Finite and Countable Compactifications

BY

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Introduction

INTRODUCTION.

In 1924, Alexandroff constructed the simplest compactification namely 1-point compactification. Since then many authors have contributed to the study of the remainder $\alpha(X)$-X, where $\alpha(X)$ is a compactification of X. This thesis is devoted to the study of finite compactifications and countable compactifications. The following four papers are taken for discussion:

(1). MAGILL, K.D., N-POINT COMPACTIFICATIONS [11]

(2).CAIN,G.L., CONTINUOUS PREIMAGES OF SPACES WITH FINITE COMPACTIFICATIONS. [3].

(3). MAGILL, K.D., COUNTABLE COMPACTIFICATIONS [12].

(4).KIMURA,T., N_0-POINT COMPACTIFICATIONS OF LOCALLY COMPACT SPACES AND PRODUCT SPACES [10].

Chapter I deals with finite compactifications. Here we discuss the results of Magill [11] and Cain [3]. Magill [11] has obtained a characterization of Hausdorff spaces which admit N-Point compactifications. He [11] has proved that each N-Star gives rise to an N-Point compactification and to each N-Point compactification there corresponds an N-Star. Also, there is a one-one correspondence between the number of N-Point compactifications and the equivalence classes of the family of N-Stars. Apart from these results, the following interesting properties are obtained:

1. If X has the property that every compact set of X is contained in a compact subset whose complement has atmost N components, then X has no M-Point compactification for M>N.

2. The inverse image of a space that has an N-point compactification under a compact, continuous mapping also has an N-Point compactification.

3. R has only one 1-point compactification only one 2-point compactification and no N-point compactification for N>2.

4. The Euclidean space E^N for N>1 has no M-Point compactification for M>1.

5. An infinite discrete space has infinitely many N-Point compactifications for each N>1.

6. Given any positive integer N, there exists a space which has an N-Point compactification but no M-point compactification for M>N.

Cain [3] has extended Magill's result and has obtained the following characterization:

"A connected locally compact Hausdorff space X has an N-Point compactification iff there is a continous mapping of X onto a compact Hausdorff space so that the singular set of the mapping consists of exactly N-points".

Chapter II is devoted to the study of countable compactifications. Magill [12] has obtained a set of equivalent conditions for a locally compact space which admit a countable compatification. Magill [12] has extended his result on N-Point compactifications and has obtained sufficient conditions under which a space has no countable compactification. He [12] has shown that no Euclidean-N space has a countable compactification. The following characterization is obtained:

"A locally compact space X has a countable compactification iff for each positive integer N, there is a N-Star".

Kimura [10] has obtained the following characterizations:

1. A locally compact space X has a countable compactification iff there is a sequence $\{U_i / i \in N\}$ of pairwise disjoint γ-open subsets of X each with noncompact closure.

2. A locally compact space X has a countable compactification iff there is a continuous mapping f of X onto a compact space Y such that the singular set S(f) is countable. He [10] has also obtained a set of necessary and sufficient conditions for the product of two locally compact spaces to admit a countable compactification.

Review of Literature

REVIEW OF LITERATURE

When we try to study the literature on compactifications we find that it is a big ocean. In 1924 Alexandroff constructed one-point compactification. From then on various aspects of compactifications have been studied by many authors. In this thesis we have made a detailed study of finite compactifications and countable compactifications as given by Magill [11,12], Cain [3] and Kimura [10]. We now present abstracts of few important papers which deal with the study of the remainder $\alpha(X)-X$.

Remainders in Hausdorff compactifications [Chandler, R.E and Fu-Chien Tzung, 1978] [6].

The authors have generalized a theorem of Steiner and Steiner and it to obtain new results concerning remainders for completely regular spaces. Both the locally compact and nonlocally compact cases are considered.

A note on a fiberwise one-point compactification. [Chung, J.M.L.H, 1981] [7].

The authors have shown that if $p:E\rightarrow B$ is a continuous surjective map of locally compact separable metric spaces, then the fiberwise one point compactification $\overline{E}$ of E is also a locally separable metric space.

Compactification with zero dimensional remainders [Dominguez., J.M, 1985][8].

The author is concerned with the lattice $K_0(X)$ of compactifications with zero dimensional remainder of a locally compact space X.

Let F(X) denote the subalgebra (and sublattice) of C(X) of all continuous real-valued functions f for which there is a compact set K with f[X-K] is finite. Denote by $C_K(X)$ all functions in C(X) with compact support. It is observed that if $\alpha(X)$ is a compactifications of X with a zero dimensional remainder, then the functions in C(X) which admit an extension to $\alpha(X)$ form a subalgebra F(X) of C(X) which contains $C_K(X)$ and determines the compactification $\alpha(X)$. Conversely, a subalgebra A of F(X) which contains $C_K(X)$ determines of compactiifications $\alpha(X)$ which zero dimensional reminder for which $A=F_\alpha(X)$

On the Lattice of one-point near compactifications [Raghavan, T.G., 1985] [15].

The author claims to have discovered a new one-point T_2 near compactification for a locally nearly compact T_2 non-H closed space (X,τ) which is the projective maximum within the class L of all one-point T_2 near compactifications of (X,τ). The author has also showed that the class L is the same as (isomorphic to) the class of all one-point H-closures of (X,τ).

One-Point T_1-bicompactifications [Osmatesca, P.C., 1987] [14].

The author presents a systematic method for constructing all one-point T_1 compactifications of a non compact T_1 space. To do this the author introduces special bases of closed sets, called LA-bases. The procedure is similar to that used to construct Wallman compactifications from Wallman bases.

Chapter I

Finite Compactifications

CHAPTER-I

FINITE COMPACTIFICATIONS

This chapter is devoted to the study of finite compactifications. Magill [11] has introduced the notion of N-Star and has proved that each N-star gives rise to an N-point compactification and to each N-point compactification there is an N-star. He [11] has shown that there is a one -one correspondence between the number of N-point compactifications and the number of equivalence classes of N-stars of X. He has also obtained necessary and sufficient condition for a space to have exactly one N-point compactification. Using the results obtained in this paper he [11] has shown that the space E^N has no M-Point compactification for M>1 and R has exactly one 1-point compactification and one 2-point compactification and no M-point compactification for M>2. Some more interesting examples are also given here. Section 2 deals with these results. In section 3 we discuss the results of Cain [3]. He has given an extension of Magill's result and has obtained a characterization of those spaces that have an N-point compactification in terms of the existence of certain mappings into compact spaces. First let us give the preliminary definitions and results.

SECTION 1.1.

Definition 1.1.1. A compactification of a topological space X is a compact Hausdorff space Y containing X such that X is dense in Y (ie) $\overline{X} = Y$. A compactification of a space X is denoted by α (X).

Note A necessary and sufficient condition for a space to have a compactification is that it should be completely regular.

Constructing a compactification is equivalent to finding a dense imbedding of a space into a compact Hausdorff space.

Definition 1.1.2. Let X and Y be to top spaces. A map $h:X \rightarrow Y$ is an imbedding provided the map $X \rightarrow h(X)$ is a homeomorphism. Choose an imbedding $h:X \rightarrow Z$ of X into a compact Hausdorff space Z.

Let X_o denote the subspace h(X) of Z and let Y_o denote its closure in Z. Then Y_o is a compact Hausdorff space and $\overline{X_o} = Y_o$. Therefore Y_o is the compactification of X_o induced by h.

Definition.1.1.3. Two compactifications $\alpha_1(X)$ and $\alpha_2(X)$ of X are said to be **equivalent** if there is a homeomorphism $h: \alpha_1(X) \rightarrow \alpha_2(X)$ such that $h(x)=x$, for every $x \in X$.

Definition 1.1.4. Let X be a locally compact Hausdorff space. Take some object outside X, denote by the symbol ∞ for convenience and adjoin it to X forming the set $Y=XU\{\infty\}$. Topologize Y by defining the collection of open sets in Y to be all sets of the following types.

i) U, where U is an open subset of X.

(ii) Y-C, where C is a compact subset of X.

The space Y is called the **1- point compactification** of X.

The above construction of 1-point compactification is given by Alexandroff in the year 1924. Hence it is also termed as **Alexandroff compactification**.

Note: It is easily seen that any two 1-point compactifications are equivalent.

Example.1.1.5 The 1-point compactification of a real line R is homeomorphic with the circle. Similarly the 1-point compactification of R^2 is homeomorphic to S^2.

Example.1.1.6. Take the unit circle S^1 in R^2 and h: $(0,1)\rightarrow S^1$ be the map

$h(t) = (\cos 2\pi t, \sin 2\pi t)$. The compactification induced by the imbedding h is equivalent to the 1- point compactification of (0,1).

Example 1.1.7. Let Y be the space [0,1]. Then Y is a compactification of (0,1). It is obtained by adding one point at each end of (0,1).

Definition 1.1.8 If $f:X\rightarrow Y$ is a continuous mapping of one locally compact space onto another, then the **singular set** S(f) is the collection of all points $p\in Y$, such that, in every neighbourhood of p there is a compact set with non-compact inverse image.

Definition 1.1.9. D is said to be **upper semicontinous** if for every $d\in D$ and U an open subset of α (Z) containing d, there exists an open $V\subset U$ containing d such that if $e\in D$ and $V\cap e \neq \phi$, then $e\subset V$.

Notation:

1) By X, we mean a topological space (X,τ).

2) For $A\subset X$, $e_x A$ denotes the complement of A with respect to X. If no confusion arises, we use the simpler notation eA.

SECTION 1.2

In this section we discuss Magill's [11] results in detail. All topological spaces discussed in this section are Hausdroff spaces.

Definition 1.2.1. A compactification $\alpha(X)$ of a space is called an **N-point compactification** if $\alpha(X)$-X contains N-Points, in which case we shall use the notation $\alpha_N(X)$.

The following theorem gives a characterization of Hausdroff spaces which admit an N-point compactification.

Theorem 1.2.2. The following statements concerning a space are equivalent:

(i) X has an N point compactification.

(ii) X is locally compact and contains a compact subset K whose complement is the union of N mutuallly disjoint, open subsets $\{G_i\}_{i=1}^{N}$ such that KUG_i is not compact for each i.

(iii) X is locally compact and contains a compact subset K whose complement is the union of N mutuallly disjoint open subsets $\{G_i\}_{i=1}^{N}$ such that KUG_i is contained in no compact subset for each i.

(iv) X is locally compact and contains N mutuallly disjoint open subsets $\{G_i\}_{i=1}^{N}$ such that $e[G_1\cup G_2\cup\ldots\cup G_N]$ is compact while for each i, $[G_1\cup G_2\ldots\cup G_{i-1}\cup G_{i+1}\cup..\cup G_N]$ (which equals X if N=1) is not compact.

