X - \{a,b\}$ for each $f \in F_1$. A set is Y_2 is open if and only if it intersects each $X \cup_f Y_1$ in an open set. We define a sequence $\{Y_n\}$ as follows:

Assuming F_{k-1} and Y_k have been defined for $k \leq n$, we define F_n to be a family of functions from $\{a,b\}$ into Y_n such that for each $y \in Y_n$ there is $f \in F_n$ with $f(a) = c$ and $f(b) = y$. Then $Y_{n+1} = \bigcup_{f \in F_n} X \cup_f Y_n$ we prove that each Y_n is Hausdorff by induction, remembering that Y_1 is Hausdorff by assumption. Assume Y_{n-1} is Hausdorff. Let $x, y \in Y_n$. Assume $x, y \in Y_{n-1}$. Therefore we choose U_1 and V_1 open in Y_{n-1} seperating X and Y

Let $E = \{ f \in F_{n-1} \,/\, f(a) \text{ or } f(b) \in U_1 \}$

Let $D = \{ f \in F_{n-1} \,/\, f(a) \text{ or } f(b) \in V_1 \}$

Choose U and V, which are disjoint open subsets of X with $a \in U$ and $b \in V$. For each $f \in F$ we let $U_f = U$ or $U_f = V$ depending on whether $f(a) \in U_1$ or $f(b) \in U_1$ correspondingly we define V_f for $f \in D$ $V_f = U$ or $V_f = V$ depending on whether $f(a) \in V_1$ or $f(b) \in V_1$. In case f(a) and f(b) are in U_1 or V_1 we let $U_f = U \cup V$ or $V_f = U \cup V$ of course, if f and g are any two functions. $U_f \cup U_g$ contains atmost one point, namely f(a) or f(b). Similarly $V_f \cap V_g$ contains at most one point. It is clear that the sets $\bigcup_{f \in E} U_f \cup U_1$ and $\bigcup_{f \in D} V_f \cup V_1$ are open in Y_n and seperate x and y. Now, if not both x,y are in Y_{n-1}, then one of them is in X and the seperation is easy.

Consider $Y = \cup Y_n$ with the union topology then this Y is countable connected Hausdorff space. To prove Y is Hausdorff.

Consider $x, y \in Y$. Since each Y_n is Hausdorff we obtain for each n for which x,y are in Y_n disjoint sets U_n and V_n open in Y_n, such that $x \in U_n$, $y \in V_n$. The sequences $\{U_n\}$ and $\{V_n\}$ are monotone increasing. To see that the sequences $\{U_n\}$ and $\{V_n\}$ can be made monotone we note that in the construction of U_n and V_n above, the sets U_1 and V_1 can be taken as U_{n-1} and V_{n-1} respectively.

$\therefore$ $\cup U_n$ and $\cup V_n$ are open in Y and seperate x and y.

Next to prove Y is connected.

Assume $Y = C \cup D$ where C and D are open. Suppose the fixed points c of Y_1 is in C choose $d \in D$ and a positive integer n such that $d \in Y_{n-1}$. $\therefore$ there is a $f \in F_{n-1}$ such that $f(a)=c$ and $f(b)=d$. Then considering U_f and V_f defined above as subsets of Y_n we have $\overline{C} \cap \overline{D} \supset \overline{C \cap Y_n} \cap \overline{D \cap Y_n} \supset \overline{U}_f \cap \overline{V}_f \neq \phi$, since $U \cap V \neq \phi$ in X.

$\therefore$ Y is connected.

Hence Y is a countable connected Hausdorff space. This completes the proof.

CHAPTER IV

CHAPTER - IV

A CONNECTED HAUSDORFF SPACE WHICH IS NOT CONTAINED IN A MAXIMAL CONNECTED SPACE.

In this chapter an example of a countable connected Hausdorff space (x, σ) is given which has the property that for every topology τ strictly larger than σ, where (x, τ) is connected, there exists a topolocy τ', strictly larger than τ such that (x, τ') is connected. There also exists uncountable connected Hausdorff spaces which have this property.

