Chapter VII

CHAPTER VII

DETERMINANT, SINGULARITY AND NON-SINGULARITY PRESERVERS FOR MATRICES OVER SEMIRINGS.

Definition: 7.1

A matrix $A \in M_n(S)$ is **$\mathcal{S}$-right singular** if $Ax = 0$ for some nonzero $x \in S^n$.

Definition: 7.2

A matrix $A \in M_n(S)$ is **$\mathcal{S}$-left singular** if $x^tA = 0^t$ for some nonzero $x \in S^n$.

Definition: 7.3

A matrix $A \in M_n(S)$ is **$\mathcal{S}$- singular** if A is either $\mathcal{S}$-left singular or $\mathcal{S}$-right singular.

Definition: 7.4

A matrix $A \in M_n(S)$ is **$\mathcal{S}$-nonsingular** if A is neither $\mathcal{S}$-left singular nor $\mathcal{S}$-right singular.

Lemma: 7.5

The following conditions are equivalent for the matrix $A \in M_n(S)$

(i) A is $\mathcal{S}$-singular

(ii) A has a zero row or a zero column.

Proof:

This follows immediately from definitions using the antinegativity of S and the fact that S has no zero divisors.

Remark: 7.6

Note that if A is $\mathcal{S}$-singular then bidet (A) = (0, 0). The inverse does not hold in general. There are $\mathcal{S}$-nonsingular matrices with the bideterminant equal to (0, 0). For example

$$\left\| \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\|^{+} = 0 = \left\| \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\|^{-}$$

Definition: 7.7

The transformation T: $M_n(S) \to M_n(S)$ is called **standard** if there exist permutation matrices P, $Q \in M_n(S)$ and invertible diagonal matrices D, $E \in M_n(S)$ such that T(X) = PDXEQ for all $X \in M_n(S)$ or $T(X) = PDX^tEQ$ for all $X \in M_n(S)$.

Definition: 7.8

We say that the transformation T: $M_n(S) \to M_n(S)$ **preserves a set** $M \in M_n(S)$ if $T(X) \in M_n(S)$ for any matrix $X \in M$.

Notation: 7.9

Let $\Delta_{m,n} = \{(i, j) \mid 1 \le i \le m, 1 \le j \le n\}$ and $E_{m,n} = \{E_{ij} \mid (i, j) \in \Delta_{m,n}\}$, where E_{ij} is the matrix whose $(i, j)^{th}$ entry is 1 and whose other entries are all 0.

Lemma: 7.10

Let T: $M_n(S) \to M_n(S)$ be a linear transformation defined by $T(E_{ij}) = b_{ij} E_{\sigma(i, j)}$ for all $(i, j) \in \Delta$, where $\sigma \in S_{n^2}$ is a permutation on the set Δ and the $b_{i,j} \in S$ are invertible elements for all $(i, j) \in \Delta$. Then the transformation T preserves the set of $\mathcal{S}$-singular matrices if and only if T is a (P, Q, B)-operator.

Proof:

(i) clearly such (P, Q, B)-operators preserve the set of $\mathcal{S}$-singular matrices. Let us prove the inverse statement.

Let B= $(b_{ij}) \in M_n(S)$ be the matrix whose (i, j)th entry is the nonzero entry of $T(E_{i,j})$, for all (i, j) $\in \Delta$.

By assumption T acts on the set of weighted cells. Let i be a fixed index $1 \le i \le n$. We are going to show that there exist an index l, $1 \le l \le n$, and a permutation $T_i \in S_n$ such that either $\sigma(i, j) = (l, T_i(j))$ for all j, $1 \le j \le n$, or $\sigma(i, j) = (T_i(j), l)$ for all j, $1 \le j \le n$.