Proof:

(i) ⇒ (ii) Assume X has an N point compactification $\alpha_N(X)$. Then there are N points $r_1,r_2\ldots r_n$ such that $\alpha_N(X) - X = \{r_1,r_2..r_n\}$.

Since, $\alpha_N(X)$ is Handsdorff, there exist mutually disjoint open subsets $\{G_i^1\}_{i=1}^{N}$ of $\alpha_N(X)$ such that $r_i\in G_i^1$ for each i. Let $G_i = G_i^1 - \{r_i\}$.

Claim: G_i is non empty.

Suppose G_i is empty, then $G_i^1 = \{r_i\}$. Hence G_i^1 is an open set containing r_i. But $G_i \cap X=\phi$. This implies $r_i \notin \overline{X}$. This is a contradictiction, because $X = \alpha_N(X)$.

Claim: G_i is an open subset of X.

This follows immediately from the fact that $G_i^1 \cap X = G_i$

Let $K = e_X [G_1 \cup G_2 \cup \ldots \cup G_N]$.

Then $K = e_{\alpha N(X)} [G_1 \cup G_2 \cup \ldots \cup G_N]$.

Hence K is a closed subset of $\alpha_N(X)$. Since close subset of a compact space is compact, we get that K is compact.

Consider, $K \cup G_i = e_X [G_1 \cup G_2 \ldots \cup G_{i-1} \cup G_{i+1} \cup .. \cup G_N]$

$$= e_{\alpha N(X)} [G_1^1 \cup G_2^1 \ldots \cup G_{i-1}^1 \cup G_{i+1}^1 \cup .. \cup G_N^1] \cap [e_{\alpha N(X)}\{r_i\}].$$

Suppose $K \cup G_i$ is compact, then it is a closed subset of $\alpha_N(X)$.

Therefore, $\{r_i\} \cup [G_1^1 \cup G_2^1 \ldots \cup G_{i-1}^1 \cup G_{i+1}^1 \cup .. \cup G_N^1]$ is an open subset of $\alpha_N(X)$. Then, $\{\{r_i\} \cup [G_1^1 \cup G_2^1 \ldots \cup G_{i-1}^1 \cup G_{i+1}^1 \cup .. \cup G_N^1]\} \cap G_i^1 = \{r_i\}$is an open subset of $\alpha_N(X)$.

This implies, $r_i \notin \overline{X} = \alpha_N(X)$, which is a contradiction. Hence $K \cup G_i$ cannot be compact. Finally, $X = \alpha_N(X). - \{ri\}_{i-1}^{N}$, and is therefore an open subset of $\alpha_N(X)$. This implies X is locally compact.

(ii)⇒(iii): Assume K as in (ii). (ie) $K = e_X [G_1 \cup G_2 .. \cup G_N]$ and $K \cup G_i = e_X[G_1 \cup .. \cup G_{i-1} \cup G_{i+1} \cup .. \cup G_N]$ is not a compact. Assume $K \cup G_i$ is contained in a compact set. Then $K \cup G_i$ is compact being a close subset of a compact set. This is a contradiction. Hence $K \cup G_i$ cannot be contained in a compact subset.

(iii)⇒(iv) This follows immediately from (iii). since, $e_X[G_1 \cup .. \cup G_{i-1} \cup G_{i+1} \cup .. \cup G_N]$ is nothing but $K \cup G_i$.

(iv)⇒(i). Let K $=e_X[G_1\cup G_2 .. \cup G_N]$. Let $\{r_i\}_{i=1}$ be any N distinct elements not in X. Let $\alpha_N(X) = X\cup\{r_i\}_{i=1}^{N}$. For any subset H of X, H^i denote the set $H\cup\{r_i\}$. We say H has property P_i if H is an open subset of X and $[K\cup G_i]\cap e_X H$ is compact

Let B= $\{H^i/H$ has property $P_i\}$

To prove *B* forms a basis for a topology on $\alpha_N(X)$. Let H_1 and H_2 have property P_i. Since H_1 and H_2 are open sets, $H_1\cap H_2$ is also an open subset of X

Consider, $[K\cup G_i]\cap eX[H_1\cap H_2] .= ([K\cup G_i]\cap e_X H_1]\cup([K\cup G_i]\cap e_X H_2)$

Since, finite union of compact sets is compact, we get that the right hand side set is compact. Therefore, *B* is a neighbourhood basis for r_i. Let τ_N be the topology on $\alpha_N(X)$ having these sets and the open subsets of X as basis.

To prove $\tau_N(X)$ is Hausdorff. Take $y_1, y_2\in\alpha_N(X)$, with $y_1\neq y_2$

Case i: $y_1=r_i$ and $y_2=r_j$

Consider for any i, $[K\cup G_i]\cap e_X G_i = K\cap e_X G_i]\cup[G_i\cap e_X G_i] = K\cap e_X G_i = K$ as $K\subset e_X G_i$.Hence $[K\cup G_i\}\cap e_X G_i$ is compact. Since G_i is also an open subset of X, we get that G_i has a property P_i. Therefore, each G_i is a basis element of τ_N. Then,

$G_i^i=G_i\cup\{r_i\}$ and. $G_j^j= G_j\cup\{r_j\}$ are the disjoint open sets containing r_i and r_j respectively.

Case ii: $y_1=x\in X$ and $y_2= r_i$

Since X is locally compact, there is a compact set K* and an open subset G* such that $x\in G^*\subset K^*$. Then $[K\cup G_i]\cap K^*$ being a closed subset of K* is compact.(ie) $[K\cup G_i]\cap e_X(e_X K^*)$ is compact. This implies $(e_X K^*)^i$ is a basis element in τ_N. Also $r_i\in(e_X K^*)^i$ and $(e_X K^*)^i\cap G^* =\phi$.

Case iii :- Both. y_1, $y_2 \in$ **X**.Since X is Hausdorff there exist disjoint open sets in X containing y_1 and y_2.

Hence in all the cases the Hausdorff condition is satisfied. Hence $\alpha_N(X)$ is Hausdorff.

To Prove $\alpha_N(X)$ is compact

Let $U=\{U_\alpha\}$ $\alpha\in\Lambda$ be an open cover of $\alpha_N(X)$. Then N of these sets $\{U_1,U_2...U_N\}$ contain respectively sets of form H_1^1, H_2^2, ...,H_N^N where for each i, H_i is an open subset of X and $[K\cup G_i] \cap eX$ Hi is compact.

Claim:- $\bigcap_{i=1}^{N} [G_1\cup...\cup G_{i-1}\cup H_i\cup G_{i+1} \cup...\cup G_N]\subset\{H_1\cup H_2... \cup H_N\}$

Let $x\in\bigcap_{i=1}^{N} [G_1\cup...\cup G_{i-1}\cup H_i\cup G_{i+1} \cup...\cup G_N]$

This implies, $x\in G_1\cup...\cup G_{i-1}\cup H_i\cup G_{i+1} \cup...\cup G_N$,for every i.

This implies, there exist G_k or H_i such that, $x\in G_k$ or $x\in H_i$.

If $x\in H_i$, $x\in H_1\cup H_2\cup...\cup H_N$.

If $x\in G_k$, then since $G_i\cap G_j=\phi$ for $i\neq j$.We have $x\notin G_1$,$x\notin G_2$... $x\notin G_{k-1}$, $x\notin G_{k+1}$... $x\notin G_N$. But $x\in G_1\cup G_2\cup...\cup G_{k-1} \cup H_k\cup G_{k+1} \cup...\cup G_N$. Hence we conclude that $x\in H_k\subset H_1\cup H_2\cup...\cup H_N$. Hence the claim.

From the claim, we get

$$e_X[H_1\cup H_2\cup\ldots\cup H_N]\subset\bigcup_{i=1}^{N} e_X[G_1\cup\ldots\cup G_{i-1}\cup H_i\cup G_{i+1}\cup\ldots\cup G_N].$$

Therefore, $e\alpha_N(X)[U_1\cup U_2\cup\ldots\cup U_N]\subset e_X[H_1\cup H_2\cup\ldots\cup H_N]$

$\subset\bigcup_{i=1}^{N} e_X[G_1\cup\ldots\cup G_{i-1}\cup H_i\cup G_{i+1}\cup\ldots\cup G_N]=\bigcup_{i=1}^{N}([K\cup G_i]\cap e_XH_i)$ which is compact. This $e\alpha_N(X)[U_1\cup U_2\cup\ldots\cup U_N]$ is a compact subset of $\alpha_N(X)$ and is covered by a finite subfamily of U Hence $\alpha_N(X)$ in turn is covered by finite subfamily of U. Hence $\alpha_N(X)$ is compact.

To prove X is a dense subset of $\alpha_N(X)$.

Let H^i be a basis open set containing r_i. Since H satisfies property P_i, we get $[K\cup G_i]\cap e_XH$ is compact. But $K\cup G_i$ is not compact. This implies, $e_XH\neq X$. Hence $H\neq\phi$. Therefore, $H^i\cap X=H\cap X=H\neq\phi$. Thus $r_i\in X$. This is true for every i. Therefore $\overline{X}=\alpha_N(X)$.Hence $\alpha_N(X)$ is an N-point compactification of X.

Definition 1.2.3. :-Let $G=\{G_i\}_{i=1}^{N}$ be mutually disjoint open family of N subsets of X with the property:-.

i). $K_G=e[G_1\cup G_2\cup\ldots\cup G_N]$ is compact .

ii). $e\,[G_1\cup G_2\cup\ldots\cup G_{i-1}\cup G_{i+1}\cup\ldots\cup G_N]$ is not compact for each i.

Then G is said to be an **N-Star** of X.

Theorem 1.2.4:- A space X has a 1-star iff it is not compact.

Proof:- Assume X has a 1-star, say,$\mathcal{G}=\{G_1\}$.Then eG_1 is compact., and $e[\phi]$ is not compact (ie) X is not compact.

Conversly assume X is not compact Let $X = \{G_1\}$ Then $eG_1=eX=\phi$ is compact and $e\phi=X$ is non compact.

Hence the theorem.

Theorem 1.2.5:- N- Star of a locally compact space X gives rise to an N-point compactification of X (called the **compactification induced by N-Star**)

Proof:- This follows from the implication (iv)=>(i) of theorem 1.2.2.