Definition 4.1

Let X be a set and let A and B be families subsets of X, then $A \overset{\bullet}{\vee} B$ denotes the topology generated by the sub base $P = \{C/C \in A \text{ or } C \in B\}$

Definition 4.2

A connected topological space (X, τ) is maximal connected if for every topology τ_1, where τ_1 is strictly larger than τ, (X, τ_1) is not connected.

Remark 4.3

Let X be a set and τ_n, $n = 1, 2, \ldots$ be a sequence of topologies on X such that (X, τ_n) is conneected for n = 1, 2,.... If τ is the topology which is the least upper bound of the sequence τ_n, n = 1, 2, ... then (X, τ) is not necessarily connected.

Definition 4.4

A topological space (X, τ) is maximal perfect if (X, τ) has no isolated points and for every topology τ_1 strictly larger than τ, (X, τ_1) has an isolated point.

Lemma 4.5

If (X, τ) is any topological space without isolated points, then there exists a topology $\tau_1 \supset \tau$ such that (X, τ_1) is maximal perfect.

Proof :

Let $\{\tau_\alpha\}_{\alpha \varepsilon A}$ be a linearly ordered family of topologies on X such that, for each $\alpha \varepsilon A$, $\tau_\alpha \supset \tau$ and (X, τ_α) has no isolated points. Let σ be the topology generated by $\{\tau_\alpha\}_{\alpha \varepsilon A}$. If there exists some $x \varepsilon X$ such that $\{x\} \varepsilon \sigma$, then, $\{x\} \varepsilon \tau_\alpha$ for some α. (Since $\{\tau_\alpha\}_{\alpha \varepsilon A}$ is linearly ordered).

$\therefore$ (X, σ) has no isolated points.

$\therefore$ By zorn's lemma which states that

"If P is a partially ordered set in which every chain has an upper bound then P possess a maximal bound".

There exist a topology $\tau_1 \supset \tau$ such that (X, τ_1) is maximal perfect.

Notation :

Let P denote a dense subset of the rationals with the topology τ inherited from R where R denote the real line with the usual topology τ.

Definition 4.6

A set $G \subset P$ is an N - set if $G = \phi$ or if for each $x \in G$ and for every $b > x$, $\{y \in G / x < y < b\}$ has non empty τ - interior.

Definition 4.7

A collection $\mathcal{G}$ of N - sets which is closed under finite intersection is an N - family.

Note 4.8

By zorn's lemma, we have every N - family is contained in a maximal N - family.

For each $x \in P$, let $I_x = \{y \in P / x \leqslant y < x + 1\}$

Put $B = \{I_x / x \in P\} \cup \{o \subset P / o \in \tau\}$

B is contained in an N - family, B_1.

Let $\mathcal{M}$ be a maximal N - family containing B_1.

If D is a subset of P, then D^c denotes the complement of D in P. Let $Q = \{D \subset P / D^c$ is τ - no where dense in P$\}$. Then Q is connected in a filter $\mathcal{F}$, where $\mathcal{F}$ is maximal with respect to properties that (i) $Q \subset \mathcal{F}$ and (ii) if $F \in \mathcal{F}$, then F is τ - dense in P. Throughout the section $\mathcal{M}$ denotes a maximal family of N - sets and $\mathcal{F}$ denote a maximal filter of dense sets constructed as above.

Lemma 4.9

Let $\sigma = \mathcal{M} \vee \mathcal{F}$ then $\tau \subset \sigma$ and (P, σ) is a maximal perfect space.

Proof :

Since $\tau \subset \mathcal{M}$, we have $\tau \subset \sigma$. Let $x \in P$ and $x \in F \cap M$, where $F \in \mathcal{F}$ and $M \in \mathcal{M}$. Since M contains a τ - open set **U** and $F \cap U$ is τ - dense in U, it follow that $F \cap M \neq \{x\}$. $\therefore$ (P, σ) is perfect.

Next to prove (P, σ) is maximal perfect. Suppose (P, σ) is not maximal perfect then there exist a topology γ on P such that γ is strictly larger than σ and (P, γ) is perfect. Let $V \in \gamma$ such thuat $V \notin \sigma$. Let x_o be an arbitrary fixed element of V. For every $b > x_o$, we have $\{x \in P/x_o < x < b\}\ V \neq \phi$ (otherwise for some $b > x_o$,

$V \cap \{x \in P/x_o - 1 < x < b\} \cap I_{x_o} = \{x_o\}$, where $I_{x_0} = \{x \in P / x_0 < x < x_0 + 1\}$ and $\{x_0\}$ would be an element of γ which is impossible).