By Lemma 7.5 the sum of $n^2 = n$ weighted cells in $\mathcal{S}$-singular if and only if it is form $(J\backslash R_k) \circ X$ or $(J\backslash C_k) \circ X$ for some k, $1 \le k \le n$, and $X \in M_n(S)$. Thus the image of $(J\backslash R_i)$ must be equal to $(J\backslash R_l) \circ B$ or $(J\backslash C_l) \circ B$ for some l. We assume at first that $T(J\backslash R_i) = (J\backslash R_i) \circ B$. Thus $(J\backslash R_i) = \sum_{(p,q)\in\Delta, p\neq i0} T(E_{p,q})$ $= \sum_{(p,q)\in\Delta, p\neq i0} b_{p,q} E_{\sigma(p,q)} = (J\backslash R_l) \circ B$. Since σ is a permutation on the set Δ it follows that for all j, j = 1,……….n, $\sigma(i, j) = (l, T_i(j))$ for some l and some permutation $T_i \in S_n$. In this $T(J\backslash R_i) = (J\backslash C_l) \circ B$ we obtain in a similar way that $\sigma(i, j) = (T_i(j), l)$ for all j = 1,………..n and some permutation $T_i \in S_n$.

Since for any i, $1 \le i \le n$ the above considerations hold, it follows that for any i, $1 \le i \le n$ there exist an index l_i. $1 \le l_i \le n$, and a permutation $T_i \in S_n$ such that either $\sigma(i, j) = (l_i, T_i(j))$ for all j = 1,…………n or $\sigma(i, j) = (T_i(j), l_i)$ for all j = 1,………n.

Similarly, it is straightforward to show that for any j, $1 \le j \le n$, there exist an index k_j, $1 \le k_j \le n$, and a permutation $\delta_j \in S_n$ such that either $\sigma(i, j) = (\delta_j(i), k_j)$ for all i, i = 1,………n or $\sigma(i, j) = (k_j, \delta_j(i))$ for all i, i = 1,……….,n.

Thus for all $i, j = 1, \ldots\ldots, n$ we have $T(C_i)$ and $T(R_j)$ are equal to $C_l \circ B$ or $R_k \circ B$ for some k, l.

Now let us show that either $\sigma(i, j) = (l_i, T_i(j))$ for all $(i, j) \in \Delta$ or $\sigma(i, j) = (T_i(j), l_i)$ for all $(i, j) \in \Delta$. Without loss of generality we may assume that $T(R_1) = R_{l_1} \circ B$ (the case $T(R_1) = C_{l_i} \circ B$ can be considered in a similar way). Suppose that there exists an index i, $2 \leq i \leq n$, such that $T(R_i) = C_{l_i} \circ B$. Thus the 2n cells whose sum is equal to $R_1 + R_i$ are mapped by T into 2n-1 weighted cells whose sum is equal to $(R_{l_i}, C_{li}) \circ B$. This contradicts the bijectivity of σ.

Similarly, either $\sigma(i, j) = (\delta_j(i), k_j,)$ for all $(i, j) \in \Delta$ or $\sigma(i, j) = (k_j, \delta_i(i))$ for all $(i, j) \in \Delta$.

Note that by the bijectivity of σ it follows that either $\sigma(i, j) = (l_i, T_i(j)) = (\delta_j(i), k_j)$ for all $(i, j) \in \Delta$ or $\sigma(i, j) = (T_i(j), l_i) = (k_j, \delta_i(i))$ for all $(i, j) \in \Delta$.

Without loss of generality we may assume that $\sigma(i, j) = (\delta_i(i), k_j)$ (the case $\sigma(i, j) = (k_j, \delta_i(i))$ can be considered in a similar way). The for any i_1, i_2, j we have that

$$\sigma(i_1, j) = (\delta_j(i_1), k_j) = \left(l_{i_1}, T_{i_1}(j)\right) \text{ and } \sigma(i_2, j) = (\delta_j(i_2), k_j) = \left(l_{i2}, T_{i_2}(j)\right), \text{ i.e,}$$

$T_{i_1}(j) = T_{i_2}(j)$ for all i_1, i_2, j. Hence, $T_1 = \ldots\ldots = T_n$.