Let $S_N(X)$ be the collection of all N-Stars of X (ie) $S_N(X)=\{\mathcal{G}_\alpha\}_{\alpha\in\Lambda}$, where $\mathcal{G}_\alpha$ is an N-Star of X. Define a relation R on $S_N(X)$ by $\mathcal{G}R\mathcal{H}$ if the elements $\{G_i\}_{i=1}^{N}$ and $\{H_i\}_{i=1}^{N}$ of $\mathcal{G}$ and $\mathcal{H}$ respectively can be ordered in such a way that $[K_{\mathcal{H}}\cup H_i]\cap eG_i$ is compact for each i.

The following theorem determines the number of N-point compactification of a locally compact Hausdorff space:-

Theorem 1.2.6.:- If X is a locally compact space, then R is an equivalence relation on $S_N(X)$ and there is a one-one correspondance between the equivalence classes of $S_N(X)$ and the different N-point compactification of X.

For the Proof of the theorem we require the following lemma:-

Lemma 1.2.7:- If G and H are elements of $S_N(X)$. Then GRH iff thecompactifications induced by G and H are equivalent.

Proof:- Let $G=\{G_i\}_{i=1}^{N}$ and $H=\{H_i\}_{i=1}^{N}$. Let $\alpha_N(X)$ and $\eta_N(X)$ be the compactification induced by G and H respectively. Denote the points of $\alpha_N(X)$-X by $\{a_i\}_{i=1}^{N}$ and those of $\eta_N(X)$-X by $\{b_i\}_{i=1}^{N}$.

First assume GRH. Then by definition for some ordering of the elements of G and H, $[K_H \cup H_i] \cap eG_i$ is compact for each i.

Claim:- If U is an open subset of X which has property P_i with respect to the N-Star G then it also has property P_i with respect to the N-Star H.

Since U has property P_i, $[K_G \cup G_i] \cap eU$ is compact

Thus $[K_H \cup H_i] \cap eG_i) \cup ([K_G \cup G_i] \cap eU)$ is compact. Hence $[K_H \cup H_i] \cap eU$ is compact being a closed subset of this compact set.

Hence the claim.

Consider the mapping, $h: \eta_N(X) \rightarrow \alpha_N(X)$ defined by, $h(b_i)=a_i$, for $i=1,2\ldots n$ and $h(x)=x$, for every $x \in X$.

Any basis element T in $\alpha_N(X)$ is of the form, T=U, U open subset of X.

or $T=U \cup \{a_i\}$, U is an open subset of X having property P_i with respect to G.

If $T=U$, then $h^{-1}(T) = h^{-1}(U) = U\in\tau_{\eta N}$.

If $T=U\cup\{a_i\}$, then $h^{-1}(T) = U\cup\{b_i\}$. By the claim, we get that U has property P_i with respect to the N-Star *H*. Hence $U\cup\{b_i\}\in\tau_{\eta N}$.

Therefore, inverse image of any basis element is open. Hence the mapping h is continuous. Since h is a continuous one-one mapping of a compact space onto a Hausdorff space $\alpha_N(X)$, h is a homeomorphism.

Hence $\alpha_N(X)$ and $\eta_N(X)$ are equivalent.

Conversely, let the compactifications induced by *G* and *H* be equivalent.

Claim:- There is an ordering (determined by the homemorphism) for the elements of *G* and *H* such that $[K_H\cup H_i]\cap eG_i$ is compact for each i. [similarly, $[K_G\cup G_i]\cap eH_i$ is compact for each i]

Since $\eta_N(X)$ and $\alpha_N(X)$ are equivalent there is a homeomorphism.

$h: \eta_N(X)\rightarrow\alpha_N(X)$ with $h(x)=x$ for $x\in X$. We reorder *G* as follows:-

Rename $h(b_i)$ by a_i. G_i is an open subset of X and, $[K_G\cup G_i]\cap eG_i=K_G$ is compact. Therefore G_i has property P_i with respect to the N-Star *G*.

Therefore, G_i^i is open in $\alpha_N(X)$. Hence $h^{-1}(G_i^i) = G_i\cup\{b_i\}$ is an open subset of containing b_i. (ie) there exists a basis element H^i, where *H* has property P_i, with respect to *H*, such that, $H^i\subset G_i\cup\{b_i\}$. (ie) $H\cup\{b_i\}\subset G_i\cup\{b_i\} \Rightarrow H\subset G_i$.

Now $[K_H \cup H_i] \cap eH$ is compact. Also, $eG_i \subset eH$. Hence $[K_H \cup H_i] \cap eG_i$ is compact being a closed subset of the compact space $[K_H \cup H_i] \cap eH$.

Therefore *GRH*.

Similarly, by considering the inverse maping h^{-1} we get that , *HRG*.

From theorem1.2.1. we get that to each N-Star there corresponds an N-point compactification and to each N-point compactification, there is an N-Star and for two N-Stars *G* and *H*, *GRH* if and only if the corresponding compactifications are equivalent.

Hence the lemma.

Proof of the theorem:- Let us now show that *R* is an equivalence relation.

i). Reflexive:- Since $[K_G \cup G_i] \cap eG_i = K_G$ is compact *GRG*.

ii) Symmetric:- Assume *GRH*..By the lemma α_N and η_N. are equivalent. Hence there exists a homeomorphism $h:\alpha_N \rightarrow \eta_N$ such that, $h(x)=x$, for every $x \in X$.By using argument in the converse part of the lemma it can be shown that $[K_G \cup G_i] \cap eH_i$ is compact for each i. Hence *HRG*.

iii) Transitive:- Let *GRH* and *HRM*. Let α_N, η_N and Γ_N be the N-point compactification induced by *G, H* and *M* respectively. By the lemma, α_N and η_N are equivalent, and η_N and Γ_N are equivalent. Since composition of homeomorphisms is again a homeomorphism we have, α_N and Γ_N are equivalent. Hence by the lemma *GRM*. Therefore *R* is an equivalence relation and from the lemma we get that there is a one-one correspondence between the equivalence classes of $\alpha_N(X)$ and the different N-point compactifications of X.

Theorem 1.2.8.:- No space has more than one1-point compactification.

Proof:-To prove this it is enough to prove that any two 1-Stars are equivalent . Let $G=\{G_1\}$and $H=\{H_1\}$be two 1-Stars . Then $K_G=eG_1$ and $K_H=eH_1$ are compact.and

(i)$[K_G \cup G_1] \cap eH_1 = eH_1$ is compact.

(ii) $[K_H \cup H_1] \cap eG_1 = eG_1$ is compact.

Hence we get that *GRH.* Since any two 1-Stars are equivalent, we get from theorem 1.2.6. that no space has more than one 1-point compactification.

Theorem 1.2.9:- X has exactly one N- point compactification iff X is locally compact, has an N-Star, and all other N-Stars are equivalent to it.

Proof:- X has exactly one N-point compactification

iff there is exactly only one equivalance class of $S_N(X)$.(by theorem 1.2.6)

iff X has an N-Star and all other N-Stars are equivalent to it.

Hence the proof.

From theorem 1.2.2. we get ,

Theorem 1.2.10:- If X has an N-point compactification, then it has an M-point compactification for every positive integer M<N.

Theorem 1.2.11. If X has the property that every compact subset of X is contained in a compact subset whose complement has atmost N components then X has no M-point compactification for M>N.

Proof:- Suppose X has an M-Star, $\{G_i\}_{i=1}^{M}(M>N)$. Let K^* be a compact subset containing $e[G_1 \cup G_2 \ldots \cup G_M]=K$, such that, eK^* has r components $H_1, H_2 \ldots H_r$ where $r<N$.(ie)

$K^* \supseteq e[G_1 \cup G_2 \cup \ldots \cup G_M]$ and $eK^* = H_1 \cup H_2 \ldots \cup H_r$. Hence $eK^* \subset G_1 \cup G_2 \ldots \cup G_M$.

(ie) $H_1 \cup H_2 \ldots \cup H_r \subset G_1 \cup G_2 \ldots \cup G_M$, Since $r < M$ and H_i is connected for each i,$[H_1 \cup H_2 \ldots \cup H_r] = \phi$ for some j.

Hence, $G_j \subset e[H_1 \cup H_2 \ldots \cup H_r]$ (ie)$G_j \subset K^*$

Therefore $K \cup G_j \subset K^*$ .Implies $K \cup G_j$ is compact being a closed subset of a compact space.. This is a contradiction to the fact that $K \cup G_j$ is not compact.

Hence X has no M-Star for M>N which in turn implies that X has no M-point compactification for M>N.

The study of the existence of N-point compactifications of a space X which is mapped under compact continuous mapping to a space having an N-point compactification is provided by the following theorem:-

Theorem 1.2.12:- If Y has an N-point compactification and is the image of X under a compact continuous mapping, then X also has an N-point compactification.

Proof:- Consider a mapping $f:X \rightarrow Y$ such that f is continuous, compact and onto. Let Y have an N-point compactification. Let $\{G_i\}_{i=1}^{N}$ be the corresponding N-Star.

Claim:- $\{f^{-1}(G_i)_{i=1}^{N}$ is the N-Star for X.

(i) Since G_i is open and f is continuous, $f^{-1}(G_i)$ is open.

Consider,$e_X[f^{-1}(G_1) \cup f^{-1}(G_2) \cup \ldots \cup f^{-1}(G_N)]$

$$=\bigcap_{i=1}^{N} e_X f^{-1}(G_i)=\bigcap_{i=1}^{N}[X- f^{-1}(G_i)]=\bigcap_{i=1}^{N} f^{-1}(Y- G_i)= f^{-1}[\bigcap_{i=1}^{N}(Y- G_i)] = f^{-1}[e_Y(\bigcup_{i=1}^{N} G_i)].$$

Since,$e_Y[G_1\cup G_2\cup\ldots\cup G_N]$ is compact and f is compact, we get $f^{-1}\{e_Y(\bigcup_{i=1}^{N} G_i)]$ is compact.

(ii) Consider, $e_X[f^{-1}(G_1) \cup f^{-1}(G_2) \cup \ldots \cup f^{-1}(G_{i-1}) \cup f^{-1}(G_{i+1}) \cup \ldots \cup f^{-1}(G_N)]$

$$= f^{-1}[e_Y(G_1\cup G_2\cup\ldots\cup G_{i-1}\cup G_{i+1}\cup\ldots\cup G_N)]$$

If this set is compact we get,

$f[f^{-1}(e_Y(G_1\cup G_2\cup\ldots\cup G_{i-1}\cup G_{i+1}\cup\ldots\cup G_N)]$ isa compact.