Suppose there exists some $b > x_0$ such that $D = \{x \in P / x_0 < x < b\} \cap V$ is τ - no where dense in

$\{x \in P / x_0 < x < b\}$. Then $D \in \gamma$ and it follows from the construction of $\mathcal{J}$ that $D^c \in \mathcal{J}$. If y is a fixed element of D, then $(D^c \cup \{y\}) \in \mathcal{J}$ and $D \cap (D^c \cup \{y\}) = \{y\} \in \gamma$, which is impossible. Therefore, for every $b > x_0$, $\{x \in P / x_0 < x < b\} \cap V$ is τ - dense in $I \cap P$, for some open interval $I \subset R$.

Let $\{J_n\}_{n=1}^{\infty}$ be a family of disjoint τ - open subsets of P whose union is the τ- interior of the τ- closure of $V \cap \{x \in P / x > x_0\}$.

For each positive integer n, there exists an $F \in \mathcal{J}$ such that $V \cap J_n = F \cap J_n$. (Suppose there exists some n such that for each $F \in \mathcal{J}$, $V \cap J_n \neq F \cap J_n$.

Since $\mathcal{J}$ is a maximal filter of τ - dense subsets of P and since $V \cap J_n$ is τ - dense in J_n, this implies that there exists some $F_1 \in \mathcal{J}$ such that $(V \cap J_n) \cap F_1$ is not τ - dense in J_n. Hence there exists some τ - open set $L \subset J_n$ such that $L \cap F_1 \cap (V \cap J_n) = \phi$. If x is a fixed element of $L \cap (V \cap J_n)$, then $(F_1 \cup \{x\}) \in \mathcal{J}$ and $(F_1 \cup \{x\}) \cap L \cap (V \cap J_n) = \{x\} \in \gamma$, which is impossible).

Put $G = \{x_o\} \cup \{\cup_{n=1} J_n\}$. G is an N - set. Also, $G \varepsilon \mathcal{M}$. (For, if $G \varepsilon \mathcal{M}$, $G \cap G_1$ is not an N - set for some $G_1 \varepsilon \mathcal{M}$. This implies there exists some $x_1 \varepsilon G \cap G_1$ and some $b > x_1$ such that $\{x \varepsilon P/x_1 < x < b\} \cap G \cap G_1$ does not contain a τ- open set. It is clear that $x_1 = x_o$. It follows as a consequence of how G was constructed from V, that $C = V \cap \{ x \varepsilon P/x_o < x < b \} \cap G_1$, is a τ - nowhere dense subset of P. Also, $C \varepsilon \gamma$. $C^c \varepsilon \mathfrak{F}$ and if $y \varepsilon c$, then $(C^c \cup \{y\}) \cap C = \{y\} \varepsilon \gamma$. If C is empty, then, sinnce $\{x \varepsilon P/x_o \leqslant x < b\} \varepsilon \mathcal{M}$, it follows that $V \cap \{x \varepsilon P/x_o \leqslant x < b\} \cap G_1 = \{x_o\} \varepsilon \gamma$. A constradiction, therefore, $G \varepsilon \mathcal{M}$).

Now, put $T = (\cup_{n=1} \{V \cap J_n\}) \cup \{\cup_{n=1} J_n\}^c$. For each τ- open set $U \subset P$, T is τ- dense in U. $\mathfrak{F}$ is a maximal filter of τ - dense subsets of P. If $T \notin \mathfrak{F}$, then there exists some $F' \varepsilon \mathfrak{F}$ such that $T \cup F'$ is not τ - dense in P. By the definition of T, this implies that $F' \cap (V \cap J_n)$ is not dense in J_n for some n. However, this is impossible since $V \cap J_n = F \cap J_n$, for some $F \varepsilon \mathfrak{F}$, and $\mathfrak{F}$ is a filter of τ- dense subsets of P. Therefore $T \varepsilon \mathfrak{F}$ clearly $x_o \varepsilon G \cap T$ and $G \cap T \subset V$. Also $G \cap T \varepsilon \sigma$, Since $G \varepsilon \mathcal{M}$ and $T \varepsilon \mathfrak{F}$. Therefore, $G \cap T \subset V$ is a σ- open neighbourhood of x_o, which contradicts the assumption that $v \notin \sigma$, since x_o is an arbitrary element of V. Hence, (P, σ) is maximal perfect and hence the lemma.