Now let us denote $T = T_i$, $i = 1, \ldots\ldots\ldots n$

Similarly $\delta = \delta_1 = \ldots\ldots = \delta_n$.

It follows now that either $\sigma(i, j) = (\delta(i), T(j))$ for all $(i, j) \in \Delta$ or $\sigma(i, j) = (T(j), \delta_i(i))$ for all $(i, j) \in \Delta$.

Thus there are permutation matrices P and Q such that either $T(X) = P(X^\circ B)Q$ for all $X \in M_n(S)$ or $T(X) = P(X^\circ B)^t Q$ for all $X \in M_n(S)$, i.e T is a (P,Q,B)-operator and the Lemma follows.

Theorem: 7.11

Let $T: M_n(S) \rightarrow M_n(S)$ be a linear operator. Then the following are equivalent:

(i) T is bijective

(ii) T is surjective

(iii) There exist a permutation $\sigma \in S_n^2$ on the set and invertible elements $b_{i,j} \in S$, $(i, j) \in \Delta$ such that $T(E_{ij}) = b_{i,j} E_{\sigma(i, j)}$ for all $(i, j) \in \Delta$.

Theorem: 7.12

Let $T: M_n(S) \rightarrow M_n(S)$ be a surjective linear operator. Then the following are equivalent:

(i) T preserves the set of $\mathcal{S}$-singular matrices,

(ii) T preserves the set of $\mathcal{S}$-nonsingular maries,

(iii) T is a (P, Q, B)-operator and the entries of B are invertible elements from S.

Proof:

Clearly such (P, Q, B)-operators preserve the set of $\mathcal{S}$-singular and $\mathcal{S}$-nonsingular matrices.

Let T is surjective and preserve the set of $\mathcal{S}$-singular matrices. By Theorem 7.11 T is a transformation defined by $T(E_{i,j}) = b_{i,j} E_{\sigma(i, j)}$ for all (i, j), where σ is a permutation on the set $\{(i, j \mid 1 \le i, j \le n)\}$, and $b_{i,j}$ are invertible for all $1 \le i, j \le n$. It follows from Lemma 7.8 that T is a (P, Q, B)-operator and all entries of B are invertible elements of S. In particular, it follows that T preserve the set of $\mathcal{S}$-nonsingular matrices.

Let T is surjective and preserve the set of $\mathcal{S}$-nonsingular matrices. By Theorem 7.10 the transformation T is bijective. Thus the inverse transformation T^{-1} preserves the set of $\mathcal{S}$-singular matrices. By the aforesaid it follows that T^{-1} is a (P, Q, B)-operator where $b_{i,j}$ are invertible elements of S. Thus T is a (P^{-1}, Q^{-1}, C)-operator, where $c_{i,j} = b_{i,j}^{-1}$ and the result follows.

Lemma: 7.13

Let T: $M_n(S) \rightarrow M_n(S)$ be the transformation defined by $T(X) = X \circ B$, where $B = [b_{i,j}] \in M_n(S)$, $b_{i,j}$ are invertible elements for all $(i, j) \in \Delta$. Assume that bidet T(X) = bidet X for any $X \in M_n(S)$. Then T(X) = DXE, where D and E are invertible diagonal matrices with bidet (DE) = (1, 0).

Proof:

We are going to show that rank (B) = 1. Assume to the contrary that rank (B) > 1. Then there exist indices i, j, k, l such that rank (B [i,j | k, l]) = 2. Up to a permutation one may assume that i = k = 1, j = l = 2. In this case, let us consider the matrix $A = E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2} + \sum_{i=3}^{n} E_{i,i}$. It is straightforward to see that bidet (A) = (1, 1). In particular, one has that $\|A\|^+ = \|A\|^-$. Thus $\|T(A)\|^+ = \|T(A)\|^-$ since T preserves the bideterminant. By the definition of T one has that

$$T(A) = \begin{bmatrix} b_{1,1} & b_{1,2} & 0 & \cdots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_{3,3} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{n,n} \end{bmatrix}.$$