(ie) $e_Y\{ G_1\cup G_2\cup\ldots\cup G_{i-1}\cup G_{i+1}\cup\ldots\cup G_N)$ is compact, which is a contradiction, hence,

$f^{1}(e_Y[G_1\cup G_2\cup\ldots\cup G_{i-1}\cup G_{i+1}\cup\ldots\cup G_N])$ is non-compact. Thus$\{ f^{1}(G_i)\}_{i=1}^{N}$ is an N-Star of X.

To prove X is locally compact.

Consider,$x\in X$, then $f(x)\in Y$.

Since Y is locally compact, there exists a compact subset T containing a neighbourhood V of f(x) (ie)$f(x)\subset V\subset T$. (ie)$x\in f^{1}(f(x))\subset f^{1}(V)\subset f^{1}(T)$

(ie)$x\in f^{1}(V)\subset f(T)$.

$f^{1}(T)$ is compact as f is compact and $f^{1}(V)$ is open as f is continuous. Thus X ix locally compact. Therefore by theorem 1.1.5 X has an N- point compactification.

The lemma and the theotern which follows it are crucial to discuss some interesting properties of special spaces.

Lemma 1.2.13. Let (X,d_1) and (Y,d_2) be two unbounded, connected metric spaces and denote their product by XxY, Let (x_0,y_0) be a point in XxY and let k be any positive number and put $K=\{(x,y)\in XxY: d_1(x,x_0)\leq k \text{ and } d_2(y,y_0)\leq k\}$. Then *e*K is a connected set.

Proof:- Let (x_1,y_1) and (x_2,y_2) be points of *e*K. Then either $d_1(x_1,x_0)>k$ or $d_2(y_1,y_0)>k$ and either $d_1(x_2,x_0)>k$ or $d_2(y_2,y_0).>k$.

Suppose $d_1(x_1,x_0)>k$ and $d_2(x_2,x_0)>k$.

Let $A_1=\{(x_1,y):y\in Y\}$. Since Y is an unbounded metric space, there exist a point $y_2\in Y$, such that, $d_2(y_3,y_0)>k$. Let $A_2=\{(x,y_3):x\in X\}$ and let, $A_3=\{(x_2,y):y\in Y\}$. Here, A_1 and A_3 are homeomorphic to Y and A_2 is homeomorphic to X.Hence A_1,A_2 and A_3 are connected since, $(x_1,y_3)\in A_1\cap A_2$, $A_1\cup A_2$ is connected.

Since,$(x_2,y_3) \in A_2 \cap A_3$, $(x_2,y_3) \in (A_1 \cup A_2) \cap A_3$. Hence $A_1 \cup A_2 \cup A_3$ is connected.

Now let us show $K \cap (A_1 \cup A_2 \cup A_3) = \phi$

Let $(x,y) \in A_1 \Rightarrow x = x_1$. Using triangle inequality, $d_1(x_1,x_0) \le d_1(x_1,x) + d_1(x,x_0)$

(ie) $d_1(x,x_0) \ge d_1(x_1,x_0) - d_1(x_1,x) = d_1(x_1,x_0) > k$. Hence $(x,y) \notin K$.(ie) $A_1 \cap K = \phi$.

Similarly it can be shown that $A_2 \cap K = \phi$ and $A_3 \cap K = \phi$. Hence $K \cap (A_1 \cup A_2 \cup A_3) =$ o (ie) $A_1 \cup A_2 \cup A_3 \subset eK$, and (x_1,y_1) and (x_2,y_2) belongs to $A_1 \cup A_2 \cup A_3$. Thus every pair of points of *e*K is contained in a connected subset of *e*K. Hence *e*K is connected.

Theorem 1.2.14: Let (X,d_1) and (Y,d_2) be two unbounded metric spaces and supppose for every real number r and points $x_0 \in x$ and $y_0 \in y$, the sets $\{x \in X: d_1(x,x_0) < r\}$ and $\{y \in Y: d_2(y,y_0) < r\}$ are compact Then XxY has no N-point compactification for N>1.

Proof:-Let $K \neq \phi$ be a compact subset of XxY. Let P_X and P_Y be the projection mapping into X and Y respectively. Since projection mappings are continuous, $P_X(K)$ and $P_Y(K)$ are compact subsets of X and Y respectively. consider, $x_0 \in P_X(K)$ and $Y_0 \in P_Y(K)$. Since compact subsets of metric spaces are bounded, there exist positive numbers r_1 and r_2 such that $Px(K) \subset \{x \in X: d_1(x_1x_0) \le r_1\}$. and $P_Y(K) \subset \{y \in Y: d_2(y,y_0) \le r_2\}$. Let $r = \max\{r_1,r_2\}$.

Then, $P_X(K) \subset \{x \in X: d_1(x,x_0) \le r\} = H_X$. and $P_Y(K) \subset \{y \in Y: d_2(y,y_0) \le r\} = H_Y$.

By hypothesis, H_X and H_Y are compact. Hence $H_X x H_Y$ is compact. Also $K \subset P_X(K) x P_Y(K)$ $\subset H_X x H_Y$. Moreover by lemma 1.2.13. e $[H_X x H_Y]$ is connected and hence $e[H_X x H_Y]$ has a single component.

Thus every compact set is contained in a compact set whose complement has atmost 1 component. By theorem 1.1.11 XxY has no N-point compactification for N>1. Hence the proof.

1.2.15. Discussion of some special spaces:-

I. The complex plane:- Let $X=(R,d_1)$ and $Y=(R,d_2)$ and, $d_1(x,y)=d_2(x,y)=|x-y|$.

Since closed and bounded subset of R is compact, we have, for any $r \in R$, $\{x \in X/|x-x_0|\leq r\}$ and $\{y \in Y/|y-y_0|\leq r\}$ are compact. Hence by theorem 1.2.14, RxR has no N-point compactification for N>1. Hence the only N-point compactification of the complex plane is the 1-point compactification (ie) the Enclidean 2- space RxR has no N-point compactification for N>1. Similarly, generalising we have E^N, the Euclidean N-space has no N-point compactification for N>1.

II. The space R of Real numbers:- Since every compact subset of R is contained in a bounded, closed interval whose complement has 2 components, we have by theorem 1.2.11 that R has no N-point compactification for N>2. By theorem 1.2.8 any two one point compactification are equivalent.

Claim:- Any two 2 point compactifications are also equivalent.

Let $G=\{G_1,G_2\}$ and $H=\{H_1,H_2\}$ be two 2-stars of the 2-point compactifications α_2 and η_2. $K_G=e[G_1 \cup G_2]$ and $K_H=e[H_1 \cup H_2]$ are compact. Hence K_G is contained in some closed interval $[a_1,b_1]$ and K_H is contained in some closed interval $[a_2,b_2]$. Let $[a,b]$ contain $[a_1,b_1]$ and $[a_2,b_2]$. Then $K_G \subset [a,b]$ and $K_H \subset [a,b]$, which implies, $(-\infty,a)$ is contained in one of G_1 or G_2 and one of H_1 or H_2.

Suppose, $(-\infty,a) \subset G_1$ and $(-\infty,a) \subset H_1$. Since eG_1 and eH_1 are not compact, we have $(b,\infty) \subset G_2$ and $(b,\infty) \subset H_2$. consider, $(K_G \cup G_1)=eG_2$.

Then, $(K_G \cup G_1) \cap eH_1=eG_2 \cap eH_1=e[G_2 \cup H_1)$. $e[G_1 \cup G_2] \subset [a,b]$. Hence $e[a,b]` G_1 \cup G_2$. (ie) $(-\infty,a) \cup (b,\infty) \subset G_1 \cup G_2$. Since, $(-\infty,a) \subset H_1$, $eH_1 \subset [a,\infty)$ and $eG_2 \subset (-\infty,b]$.

Hence, $eH_1 \cap eG_2 \subset [a,\infty) \cap (-\infty,b]=[a,b]$.

Therefore, $[K_G \cup G_1] \cap eH_1=eG_2 \cap eH_1 \subset [a,b]$.

Similarly, $[K_G \cup G_2] \cap eH_2=eG_1 \cap eH_2 \subset [a,b]$.

Hence both the left hand side sets are compact being closed subsets of the compact set $[a,b]$ Hence G and H are equivalent. Hence the claim.

Therefore R has only one 1-point compactification and one 2-point compactification and has no N-point compactification for $N>2$.

III. The "S" spaces:- The construction of "S" spaces given below provides an answer to the following question affirmatively.

"Given a positve integer N, does there exist a space which has an N-point compactification but no M-point compactification for M>N?".

Let n be a positive integer and define. $L_n=\{(x,y)\}\in RxR: Y=1/nX$ and $X>0\}$ where R is the space of real numbers. $L_n=\{x,x/n)/x>0\}$ denotes a ray of the Euclidean plane originating from the origin.

Define $S_N=\bigcup_{n=1}^{N} L_n$. Let $K=\{(0,0)\}$. $e_{SN}K=\bigcup_{n=1}^{N} L_n^{1}$, where $L_n^{1}=Ln-\{(0,0)\}$. The family $\{L_n^{1}\}_{n=1}^{N}$ is mutually disjoint and $KUL'_n = L_n$ is not compact for each n Hence from theorem 1.2.2. S_N has an N-point compactification. Let K be any compact subset of S_N. There exist a positive integer r such that, $K\subset\{(x,y)\in S_N:\sqrt{x^2+y^2}<r\}=K^*$. (ie) K^*denotes the intersection of a circular disk with S_N. Hence K^* is compact and $e_{SN}K^*$ has N components. Thus the compact subset K is contained in a compact subset K^* whose complement has atmost N components By theorem 1.2.11 S_N has no M-point compactification for M>N.

Note:- By theorem 1.2.14 $S_N X S_N$ has no M-point compactification for M>1. Hence product of two spaces having M-point compacifications need not have an M-point compacification for M>1.

IV. Infinite Discrete spaces:- Let X be an infinite discrete space and N a positive integer such that N>1. Let $\{G_{1j}\}_{j=1}^{N}$ be a mutually disjoint collection of infinite subsets whose union is whole of X, Now for each j, where 1<j<N, let $\{G_{ij}\}_{i=1}^{N}$ be a requence of sets such that,

1) $G_{i+1,j}\subset G_{ij}$ for =1,2... and 2) $G_{ij}-G_{i+1,j}$ contains infinitely many elements for i=1,2...

Define for i>1, $G_{iN}=e[G_{i1}\cup G_{i2}\ldots\cup G_{i},N-1]$. Let $\mathcal{G}_i=\{G_{i1}\cup G_{i2}\ldots\cup G_{iN}]$.