Now we given a example of a countable connected Hausdorff space (X, σ) which is not contained in any maximal connected space. The space (X, σ) is a modification of a countable connected Urysohn space which is given in example 2.7.

Example 4.10

Let $\{E_i\}_{i=+\infty}^{\infty}$ be a countable, disjoint collection of dense subsets of rational numbers indexed by the set of all integers. For each integer n Let $P_n = \{(x,n)/x \in E_n\}$ Let ω denote an ideal point. Let $X = \{\omega\} \cup \{\bigcup_{n=-\infty}^{\infty} P_n\}$. ∴ we may consider X as a collection of points in the plane lying on horizontal lines with integer ordinates together with the point ω . We now construct a neighbourhood system for the points of X.

(a) If n is even, then put a maximal perfect topology σ_n on P_n exactly as described in lemma 4.9.

(b) Let $n \neq 1$ be an odd integer and let $p = (x, n) \in P_n$ Then, for each positive integer m, let

$U_m(p) = \{(y, n+1) \in P_{n+1}/|x-y| < 1/m\} \cup \{(y, n-1) \in P_{n-1}/|x-y| < 1/m\} \cup \{p\}$.

(c) Let $n = 1$ and $p = (x, n) \in P_n$. If $x \notin (\pi/2, \pi)$, then, for each positive integer m, define $U_m(p)$ as in (b).

If $\pi/2 < x < \pi$, then for each positive integer m, let $U_m(p) = \{ (y, n-1) \varepsilon P_{n-1}/|x-y| < 1/m \} \cup \{ p\}$.

(d) If $p = \omega$, then, for each positive integer m, let $U_m(p) = \{ p\} \cup \{ (x, n)/ |n| \geqslant 2m \}$ be a neighbourhood of p.

It is clear that the neighbourhood system described above generates a topology σ on X. X is countable and (X, σ) is a Hausdorff space.

We state the following lemmas which are slight modifications of lemma 4.5, 4.9, 4.11.

Lemma 4.11

Suppose $p = (x, n)$, where n is an even integer. Let U be a basic open set containing p and let m be an integer.

(i) If $p \notin \{ (x, n)/n = 2$ and $\pi/2 < x < \pi\}$, then $(cl_\sigma U) \cap P_m \neq \phi$ if and only $|m-n| \leqslant 1$.

(ii) If $p = (x, 2)$ where $x \varepsilon (\pi/2, x)$, then $(cl_\sigma U) \cap P_m \neq \phi$, if and only if m = 2 or 3

(iii) $\omega \notin cl_\sigma U$.

Lemma 4.12

Suppose $p = (x, n)$, where n is an odd integer. Let U be a basic open set containing p and let m be on integer.

(i) If $p \notin \{(x, n)/n = 1 \text{ or } 3 \text{ and } \frac{\pi}{2} < x < \pi\}$ then $(\mathrm{cl}_{\sigma} U) \cap P_m \neq \phi$, iff $|m - n| \leqslant 2$.

(ii) If $p = (x, 1)$ where $x \in (\frac{\pi}{2}, \pi)$ then $(\mathrm{cl}_{\sigma} U) \cap P_m \neq \phi$, if and only if, $m = 1, 0$ or -1.

(iii) If $p = (x, 3)$, where $x \in (\frac{\pi}{2}, \pi)$ then $(\mathrm{cl}_{\sigma} U)\ P_m \neq \phi$ if and only if $m = 2, 3, 4$ or 5.

(iv) $\omega \notin \mathrm{cl}_{\sigma} U$.