Since all entries $b_{i,j}$ are invertible it follows from $\|T(A)\|^{+} = \|T(A)\|^{-}$, by cancellation of $b_{3,3} \ldots\ldots\ldots b_{n,n}$, that $b_{1,1}b_{2,2} = b_{1,2}b_{2,1}$. Therefore

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix} \begin{bmatrix} 1 & b_{1,1}^{-1}b_{1,2} \end{bmatrix}$$

which contradicts the fact that

$$\operatorname{rank} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = 2.$$

Hence rank(B) = 1 and therefore we obtain that $T(X) = DXE$ for all $X \in M_n(S)$ or $T(X) = DX^tE$ for all $X \in M_n(S)$.

Now,

$$(1, 0) = \text{bidet}\,(I) = \text{bidet}\,(T(I)) = \text{bidet}\,(DIE) = \text{bidet}\,(DE).$$

This observation concludes the proof.

Definition: 7.14

Let $P \in M_n(S)$ be a permutation matrix. The **parity $\pi(P)$** is the parity $\pi(\sigma)$ of the permutation σ corresponding to the matrix P.

Note: 7.15

Since the parity function provides a homomorphism from S_n to the cyclic group of order two, we have the usual multiplication rule $\pi(PQ) = \pi(P)\,\pi(Q)$ for any permutation matrices $P, Q \in M_n(S)$.

Definition: 7.16

Let $\overline{X} = [\overline{x}_{i,j}] \in M_n(\mathbb{Z})$ be a (0, 1)-matrix which corresponds to (0, 1)-matrix $X = [x_{i,j}] \in M_n(S)$as follows:

- $\overline{x}_{i,j} = 1 \in \mathbb{Z}$ iff $x_{i,j} = 1 \in S$.
- $\overline{x}_{i,j} = 0 \in \mathbb{Z}$ iff $x_{i,j} = 0 \in S$.

Lemma: 7.17

The following three conditions are equivalent for permutation matrices P and Q $\in M_n(S)$

(i) $\pi(P) = \pi(Q)$;

(ii) bidet(PQ) = (1, 0);

(iii) det $(\overline{P}\,\overline{Q}) = 1$, where the determinant is evaluated as the ordinary determinant over the field of real numbers.

Lemma: 7.18

Let T: $M_n(S) \rightarrow M_n(S)$ be a linear transformation defined by $T(E_{i,j}) = b_{i,j} E_{\sigma(i,\ j)}$ for all (i, j) $\in \Delta$ where σ is a permutation on the set Δ, and $b_{i,j} \in S$ are invertible elements for all (i, j) $\in \Delta$. Assume that T preserves the bideterminant. Then T is standard,

Proof:

For any i, $1 \le i \le n$, let us show that there exist an index l, $1 \le l \le n$, and a permutation $T_i \in S_n$ such that either $\sigma(i, j) = (l, T_i(j))$ for all j, $1 \le j \le n$ or $\sigma(i, j) = (T_i(j), l)$ for all j, $1 \le j \le n$. If n = 1 then all operators are (P, Q, B)-operators with Q = P= 1, we may assume that $n \ge 2$.