Then each $\mathcal{G}_i$ is a collection of N mutually disjoint subjects. Since $G_{ij}=G_{H1,j}$ contains infinitely many elements for i=1,2 ... We get each G_{ij} has infinite number of elements. Moreover,

$\bigcup_{j=1}^{N} G_{ij}=G_{i1}, \cup G_{i2}\cup\ldots\cup G_{iN-1}\cup G_{i,}N = G_{i,1}\cup G_{i,2}\cup\ldots\cup G_{i,N-1}\cup e[G_{i1}\cup\ldots\cup G_{iN-1}] = X$ for every i.

Hence, (i) $e\,[\bigcup_{j=1}^{N} G_{ij}]=\phi$. is compact. and

(ii) $e\ [G_{i,1}\cup G_{i,2}\cup\ldots\cup G_{iK-1}\cup G_{i,K+1}\cup\ldots\cup G_{i,N}]=e[\bigcup_{j=1}^{N} G_{ij}]\ \cup G_{iK}=\phi\ \cup G_{i,K}=G_{i,K}$. Since G_{i},k contains infinitely many elements and the space is discrete, we get $G_{i;K}$ is not compact, Hence $\mathcal{G}_i$ is an N-Star.

Claim:- If $i_1\neq i_2$, then $\mathcal{G}_{i1}$ is not equivalent to $\mathcal{G}_{i2}$. It is sufficient to prove for $i_1 < i_2$ and $j_1\neq N$, that, $e[G_{i,1}\cup G_{i,2}\cup\ldots\cup G_{i1,\,j1-1}\cup G_{i1,j1+1}\cup\ldots\cup G_{i,N}]\cap eG_{i2j2}$. is not compact (ie) we will have to show that it contains infinitely many elements.

Case i : $j_1=j_2$.

Then, $e[G_{i,1}\cup G_{i,2}\cup\ldots\cup G_{i1,\,j1-1}\cup G_{i1,j1+1}\cup\ldots\cup G_{i,N}]\cap eG_{i2j1}=G_{i1,j1,}\cap eG_{i2,j1}$ which is infinite, since $G_{i,j}-G_{i+1,j}$ has infinitely many elements.

Case ii : If $j_1\neq j_2$.

Then, $G_{i1,j1}\cap eG_{i2,j2}\ G_{i2,j1}\cap eG_{i2,j2}=G_{i2,j1}$ is infinite Hence the claim.

Thus we get that no two N-Stars are equivalent. Hence by theorem 1.2.6 X has infinitely many N point compactifications for each $N>1$.

By a suitable modification of the argument used in the above example, we get a more general result as stated in the following theorem:-

Theorem 1.2.16: If X is locally compact and the union of an infinite number of mutually disjoint, open sets then for each $N>1$, X has infinitely many N-point compactifications.

SECTION 1.3

The results due to Cain [3] are presented in this section. The fact which is used in proving almost all the theorems in this section is given in the following lemma.

Lemma 1.3.1. Suppose C is a component of a compact Hausdorff space Y and W is an open neighbourhood of C. Then there exist an open neighbourhood U of C such that $C\subset U\subset W$ and $Fr(U)=\phi$.

Proof:- Define $P\subset Y$ by

$P=\{y\in Y/\text{there is a separation } Y=R\cup T. \text{ with } C\subset R \text{ and } y\in T\}$

Take any $y\in P$, then there is an open T,such that $y\in T\subset P$. This implies y is an interior point of P Hence P is an open set and $C\subset Y-P$. Therefore $Q=Y-P$ is closed.

Claim:- Q is connected.

Suppose Q is not connected,then there is a separation $Q=A\cup B$. The component $C\subset Q$. Hence $C\subset A$ or $C\subset B$.

Assume $C\subset A$. Since the sets A and B are compact, there are disjoint open neighbourhoods W_A and W_B for which $A\subset W_A$ and $B\subset W_B$, and $FrW_A\subset P$. Take any $p\in FrW_A$. This implies $p\in P$. This implies there is a separation $Y=U_P\cup V_P$ such that $C\subset U_P$ and $p\in V_P$,

Consider the collection$\{V_P\}$. Then this collection is an open cover for the compact set FrW_A. Hence there is a finite subcollection $\{V_i/i=1,2...n\}$ which also covers FrW_A. Each Vi is open and closed. Hence $V=\bigcup_{i=1}^{N}V_i$ is both open and closed. Let $U^1=W_A-V$. Then U^1 is both open and closed, open since V is closed and closed since V is open. Also $FrW_A\subset\cup V_i\subset V$. $Y= U^1\cup(Y- U^1)$. Since $C\subset W_A$ and $C\cap V=\phi$, we get $C\subset W_A-V= U^1$, $B\cap W_A=\phi$. This implies $B\cap(W_A-V)= \phi$ which implies $B\cap U^1=\phi$. Hence $B\subset Y- U^1$. Therefore we have a separation of $Y= U^1\cup Y- U^1$ with $C\subset U^1$. Hence by the definition of P, $(Y- U^1)\subset P$. Hence $B\subset P$, a contradiction as $B\subset Q=Y-P$. Therefore Q is connected.

Since $C\subset Q$ and C is a maximal connected set, we must have $Q=C$. Since $Y-W\subset P$ for every $x\in Y-W$, there is a separation $Y=R_XUT_X$ with $C\subset R_X$, $x\in T_X$. Since Y-W is compact being closed, there is a finite subcover, $\{T_i:i=1,2...m\}$ of $\{T_X\}$. $T=\cup\{T_i\}$ is both open and closed with $C\subset Y-T$. Thus $U=Y-T$ is a neighbourhood of C such that $U\subset W$ and $FrU=\phi$.

Theorem 1.3.2:- Suppose Z is a connected Hansdorff space with an N point compactification and $X\subset Z$ is open and such that Z-X has atleast k compact components.

If all components of Z-X are compact, then X has an (N+k) point compactification. Otherwise X has a (k+1)-point compactification.

Proof:- Let $\alpha(Z)$ be an N-point compactification of Z and let $K=\alpha(Z)-X$. Define a decomposition *D* of $\alpha(Z)$ as $D=\{\{x\}/x\in X\}\cup\{\text{components of } K\}$.

Claim:- *D* is upper semicontinuous.

Case i : $d=\{x\}$, $x\in X$.

Consider an open subset U such that $d\subset U$. Let $V=U\cap X$. Then $d\subset V\subset U$. Take $e\in D$ with $V\cap e\neq\phi$. This implies e cannot be a component of K and $e=\{y\}$ for some $y\in X$. Hence $y\in V$, which implies $e\subset V$.

Case ii. If d is a component of K.

Let U be an open set containing d. By lemma 1.3.1, there is an open set V such that $d\subset V\subset U$ and $Fr(V)=\phi$. This implies $d\subset V\cap K\subset U\cap K$, and $(FrV)\cap K=\phi$.

Take an $e\in D$ with $V\cap e\neq\phi$.Since $Fr(V)=\phi$, V is both open and closed. Thus there is a separation for $\alpha(Z)$.(ie)$\alpha(Z)=VU(\alpha(Z)-V)$. Since e is a component and $V\cap e\neq\phi$, we must have $e\subset V$.

Thus *D* is upper semicontinous.

Let $\hat{X}$ be the decomposition space induced by *D* and let h denote the natural map of $\alpha(Z)$ onto $\hat{X}$. It follows from the upper semicontinuity of *D* that $\hat{X}$ is a compact Hausdorff space.

The continuous map h is one to one on X and $X=h^{-1}h(X)$ is an inverse set. Hence h/X is a homeomorphism. h(X) is identified with X and X is referred to as a subspace of $\hat{X}$, X is open in $\hat{X}$. Therefore $\hat{X}$-X is compact.

Suppose A is a connected subset of the remainder $\hat{X}$-X,. then $h^{-1}(A)$ is a connected subset of K. This implies $h^{-1}(A)$ is contained in one component of K. Hence A={p} is a single point. Implies that $\hat{X}$-X is totally disconnected. The space $\alpha(Z)$ is connected hence $\hat{X}$ is connected.

Claim:- X is dense in $\hat{X}$.

Suppose p is a point in the interior of $\hat{X}$-X. The points of $\hat{X}$-X are components, Hence p is a component. Suppose p $\notin$X, there exists a neighbourhood W of p such that $W\cap X=\phi$. Implies that $p\subset W\subset\hat{X}-X$. Hence by lemma 1.3.2 there is an open set U is such that $p\subset U\subset W\subset\hat{X}-X$ and FrU=ϕ. Hence U is both open and closed. Therefore there is a separation for $\hat{X}$.A contradiction, since $\hat{X}$ is connected. Hence X is dense in $\hat{X}$. Therefore $\hat{X}$ is a compactification of X. Suppose C is a compact component of K=Z-X and let N be a compact neighbourhood of C such that $N\cap(\alpha(Z)-Z)=\phi$. Implies $N\subset Z$. Then $C\subset K$=Z-X.implies that $C\subset N\cap K\subset N$. By lemma 1.3.1. there is a neighbourhood V of C such that $V\subset N$ and $(FrV)\cap K=\phi$. This implies that C is a component of $\alpha(Z)$-X. Hence the number of component of $\alpha(Z)$-X is atleast as large as the number of compact components of K=Z-X.

Suppose if all the components of K are compact, no point of $\alpha(Z)$-X is an accumulation point of a component of K. Thus each $r\in\alpha(Z)-Z$ is a component of $\alpha(Z)$-X. Thus we have shown that if all components of Z-X are compact, there are atleast (N+k) components of $\alpha(Z)$-X. Hence either $\hat{X}$-X is finied and contains atleast (N+k) points or it is infinite and totally disconnected. In either case, we conclude that there is an (N+k)-point compactification of X. [5].

Next, suppose Z-X has atleast one noncompact component C. Then that α (Z)-X has atleast k+1 components or $\hat{X}$-X contains atleast k+1 points.

The following theorem extents Magill's results.(Theorem 1.2.12.)

Theorem 1.3.3. Suppose Y has an N-Point compactification, X is connected and f: X $\rightarrow$ f(X)=Y is a continuous maping for which the singular set S has atleast k compact components. If all components of S are compact, then X has an (N+k)-point compactification Otherwise, **X** has a (k+1)-point compactification.

Proof:- Let (X_W,F_W) be the Whyburn compactification of the map f. That is f_W:$X_W \rightarrow f(X_W)$=Y with fw is compact and continous. X_W is locally compact Hausdorff and contains X as a dense subspace, f_W/X=f and f_W/(X_W-X) is a homeomorphism onto S.([2] and [17]).

By theorem 1.2.12. **X_W** has an N-Point compactification. Since X_W-X and S are homeomorphic the conclusions of the theorem follow directly from theorem 1.3.2.