Lemma 4.13

Suppose $p = \omega$, m is a positive integer greater than 1, and n is an integer, then

(i) $\mathrm{cl}_{\sigma}\ U_m(p) \supset P_n$, if $|n| \geqslant 2m-1$, and

(ii) $\{\mathrm{cl}_{\sigma}\ U_m(p)\} \cap P_n = \phi$, if $|n| < 2m - 1$.

Lemma 4.14

(X, σ) is connected.

Note:

Let γ be any topology on X such that γ is larger than or equal to σ and (X, γ) is connected. We will now show that there exists a topology γ' on X such that γ' is strictly larger than γ and (X, γ') is connected.

Let I be any open interval in R and, for each integer n, let $I_n = \{(x, n) \epsilon P_n / x \epsilon I\}$. For each odd integer n, let $U(I_n) = \{x \epsilon I / (x, n) \epsilon I_n, (x, n) \epsilon cl_\gamma P_{n+1}$ and $(x, n) \epsilon cl_\gamma P_{n-1}\}$. Throughout I will denote an open interval of the real line with the usual topology. For each integer n, I_n is a subset of X and $U(I_n)$ is a subset of I for each odd integer n.

Lemma 4.15

If I is any open interval contained in $(\frac{\pi}{2}, \pi)$ then $U(I_n)$ is dense in I for all odd integers n, except for n = 1.

Proof:

It follows from condition (c) in the definition of the neighbourhood base for the topology σ on X, that for every open interval $I \subset (\frac{\pi}{2}, \pi)$, $U(I_1) = \phi$. Suppose there exists some interval $I \subset (\frac{\pi}{2}, \pi)$ and an odd integer $n \neq 1$ such that $U(I_n)$ is not dense in I We may assume without loss of generally that I = (a, b), where a and b are irrationals, $U(I_n) = \phi$ and $n > 1$. put $H_n = \{p \epsilon I_n / p \notin cl_\gamma P_{n+1}\}$ Let $H = H_n \cup \{\bigcup_{k=2}^{n-1} I_k\}$. We will show that H is both γ-open and γ- closed

Suppose $p \epsilon H_n$, then $p \notin cl_\gamma P_{n+1}$, So there exists a γ- open neighbourhood, V, of p which consists of p and a subset of I_{n-1}. Hence $V \subset H$. clearly, if $p \epsilon I_k$, k = 2, 3, ... n - 1, there exists a σ- open set containing p and contained in $\bigcup_{k=2}^{n-1} I_k$. Therefore, H is γ- open.

If $p \epsilon I_{n+1}$, then p is not a σ- limit point of H and since $\sigma \subset \tau$, it follows that $p \epsilon cl_\gamma H$. If $p \epsilon I_n$ and $p \notin H$, then p is a γ - limit point of I_{n+1} and since $U(I_n) = \phi$, p must have a γ- open neighbourhood which consists of p and a subset of I_{n+1}. Hence $p \notin cl_\gamma H$. It follows from the construction of σ, that if $p \in P_1$ or $p \in P_0$, then $p \notin cl_\gamma H$. Also, if $p \in P_k$, for k = 2, ... n - 1, and $p \notin H$, there exists a σ - open set containing p which does not meet H. Therefore, H is γ - closed. This contradicts the assumption that (X, γ) is connected. Hence the lemma is established.

We use the previous lemma to put a topology on X such that γ' is strictly larger than γ. Let J be a fixed open interval contained in $(\frac{\pi}{2}, \pi)$. It follows from Lemma 4.15 that $U(J_n)$ is dense in J for all odd integers, except for n = 1. Let y_0 be a fixed element of $U(J_3)$ and put $y = (y_0, 3)$. Let V be an open neighbourhood of y in γ. It follows that $V \cap P_4$ and $V \cap P_2$ are both non - empty. Put $G = \{y\} \cup \{V \cap P_2\}$. Clearly $G \notin \gamma$. Put $\gamma' = \gamma \vee G$. We will show that (X, γ') is connected.

Lemma 4.16

If n is an integer and $V \subset P_{2n}$, where $V \in \gamma'$, then there exists some interval $I \subset R$ such that $\{x/(x, 2n) \in V\}$ is dense in I.