Assume that there exist indices i, j, k, $j \neq k$, such that the nonzero entries of the weighted cells $T(E_{i,j})$ and $T(E_{i,k})$ are not in one row or one column. Thus there exist cells $E_{i_1,j_1},\ldots\ldots\ldots E_{i_{n-2},j_{n-2}}$ such that $A = b_{i,j}^{-1}T(E_{i,j}) + b_{i,k}^{-1}T(E_{i,k}) + E_{i_1,j_1} + \ldots\ldots\ldots + E_{i_{n-2},j_{n-2}}$ is a permutation matrix. Then either bidet (A) = (1, 0) or bidet (A) = (0, 1). By the assumptions on T, there exists the transformation T^{-1} and the matrix $T^{-1}(A) = b_{i,j}^{-1}T(E_{i,j}) + b_{i,k}^{-1}T(E_{i,k}) + T^{-1}(E_{i_1,j_1}) + \ldots\ldots\ldots + T^{-1}(E_{i_{n-2},j_{n-2}})$ has both entries $b_{i,j}^{-1}$ and $b_{i,j}^{-1}$ in one row. Also by the assumptions on T the transformation T^{-1} maps weighted cells to weighted cells. Hence $T^{-1}(E_{i_l,j_l})$ has only one nonzero entry for each $l = 1,\ldots\ldots\ldots,n-2$. Thus it follows by the pigeonhole principle that $T^{-1}(A)$ has a zero row. Hence bidet $T^{-1}(A)$ = (0, 0). This contradiction shows that T transforms the set of n cells whose nonzero entries lie in one row into the set of n cells whose nonzero entries lie in a row or a column.

The same arguments prove that for any j, $1 \leq j \leq n$, there exist an index k, $1 \leq k \leq n$, and a permutation $\delta_j \in S_n$ such that either $\sigma(i, j) = (\delta_j(i), k)$ for all i, $1 \leq i \leq n$ or $\sigma(i, j) = (k, \delta_j(i))$ for all i, $1 \leq i \leq n$.

Similar to the proof of Lemma 7.8, it follows that T is a (P, Q, B)-operator.

By Lemma 7.11 it follows that rank (B) = 1. Then by the results which states that "If $T(X) = X \circ B$ for all $X \in M_n(S)$ and rank (B) = 1 then there exist diagonal matrices D and E such that $T(X) = DXE$ for all $X \in M_n(S)$" concludes the proof.

Theorem: 7.19

Let T: $M_n(S) \to M_n(S)$ be a surjective linear transformation. Then bidet T(X) = bidet X for all $X \in M_n(S)$ if and only if T is standard with P and Q of the same parity and bidet (DE) = (1, 0). Here the matrices P, Q are defined uniquely, and the matrices D, E are defined uniquely up to an invertible scalar factor.

Proof:

It is straightforward to check that the transformations under consideration satisfy the condition bidet T(X) = bidet X.

If T preserves the bideterminant then by Lemma 7.15 the transformation T is standard.

Let us show that the permutations corresponding to P and Q are either both even or both odd. Indeed, bidet (I) = (1, 0). Thus, (1, 0) = bidet (I) = bidet (T(I)) = bidet (P(I°B)Q) = $b_{1,1} \ldots\ldots b_{n,n}$ bidet (PQ), and hence, the parity of PQ is even. It follows that P and Q are of the same parity.

It remains to prove the uniqueness. The general situation splits into the following three cases:

(i) Suppose that for any matrix $X \in M_n(S)$ we have that PDXEQ = P'D'XE'Q' for some permutation matrices P, Q, P', $Q' \in M_n(S)$ and invertible diagonal matrices D, E, D', $E' \in M_n(S)$. Thus $(D')^{-1} (P')^{-1}$ PDX = X E' $Q'Q^{-1}E^{-1}$. Then for the matrix X = I one has $(D')^{-1} (P')^{-1}$ PD = E' $Q'Q^{-1}E^{-1}$. We denote F = $(D')^{-1} (P')^{-1}$ PD = E' $Q'Q^{-1}E^{-1}$. Thus FX = XF for any $X \in M_n(S)$. It follows that F is a scalar matrix with an invertible element on the diagonal. Since P, P', Q, Q' are permutation matrices it follows that D and E are defined uniquely up to an invertible scalars factor.

(ii) The case $PDX^tEQ = P'D'X^tE'Q'$ for any matrix $X \in M_n(S)$ can be considered analogously.

(iii) The case $PDXEQ = P'D'X^tE'Q'$ for all $X \in M_n(S)$ is impossible since there do not exist nonzero matrices F such that $FX = X^tF$ for every matrix $X \in M_n(S)$.