This section is concluded with the following characterization of spaces having N-point compactifications.

Characterization in terms of mappings onto compact spaces.

Theorem 1.3.4:- A connected locally compact Hausdorff space X has an N-Point compactification iff there is a continous mapping of X onto a compact Hausdorff space so that the singular set of the mapping consists of exactly N points.

Proof:- Suppose there is a continous $f:X \to f(X)=Y$ with Y compact and S having exactly N points. Then the existence of an N-point compactification of X follows from theorem 1.3.3.

Conversely, suppose there is an N-point compactification $\alpha(X)$ of X. Let $U_1,U_2\ldots U_N$ be disjoined compact neighbourhoods in $\alpha(X)$ of the points of $\alpha(X)-X$. Define the decomposiition *D* of $\alpha(X)$ by taking $U_1,U_2\ldots U_N$ to be the members of *D* and the individual points of $\alpha(X)-\cup U_i$ as members of *D*. Clearly, *D* is upper semicontinuous, so the decomposition space Y induced by *D* is compact Hausdorff. Let h denote the natural map of $\alpha(X)$ onto Y and define $f:X \to Y$ by $f=h/X$. The map f carries X onto Y, since each U_i meets X, and the singular set $S=\{h(U_i),i=1,2\ldots N\}$.

Chapter II

Countable Compactifications

CHAPTER-II

COUNTABLE COMPACTIFICATIONS

In this chapter we discuss the contributions of Magill [12] and Kimura [10] to the study of countable compactification.Magill [12] has characterised Hausdorff spaces that are locally compact and have countable compactfications. He has also shown that no Eucidean N-space has countable compactification but every infinite discrete space has a countable compactification.

Kimura [10] has obtained some characterizations of locally compact spaces which admit countable compactification.He [10] has obtained a set of sufficient conditions for the product of two locally compact spaces to admit a countable compactification. The preliminary definitions and results needed are given in the first section.

SECTION 2.1

Definition 2.1.1. A compactification $\alpha(X)$ of a space X is called a **countable compactification** or **N_0-point compactification** (we say X has an N_0CF) if the remainder $\alpha(X) - X$ is countable.

Definition 2.1.2. A open set of U of a space X is γ **-open** if Bd_xU is compact.

Definition 2.1.3. A space X is called **rim-compact** if it has a base consisting of γ-open sets.

Remark 2.1.4. Locally compact space are rim compact.

Definition 2.1.5. Freudenthal compactification $\gamma(X)$ of a ~~rim~~-compact Hausdorff space X is described as the completion of X with respect to the uniformity having as a base the collection

$\{\cup(A_i x A_i)\}$ where X is covered by the A_i's, each A_i is an open subset of X and $Fr(A_i) = clA_i - A_i$ is compact for each i.

The following lemma regarding Freudenthal compactification is proved by Terada [16].

Lemma 2.1.6. $Cl_{\gamma(X)} U \cap (\gamma(X)-X) = (\gamma(X)-Cl_{\gamma(X)}(X-U)) \cap (\gamma(X)-X)$

$(\gamma(X)-Cl_{\gamma(X)}(X-U)) \cup (\gamma(X)-Cl_{\gamma(X)}U) \supset \gamma(X)-X$ and

$(\gamma(X)-Cl_{\gamma(X)}(X-U)) \cap (\gamma(X)-Cl_{\gamma(X)}U) = \phi$

SECTION 2.2

Results of Magill [12].

In this section all topological spaces are assumed to be Hausdorff.

Theorem 2.2.1. The following statements concerning a space X are equivalent:

i) X is locally compact $\beta(X)-X$ has an infinite number of components (maximal connected sets).

ii) X is locally compact and there exist a compactification $\alpha(X)$ of X such that $\alpha(X)-X$ is infinite and locally disconnected.

iii) X is locally compact and has a countable compactification.

iv) X has N-point compactification for each positive integer n.

Proof

i ⇒ ii. Let $\beta(X)-X = \cup\{H_a : a \in \Lambda\}$, H_a is a component of $\beta(X)-X$. Let $\alpha(X) = X \cup \Lambda$ and define a function h: $\beta(X) \rightarrow \alpha(X)$ by

$h(P) = p$ if $p \in X$.

$= a$ if $P \in H_a$.

Endow $\alpha(X)$ with the quotient topology induced by h. Since $\alpha(X)$ is the continuous image of compact space it is compact.

Claim:- $\alpha(X)$ is Hausdorff.

To prove this consider the following three cases for distinct points p and q .

1) Both p and q belongs to X.

2) $p \in X$ and $q \in \alpha(X) - X$.

3) Both p and q belongs to $\alpha(X)-X$.

Case 1. Both p and q belongs to X.

Since X is locally compact, it is an open subset of any compactification and hence of $\beta(X)$. Let U be any subset of X. Then U is open in $\beta(X)$.

To prove U is an open subset of $\alpha(X)$.

Now $h^{-1}(U) = U$, is open in (X).

Hence by the definition of quotient topology, U is open in $\alpha(X)$. Hence the claim.

If $p,q \in X$, as X is Hausdorff there exists disjoint open sets U and V in X such that $p \in U$ and $q \in U$ and $U \cap V = \phi$. By the claim U and V are open in $\alpha(X)$ and hence the Hausdorff condition for $\alpha(X)$ is satisfied in this case.

Case 2: $p \in X$ and $q \in \alpha(X)-X$.

Since X is locally compact, there exist a compact subset K of X which contains an open subset G of X such that $p \in G \subset K \subset X$. Since K is a compact set in X, it is closed in X and hence $\alpha(X) - K$ is open in $\alpha(X)$. Hence G and $\alpha(X) - K$ are the disjoint open subsets of p and q respectively.

Case 3: p and q belong to $\alpha(X) - X \Rightarrow p,q \in \Lambda$. Hence there exist disjoint closed components H_p and H_q which is compact. Therefore, H_p and H_q are disjoint, closed subsets of $\beta(X)$ and there are disjoint, open subsets G_p and G_q of (X) containing H_p and H_q respectively. By the result, "The component of a point in a compact space is the intersection of all open and closed sets containing it"[9]. We have, H_p is the intersection of all open and closed sets (relative to $\beta(X) - X$) containing it. Now $\beta(X) - X$ is compact and $Gp \cap [\beta(X)-X]$ is an open subset of $\beta(X)-X$ containing H_p. Hence the intersection of a finite number of open and closed sets contained in $G_p \cap [\beta(X)-X]$. Let V_p be this finite intersection. Then V_p is an open and closed subset of $\beta(X)$-X and $H_p \subset V_p \subset G_p \cap [\beta(X)-X]$.Since V_p is both open and closed it is the union of all H_a contained it. Also $V_p = V_p^* \cap [\beta(X)-X]$ for some open set $V_p^*$of $\beta(X)$, where $V_p^* \subset G_p$. This implies $V_p^* = V_p U [V_p^* \cap X]$. Similarly there exists sets Vq and Vq* related to Hq. Hence Vp* and Vq* are disjoint.

Let $U_p = [V_p^* \cap X]\ U\ \{a: H_a \subset V_p\}$ and $U_q = [V_q^* \cap X]\ U\ \{a:H_a \subset V_q]$. Then $h^{-1}(U_q) = V_p^*$ and $h^{-1}(U_q) = V_q^*$. Since V_p^* and V_q^* are open subsets sof $\beta(X)$ by the definition of quotient topology we have U_p and U_q are open, disjoint sets of $\alpha(X)$ containing p and q respectively. Implies $\alpha(X)$ is Hausdorff. Hence the claim.

It is easily seen that **X** is dense in $\alpha(X)$. Hence $\alpha(X)$ is a compactification of X.

Claim: $\alpha(X)$-X is totally disconnected.

$U_p \cap [\alpha(X) - X]$ is open in $\alpha(X)$-X and $h^{-1}[U_p \cap [\alpha(X) - X\ ]] = V_p^* \cap [\beta(X)-X] = V_p$.

Since V_p is a closed subset of $\beta(X)$, $U_p \cap[\alpha(X)-X]$ is a closed subset of $\alpha(X)$ and is hence the closed subset of $\alpha(X)-X$. Hence p and q does not belong to the same component. Since p and q are distinct points of $\alpha(X)-X$, we get $\alpha(X)-X$ is totally disconnected.

ii $\Rightarrow$iii: Given X is locally compact and $\alpha(X)-X$ is infinite and totally disconnected. Hence using the result, "the component of a point in a compact space is the intersectio of all open and closed sets containing it,"[9] we get X has a basis of open and closed sets. Therefore there exists a countable family $\{H_n\}_{n=1}^{\infty}$ of non empty, mutually disjoint subsets of $\alpha(X)-X$ which are both open and closed in $\alpha(X)-X$.

Let $H_0= [\alpha(X)-X] -\cup\{H_n\}_{n=1}^{\infty}$. Since $\alpha(X)$ is compact,$H_0\neq\phi$ .Define a function $h:\alpha(X)$ $\rightarrow \Gamma(X)$, where $\Gamma(X) =XU\{n\}_{n=0}$, such that

$h(p) = n$ for $p\in H_n$.

$= p$ for $p\in X$.

Endow $\Gamma(X)$ with the quotient topology induced by h. As in the previous discussion it can be shown that $\Gamma(X)$ is Hausdorff and X is dense in $\Gamma(X)$. Hence $\Gamma(X)$ is the countable compactification of X.

iii $\Rightarrow$ iv: Suppose X is locally compact and $\Gamma(X)$ is a countable compactification of X. Assume p and q are distinct points belonging to some connected subset H of $\Gamma(X)-X$. Since $\Gamma(X)-X$ is locally compact Hausdorff space it is completely regular. Hence there exists a continuous function f: $\Gamma(X)-X \rightarrow I = [0,1]$ such that $f(p) = 0$ and $f(q)=1$. Then f[H] is connected. Also f[H] =[0,1] suppose not, then there is a point x $\in[0,1]$ but $t\neq f[H]$. Then $U=f[H] \cap[0.t)$ and $V=(t,1] \cap f[H]$ forms a separation of f[H] which is a contradiction. Hence f[H] =I.Since $\Gamma(X)$ is a countable compactification of X, it has countable elements, but I is uncountable hence $f[H] \neq I$. This contradiction is due to our assumption that p and q belong to the same connected subset H of $\Gamma(X)$ -X. Hence the only connected subsets of $\Gamma(X)$-X are the one point sets. Hence $\Gamma(X)$-X is totally disconnected.