Proof:

Suppose not, then there exists some integer n and a γ'- open Set $V \subset P_{2n}$ such that $\{x/(x, 2n) \in V\}$ is not dense in any sub interval of R. Since $P_{2n} \in \sigma$ and $(P_{2n}, \sigma / P_{2n})$ is maximal perfect, it follov that $V \in \sigma$(otherwise, some point of V would be an isolated point in γ'). Put $F = \{ (x, 2n) \in P_{2n} / (x, 2n) \notin V\}$. Since $\{ x/(x, 2n) \in V\}$ is not dense in any subinterval of R, it follows from the definition of the maximal filter $\mathfrak{F}$ on P_{2n} that $F \in \mathfrak{F}$. If $p \in V$, then $F \cup \{p\} \in \mathfrak{F}$ and

$(F\cup\{p\})\in\sigma$. Therefore, $(F\cup\{p\})\ V = \{p\}$ and $\{p\}\in\sigma$. This is a contradiction. Hence the lemma.

Lemma 4.17

Suppose n is an integer and V and U are open subset of P_{2n} If $\{x/(x, 2n)\in V\}$ and $\{x/(x, 2n)\in U\}$ are both dense in some open interval I, then $V\cap U \neq \phi$.

Proof:

Suppose $U\cap V = \phi$ since $\{x/(x, 2n)\in V\}$ and $\{x/(x, 2n)\in U\}$ are both dense in I, we may assume that $V = I_{2n}\cap F_1$ and $U = I_{2n}\cap F_2$ where $F_1, F_2\in\mathfrak{F}$ and $\mathfrak{F}$ is the maximal filter of dense sets used to construct the neighbourhood system of P_{2n}. If $p\in v$, then $F_2\cup\{p\}\in\mathfrak{F}$. Hence $U\cap\{p\} = (F_2\cap\{p\})\cap I_{2n}$ is a σ - open subset of P_{2n}. However $(U\cap\{p\})\cap V = \{p\}$ and $\{p\}\in\sigma$. This is a contradiction. Hence the lemma

Lemma 4.18

Let U be a γ'- open neighbourhood of ω. If I is any open interval contained in J, then there exists an integer $n\geq 2$ such that $I_n\cap U \neq \phi$.

Proof:

Suppose for some γ'- open set U containing ω, there exists an interval $I\subset J$, $I = (a, b)$ where a and b are irrational numbers, Such that $I_n \cap U = \phi$, for each integer $n \geq 2$. Since every γ'- open set containing ω contains a γ- open neighbourhood of ω, we may assume that $U \varepsilon \gamma$. Put $H = \bigcup_{n \geq 2}^{\infty} I_n$. It follows that H is γ- open. H is also γ- closed. For, if $p \varepsilon I_1$, p has a σ- open neighbourhood which does not meet H by assumption, ω has a γ- open neighbourhood which does not meet H. This contradicts the assumption that (X, γ) is connected and hence the lemma.

From the above lemma we get the following corollary.

Corollary 4.19

Let $I \subset J$. If U is any γ'- open set containing ω, then there exists a infinite set of positive integers, N, such that $I_n \cap U \neq \phi$ for each $n \varepsilon N$.

Lemma 4.20

Let n be an integer such tht $n \geq 1$. Let v be an open subset of P_{2n} such that $\{x/(x, 2n) \varepsilon V\}$ is dense in an open interval $I \subset J$. If $V \subset K$, where K is an open and closed set in (X, γ') then $I_k \subset K$, for all $k > 2n$.

Proof:

First we will show that I_{2n+1} and I_{2n+2} are both contained in K. $U(I_{2n+1}) = \{x \varepsilon I/p = (x, 2n+1) \varepsilon I_{2n+1}$ and p is a γ- limit point of both I_{2n} and $I_{2n+1}\}$. It follows from Lemma 4.15 and the choice of J, that, since $I \subset J$, $U(I_{2n+1})$ is dense in I. Let $p = (x, 2n+1)$, where $x \varepsilon U(I_{2n+1})$. Let u be a γ'- open set containing p (we may, without loss of generality, assume that $U \varepsilon \gamma$). Since $U \cap I_{2n} \neq \phi$, it follows from Lemma 4.16 that there exists some open interval $I' \subset I$ such that $\{x/(x,2n) \varepsilon U\}$ is dense in I'. Also, $\{x/(x,2n) \varepsilon V\}$ is dense in I'. Therefore by lemma 4.17, $V \cap U \neq \phi$. Since $V \subset K$, $U \cap K \neq \phi$ and p is therefore a limit point of K. Since K is γ'-closed $p \varepsilon K$. Therefore, $\{p/p=(x,2n+1)$ and $x \varepsilon U(I_{2n+1})\} \subset K$.