$\mathcal{R}$-singularity

In this section we assume that the semiring S is also a subsemiring of a commutative associative ring $\mathcal{R}$ without zero divisors.

Now for any $A = [a_{ij}] \in M_n(S)$ we have that $A \in M_n(\mathcal{R})$. Therefore,

$$\det(A) = \sum_{\sigma \in S_n} (-1)\sigma_{a_{1,\sigma(1)}} \ldots\ldots\ldots a_{n,\sigma(n)} = \|A\|^+ - \|A\|^-$$ is well defined.

Definition: 7.20

We say that a matrix $A \in M_n(S)$ is $\mathcal{R}$ **-right singular** if $Ax = 0$ for some nonzero $x \in \mathcal{R}^n$.

Definition: 7.21

We say that a matrix $A \in M_n(S)$ is $\mathcal{R}$**-left singular** if $x^tA = 0$ for some nonzero $y \in \mathcal{R}^n$.

Lemma: 7.22

For a matrix $A \in M_n(S)$ the following conditions are equivalent:

(i) A is $\mathcal{R}$-right singular;

(ii) A is $\mathcal{R}$ -left singular;

(iii) $\det A = 0$.

Definition: 7.23

We say that a matrix $A \in M_n(S)$ is $\mathscr{R}$**-singular** if one of the equivalent conditions from Lemma 7.19 is satisfied.

Definition: 7.24

A matrix $A \in M_n(S)$ is $\mathscr{R}$**-non singular** if A is not $\mathscr{R}$-singular.

Lemma: 7.25

Let $T: M_n(S) \rightarrow M_n(S)$ be a linear transformation defined by $T(E_{i,j}) = b_{i,j} E_{\sigma(i,\ j)}$ for all $(i, j) \in N$, where σ is a permutation on the set Δ, and $b_{i,j} \in S$ are invertible elements for all $(i, j) \in \Delta$. Assume that T preserves the set of $\mathscr{R}$-nonsingular matrices. Then T is standard.

Theorem: 7.26

Let $T: M_n(S) \rightarrow M_n(S)$ be a surjective linear transformation. The following conditions are equivalent:

(i) T preserves the set of $\mathscr{R}$-nonsingular matrices;

(ii) T preserves the set of $\mathscr{R}$-singular matrices.

(iii) T is standard.

Here the matrices P, Q are defined uniquely, and the matrices D, E are defined uniquely up to an invertible scalar factor.

Proof:

It is straightforward to check that transformations under consideration preserve the set of $\mathscr{R}$-singular and $\mathscr{R}$ -nonsingular matrices.

Let T preserve the set of $\mathscr{R}$-nonsingular matrices. By Theorem 7.9 there exist a permutation σ on the set Δ, and invertible scalars $b_{i,j} \in S$

$(i, j) \in \Delta$ such that $T(E_{i,j}) = b_{i,j} E_{\sigma(i,\ j)}$ for all $(i, j) \in \Delta$. Thus by Lemma 7.22 the transformation T is standard. It follows that T preserves $\mathcal{R}$-singular matrices.

Let T preserve the set of $\mathcal{R}$-singular matrices. By Theorem 7.9 the transformation T is bijective. Thus T^{-1} preserves the set of $\mathcal{R}$-nonsingular matrices. By the aforesaid it follows that T^{-1} is standard. It is straight forward to check that the inverse of a standard transformation is a standard transformation.

The uniqueness can be established as in Theorem 7.16.

Corollary: 7.27

Let S be an arbitrary antinegative subsemiring of a commutative ring R without zero divisors, $T: M_n(S) \to M_n(S)$ be a surjective linear transformation. Then $\det T(X) = \det X$ for all $X \in M_n(S)$ if and only if T is standard where P and Q are of the same parity and $\det(DE) = 1$. Here the matrices P, Q are defined uniquely, and matrices D, E are defined uniquely up to an invertible scalar factor.