Since X is locally compact, $\Gamma(X)$-X is compact we get $\Gamma(X)$-X has a basis of open and closed sets [9]. Hence for any positive integer n there are n nonempty mutually disjoint subsets of $\Gamma(X)$-X that are both open and closed and whose union is all of $\Gamma(X)$-X. Let these sets be denoted by $\{H_i\}_{i=1}$ and define a function $h:\Gamma(X) \rightarrow X \cup \{1,2....n\} = \alpha_N(X)$ by $h(p) = i$ if $p \in H_i$

$= p$ if $p \in \mathbf{X}$.

Endow$\alpha_N(X)$ with the quotient topology induced by h. Then as in the proof of claim it can be shown that $\alpha_N(X)$ is a Hausdorff and a compactification of X.

iv ⇒ i: Take any positive integer N, and let $\alpha_N(X)$ be an N point compactification of X. Therefore there is a continuous function,

f: $\beta(X)$ -X $\rightarrow \alpha_N(X)\text{-}X = \{x_1,x_2...x_N\}$ hence$\alpha_N(X)$-X has N components. Implies $\beta(X)$-X has atleast N components. This is true for every positive integer hence $\beta(X)$-X has infinitely many components. Since X has N-point compactification it is locally compact. Hence the proof.

Corollary 2.2.2. Suppose X is locally compact and there exists a positive integer N such that every compact subset of X is contained in a compact subset whose complement has atmost N components. Then X has no countable compactification.

Proof: Suppose X has countable compactification by theorem 2.1.1 X has an N-point compactification for each positive integer n. ...(1)

From theorem 1.2.11 we get that X has no N-point compactification

for n>N ...(2)

(1) and (2) are contradictory. Hence X has no countable compactification.

Corollary 2.2.3: Let (X,d_1) and (Y,d_2) be two unbounded connected locally compact metric spaces and suppose that for all points $x_0 \in X$ and $y_0 \in Y$ and every positive number r, the sets $\{x \in X : d_1(x,x_0) < r\}$ and $\{y \in Y : d_2(y,y_0) < r\}$ are compact then XxY has no countable compactification.

Proof: As in the above corollary, the proof follows from theorem 1.2.14 and theorem 2.1.1.

Remark 2.2.4. From the above corollaries it follows that no Euclidean N-space has a countable compactification.

Corollary 2.2.4. A locally compact space X has a countable compactification iff for each positive integer n it contains a compact subset K_n whose complement is the union of n mutually disjoint open subset $\{G_{n,i}\}_{i=1}^{n}$ with the property that $K_n \cup G_{n,i}$ is not compact for each i.

Proof: X has a countable compactification

iff X has an N-point compactification for every N (by theorem 2.1.1).

iff the given condition in the statement is satisfied. (by theorem 1.2.2).

We conclude the section with the following corollary.

Corollary 2.2.6: Suppose X is locally compact and is the union of an infinite number of mutually disjoint open subsets. Then X has a countable compactification. In particular every infinite discrete space has a countable compactification.

Proof: X can be considered as the union of the n mutually disjoint open subsets which are not compact. For any positive integer let $K_n = \phi$. Hence using the previous corollary we get X has a countable compactification.

SECTION 2.3

Results of Kimura [10]

All spaces are assumed to be completely regular and T_1 and by $|A|$, we mean the cardinality of the set A.

The following lemma is useful in obtaining characterizations: –

Lemma 2.3.1: Let A be a countably infinite subset of a space X. Then there is a countably infinite subset $B = \{b_i / i \in N\}$ of A and a sequence $\{U_i / i \in N\}$ of pairwise disjoint open subset of X such that $b_i \in U_i$ and $Bd_X U_i \cap A = \phi$ for each $i \in N$.

Proof: Since X is assumed to be completely regular it is regular space. Hence there exists a point $b_1 \in A$ and an open subset V_1 of X such that $b_1 \neq Cl_X V_1$ and $|V_1 \cap A| = N_0$. Similarly there exists $b_2 \neq Cl_X V_2$ and $|V_2 \cap V_1 \cap A| = N_0$. Proceeding like this we get a subset $B = \{b_i / i \in N\}$ of A and a sequence $\{V_i / i \in N\}$ of open subset of X. Let $V_0 = X$ and $W_i = V_i\text{-}1\text{-}Cl_X V_i$ for every $i \in N$.

$b_i \neq Cl_X V_i$ and $b_i \in V_{i-1} \cap A$. Implies $b_i \in W_i$ and $b_i \in A$ (ie) $b_i \neq X - W_i$ and $b_i \in A$. Since X is completely regular, there is a continuous mapping $f_i : X \rightarrow I = [0.1]$ such that $f_i(b_i) = 0$ and $f_i(X - W_i) = \{1\}$. Since $f_i(A)$ is countable, but $[0,1]$ is uncountable hence $f_i(A) \neq [0,1]$. Therefore there exists $r_i \in I$ such that $r_i \neq f_i(A)$. Let $U_i = f_i^{-1}[0, r_i)$.

Claim: $Bd_X(U_i) \subset f_i^{-1}(r_i)$

Let $x \in Bd_X(U_i)$. Implies $x \in \overline{U_i} \cap \overline{X - U_i}$. Implies

$f(x) \in f_i(\overline{U_i}) \cap f_i(\overline{X\text{-}U_i}) \in \overline{f_i(U_i)} \cap \overline{f_i(X\text{-}U_i)} \in [0,r_i] \cap [r_i,1] = r_i$. Implies $x \in f_i^{-1}(r_i)$. Therefore $Bd_X(U_i) \subset f_i^{-1}(r_i)$.

Hence the claim.

Since $r_i \neq f_i(A)$, $f_i^{-1}(r_i) \cap A = \phi$. Hence $Bd_X(U_i) \cap A = \phi$.

Characterization in terms of γ-open sets.

Theorem 2.3.2 A locally compact space X has a countable compactification iff there is a sequence $\{U_i/i \in N\}$ of pairwise disjoint γ-open subsets of X with non compact closure.

Proof: Let Y be a countable compactification of X, then $|Y\text{-}X| = N_0$ is countable. Hence using lemma 2.3.1, there are countble infinite subsets $\{a_i/i \in N\}$ of Y-X and a sequence $\{W_i/i \in N\}$ of pairwise disjoint open subsets of Y such that $a_i \in W_i$ and $Bd_Y \cap (Y\text{-}X) = \phi$ for $i \in N$. Let $U_i = W_i \cap X$ for every $i \in N$. Then $Cl_X U_i$ is non compact. Now $U_i = W_i \cap X \subset W_i$ implies $Bd_X U_i \subset Bd_X W_i \subset Bd_Y W_i \subset X$.

Since Y is compact, $Bd_Y W_i$ is closed, we get $Bd_Y W_i$ is compact. Hence $Bd_X U_i$ is compact being the close subset of the compact space $Bd_Y W_i$. Therefore U_i is γ-open set. Hence $Bd_X U_i$ is compact. Thus the sequence $\{U_i/i \in N\}$ satisfies the required properties.

Conversely, let $\{U_i/i \in N\}$ be a family of pairwise disjoint γ-open subset of X each with non compact closure. Let $\gamma(X)$ be the Freudenthal compactification of X. Define $F_i = Cl_{\gamma(X)} U_i$ $(\gamma(X)\text{-}X)$ and $F_0 = (\gamma(X)\text{-}X) - \cup\{F_i/i \in N\}$. Then $D = \{F_i/i = 0.1.2...\} \cup \{\{x\}/x \in X\}$ is an upper semicontinuous decomposition of $\gamma(X)$. It is clear that D is a composition of $\gamma(X)$.

Claim: D is upper semicontinuous.

Take $d \in D$, then $d = \{x\}$ or $d = F_i$ for some i.

Case i: $d = \{x\}$.

Let U be an open set in $\gamma(X)$ containing x. The $W=U\cap X$ is an open neighbourhood of x in $\gamma(X)$. Suppose $F^1\in D$ such that $F^1\cap W\neq\phi$, then to prove $F^1\subset W$. Since $W\subset X$ and $F^1\cap W\neq\phi$, we have $F^1=\{y\}$ for some $y\in X$. Hence $y\in W$. Therefore $F^1=\{y\}\subset W\subset U$.

Case ii: Let $d=F_i$ for some $i\in N$.

Let U be an open set of $\gamma(X)$ containing d. Let $F=F_i$. By lemma 2.1.6 F is open in $\gamma(X)$-X. Hence there is an open subset V of $\gamma(X)$ such that $F=V\cap(\gamma(X)-X)$. Define $W=U\cap V$, open in $\gamma(X)$ and $W\subset U$. Then $F\subset W\subset U$. Take any element F^1 of D such that $F^1\cap W\neq\phi$. If $F^1=\{x\}$ then $F^1\cap W\neq\phi$ implies $x\in W\subset U$, which implies $F^1\subset W\subset U$.

If $F^1=F_j=Cl_{\gamma(X)}U_j\cap(\gamma(X)-X)$,then $F'\cap W\neq\phi$ implies $F'\subset W\subset U$.

Case iii: $d=F_0$

Let U be an open subset of $\gamma(X)$ containing F_0. Let D^1 be the collection of the elements of $\{F_i/i\in N\}$ not contained in U.Since $\gamma(X)$-X is compact and F_i is open in $\gamma(X)$-X, D^1 is finite. Thus $W=U-\cup\{F/F\in D^1\}$ is open in $\gamma(X)$. For any element F^1 of D such that $F'\cap W\neq\phi$ we have $F^1\subset W\subset U$. Hence $F_0\subset W\subset U$. Here D is an upper semicontinuous decomposition of $\gamma(X)$.

Let $Y=\gamma(X)/D$ be the quotient space of $\gamma(X)$ determined by the upper semicontinuous decomposition. As in theorem 2.3.2 it can be shown that Y is a countable compactification of X. Hence the proof.

Cain proved that a connected locally compact space X has an N-point compactification iff there is a continuous mapping f of X onto a compact space Y so that S(f) consist of exactly N points (theorem 1.3.4.). Cain, Chandler and Faulkner [4] strengthened this theorem so it remains true without connectedness. Chandler and Tzung [6] defined the remainder induced by f to be $C(f) = \cap\{Cl_Y f(X-F)/F$ is a compact subset of $X\}$ and proved that (whenever Y is compact) there is a compactification of $\alpha(X)$ with $\alpha(X)-X$ homeomorphic to $C(f)$. Their "remainder induced by f" is the same as the singular set [4]. Thus the if part of Cain's theorem holds for any cardinality of S(f).

The following theorem which is anologous to Cain's theorem gives characterization using singular sets.

Theorem 2.3.3:- A locally compact space X has a countable compactification iff there is a continuous mapping f of X onto a compact space Y such that S(f) has cardinality N_0.