Put $Q = \{x \varepsilon I/q=(x,2n+2) \varepsilon I_{2n+1}$ and $q \varepsilon K\}$. Since $U(I_{2n+1})$ is dense in I, Q is also dense in I. Let $z \varepsilon I_{2n+2}$. It will be shown that z is a γ'-limit point of K. Let H be a γ'-open set containing z. We may assume that $H \subset P_{2n+2}$. Again, by lemma 4.16, there exists an interval $I' \subset I$ such that $\{x/(x, 2n+2) \varepsilon H\}$ is dense in I'. Since Q is dense in I, $K \cap I'_{2n+2} \neq \phi$. This implies that $K \cap I'_{2n+2} \varepsilon \gamma'$ and by lemma 4.16, we may aassume without loss of generality that $K \cap I'_{2n+2}$ is dense in I'_{2n+2}. By lemma 4.17, $(K \cap I'_{2n+2}) \cap H \neq \phi$. Therefore, z is a γ'-limit point of K and since K is γ'-closed, $z \varepsilon K$. Hence $I_{2n+2} \subset K$.

Now, suppose there exists some $p \varepsilon I_{2n+1}$ such that $p \varepsilon K$. Then there exists a γ'-open set L containing p such that $L \cap I_{2n+2} = \phi$ and $L \cap V = \phi$. By lemma 4.16, 4.17, it is impossible. Hence $I_{2n+1} \subset K$. It can now be shown by induction that, for all $k > 2n$, $I_k \subset K$.

Theorem 4.21

(X, γ') is connected.

Proof:

Let y be that element of J_3, where $J \subset (\frac{\pi}{2}, \pi)$ such that $\gamma' = \gamma \vee G$, $y \in G$ and $G \notin \gamma$. Suppose there exists a set $K \subset \gamma$ such that k is both open and closed in (X, γ'). Assume $y \notin K$. This implies that $K \in \gamma$. Since (X, γ) is connected, K is not closed in (X, γ). Therefore, if U is a γ-open neighbourhood of y, $U \cap K \neq \phi$. Hence $U \cap (P_2 \cap K) \neq \phi$ or $U \cap (P_4 \cap K) \neq \phi$. Since $G = \{y\} \cup (V \cap P_2)$, where V is a fixed γ-open neighbourhood of y, and y is not a γ'-limit point of K, it follows that $U \cap (P_4 \cap K) \neq \phi$. By lemma 4.16, there exists an γ'-open set $U_4 \subset U \cap (P_4 \cap K)$ such that $\{x/(x,4) \in U_4\}$ is dense in an open interval $I \subset J$. Lemma 4.20 implies that, for all $k > 4$, $I_k \subset K$. Then by corollary 4.19 following lemma 4.18, $\omega \in K$.

Since $y \notin K$, $y \in K^c$ and K^c is also open and closed in (X, γ'). Let H be a γ'-open neighbourhood of y such that $H \subset K^c$ then $H \cap P_2 \neq \phi$. This implies, by lemma 4.16, that there exists a γ'-open set $H_2 \subset H \cap P_2 \cap K^c$ such that $\{x/(x,2) \in H_2\}$ is dense in some open interval $I' \subset J$. Again it follows from lemma 4.18 and 4.20 that $\omega \in K^c$. This contradicts the fact that $K \cap K^c = \phi$. Therefore (X, γ') is connected.

From the following theorem we get (X, σ) cannot be embedded into any maximal connected space.

Theorem 4.22

Every connected subspace of a maximal connected space is maximal connected.

Remark: 4.23

If for each n, E_n is an uncountable dense subset of the irrationals, it follows that one can then construct an uncountable connected Hausdorff Space which cannot be embedded into any maximal connected space.

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