Proof:- In view of the discussion preceeding this theorem, it is enough if we prove the only if part. Suppose X has a countable compactification. By theorem 2.3.2 there is a sequence $\{U_i/i\in N\}$ of pairwise disjoint γ-open subsets of X each with noncompact closure. For each i=2,3...choose a point $x_i\in U_i$ and put $F_i=(Cl_{\gamma(X)}U_i\cap(\gamma(X)-X))\cup\{x_i\}$, where $\gamma(X)$ is the Freudenthal compactification and

let $F_1= \gamma(X)- \cup\{\gamma(X)-Cl_{\gamma(X)}(X-U_i)/i=2,3,...\}$

Let $F=\{F_i/i\in N\}\cup\{\{x\}/x\in \gamma(X)- \cup\{F_i/i\in N\}\}$.

Then F is an upper semicontinuous decomposition of $\gamma(X)$. Thus $Y= \gamma(X)/F$ is a compact Hausdorff space. Let $g:X \rightarrow \gamma(X)$ be an embedding and $q: \gamma(X) \rightarrow Y$ be a quotient mapping. Then the composition f=qog is a continuous mapping of X onto Y.

Let $\{a_i\}=q(F_i)$ for each $i\in N$ and $A=\{a_i/i\in N\}$.

Then $q^{-1}(A) = \gamma(X)-\cup\{U_i-\{x_i\}/i=2,3..\}$. As X is locally compact, it is open in $\gamma(X)$. Therefore $q^{-1}(A)$ is closed in $\gamma(X)$. Since q is quotient mapping A is closed in Y. Then U=Y-A is open in Y. Since the restriction of $f:X\rightarrow Y$ to $U\subset Y$, $f_u:f^1(U)\rightarrow U$ is a homeomorphism for every compact subset K of U,$f^1(K)$ is compact.

Hence for every $y\in U$, we have $y\neq S(f)$ and for each a_i and a neighbourhood V_i of a_i there exists an open subset W_i such that $a_i\in W_i\subset Cl_YW_i\subset V_i$. Then Cl_YW_i is compact, but $f^{-1}(Cl_YW_i)$ is noncompact,hence we have $a_i\in S(f)$. Therefore, S(f) has cardinality N_0.Hence the theorem.

Countable compactifications of product spaces:-

First we discuss the folloowing lemmas which are used to prove the main theorem on product space.

Lemma 2.3.4:- If X and Y are noncompact connected spaces then XxY has no countable compactification.

Proof:- Suppose XxY has a countable compactification. Then by theorem 2.3.2, there is a sequence $\{U_i/i\in N\}$ of pairwise disjoint γ-open subsets of XxY each with noncompact closure. Let $p_X:XxY\rightarrow X$ and $p_Y:XxY\rightarrow Y$ be projections and let (i,j) be a pair of distinct positive integers. Let $A=p_X(Bd_{XxY}U_i)xp_Y(Bd_{XxY}U_i)$. Then A is compact. Since X and Y are noncompact and connected, Z=XxY-A is connected. Since $Bd_{XxY}U_i\subset A$ we have $Bd_{XxY}U_i\cap Z=\phi$

Therefore $Z=(U_i\cap Z)\cup(XxY-U_i)\cap Z=(U_i\cap Z)\cup((XxY-Cl_{XxY}U_i)\cap Z)$

(ie) $Z\subset U_i\cap Z$ or $Z\subset((XxY-Cl_{XxY}U_i)\cap Z$

If $Z\subset Ui\cap Z$ then $Z\subset Cl_{XxY}U_i$ and if $Z\subset((XxY-Cl_{XxY}U_i)\cap Z)$ then $Z\cap U_i=\phi$. Suppose $Z\subset Cl_{XxY}U_i$. Since $Cl_{XxY}U_i\cap U_j=\phi$,we have $U_j\subset A$. This implies that $Cl_{XxY}U_j$ is compact and we have a contradiction. Suppose $U_i \cap A = \phi$. Then $U_i \subset A$. This implies that $Cl_{XxY}U_i$ is compact and we have a contradiction. Hence XxY has a countable compactification

Lemma 2.3.5:- If X is a compact connected space and Y has no countable compactification then XxY has no countable compactification.

Proof:- Suppose XxY has countable compactification then by theorem 2.3.2 there is a sequence $\{U_i/i \in n\}$ of pairwise disjoint γ-open subsets of XxY each with noncompact closure.

Let p_Y:XxY → Y be the projection onto Y and let $V_i = Y - p_Y(XxY - U_i)$. Since X is compact, p_Y is closed and hence V_i is open in Y

To prove $Cl_Y V_i$ is noncompact. Suppose not, (ie) $Cl_Y V_i$ is compact. Let $A = Xx(Cl_Y V_i \cup p_Y(Bd_{XxY}U_i))$. Then A is compact. We claim that $Ui \subset A$.Take any $(x,y) \in U_i$

Case i $Xx\{y\} \subset Ui$. Then $pY(Xx\{y\}) \subset pY(U_i)$. Implies $y \in pY(U_i)$ which implies $y \neq p_Y(XxY-U_i)$. This implies $y \in V_i$. Hence $(x,y) \in A$.

Case ii $Xx\{y\} \not\subset U_i$. Since $Xx\{y\}$ is connected, by using the result "if $A \subset X$ and if C is a connected subset of X that intersects both A and X-A, then C intersects BdA" [13] we get $Xx\{y\} \cap Bd_{XxY}U_i \neq \phi$, hence there exists $(x_0,y) \in Bd_{XxY}U_i$. This implies $p_Y(x_0,y) \in pY(Bd_{XxY}U_i)$ which implies $y \in pY(Bd_{XxY}U_i)$. Hence $(x,y) \in A$.

Therefore in anycase $U_i \subset A$. Implies $Cl_{XxY}U_i$ is compact, a contradiction as U_i has noncompact closure. Therefore $Cl_Y V_i$ is not compact.

Clearly for positive integers $i \neq j$, $V_i \cap V_j = \phi$ and the collection $\{V_i/i \in N\}$ is a collection of pairwise disjoint γ-open subsets of Y with noncompact closure. By theorem 2.3.2 Y has a countable compactification. This is a contradiction. Hence our assumption is wrong. Hence XxY has no countable compactification.

Lemma 2.3.6: Let X be a space which can be represented as the finite topological sum $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$. Then X has a countable compactification iff ther is a positive integer $(1<)i(<n)$ such that X_i has countable compactification.

Main theorem:

Theorem 2.3.7: A product space XxY has a countable compactification iff the following conditions are satisfied:

a) either X or Y has an infinite number of components and the other is noncompact.

b) either X or Y has a compact component and the other has a countable compactification.

Proof: First assume XxY has a countable compactificaiton.

Case i: X and Y have atmost finite number of components. Let $X=\bigcup_{i=1}^{K} X_i$, where X_i is a component and $Y=\bigcup_{j=1}^{m} Y_j$ where Y_j is a component.

Then $XxY = (X_1xY) \oplus (X_2xY) \oplus \dots \oplus (X_kxY)$. By lemma 2.3.6 then is a integer i such that X_ixY has a countable compactificatiton.

$X_ixY=(X_ixY_1) \oplus (X_ixY_2) \oplus \dots \oplus (X_ixY_m)$. Once again by using lemma 2.3.6, there is an integer j such that X_ixX_j has a countable compactification. Since X_i and Y_i are connected, by lemma 2.3.4. either X_i or Y_i is compact.

Assume X_i is compact: By lemma 2.3.5. as X_ixY has a countable compactification. We conclude that Y has a countable compactification.

The other case can be dealt similarly. Thus, condition (b) holds.

Case ii. If both X and Y have an infinite number of components then (a) holds.

Casi iii. X has an infinite number of components and Y has atmost a finite number of components.

If Y is non compact then (a) holds.

If Y is compact, let $Y=\bigcup_{i=1}^{m} Y_i$, where Y is a component. By lemma 2.3.6., there is a positive integer j such that $X\mathrm{x}Y_j$ has a countable compactification. Hence by lemma 2.3.5., X has a countable compactification. Hence (b) holds.

Conversely, let X and Y satisfy (a). Suppose X has an infinite number of components and Y is non compact. Then there is a sequence $\{U_i/i\in N\}$of pairwise disjoint open and closed subsets of X. Now let $V_i=U_i\mathrm{x}Y$ for each $i\in N$. Then $\{V_i/i\in N)$ satisfies the conditions of theorem 2.3.2. Hence XxY has a countable compactification.

Next, let X and Y satisfy condition (b). Suppose X has a compact component X_0 and Y has a N_0CF. If X has an infinite number of components, then X and Y satisfy (a). Hence, XxY has an N_0CF. If X has atmost a finite number of components, then X_0 is open and closed in X. Hence, $X_0\mathrm{x}Y$ has an N_0CF. Since $X\mathrm{x}Y = (X_0\mathrm{x}Y) \oplus ((X-X_0)\ \mathrm{x}Y)$, and by lemma 2.3.6. XxY has an N_0CF. Hence the theorem.

Corollary 2.3.8:- Let X and Y be connected. Then XxY has a countable compactification iff either X or Y is compact and the other has a countable compactification.

Proof: Follows easily from the above theorem.

Summary and Conclusion

SUMMARY AND CONCLUSION

It is interesting to study various aspects of compactifications. Eminent topologists who have contributed to the study of compactifications include Alexandroff , Stone, Cech, Wallman, Smirnov, Leader, Henriksen, Isbell, Gall, Frink, Magill, Cain, Kimura et al. In this thesis we have made an attempt to study finite and countable compactifications. The results are collected from the contributions of Magill [11,12], Cain[3], and Kimura [10]. Characterizations of spaces that admit finite compactifications are given in terms of

(1) N-Stars [11] and

(2) Existence of certain mappings onto compacts spaces [3].

The construction of "S" spaces by Magill [11] provides an answer to the following question:

"Given a positive integer N, does there exist a space which has an N-point compactification but no M-Point compactification for M>N?".

Regarding countable compactifications, Magill [12] and Kimura [10] have provided interesting characterizations. Regarding product compactifications, Kimura [10] has proved the following:

A product space XxY has an N_0CF if and only if one of the following condition is satisfied:

a) either X or Y has an infinite number of components and the other is non-compact;

b) either X or Y has a compact component and the other has an N_0CF.

We study the literature on fuzzy toplogical spaces, we find that the concept of compactification is extended to fuzzy topological spaces also and some interesting development has taken place. It will be an interesting research problem to study the results discussed in this thesis in the case of fuzzy topological spaces.

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