Chapter 4

Separation Axioms using δP_S-open sets

4.1 Introduction

The topological spaces usually have two features which do not seem to reconcile each other usually. One thing is any two points can be separated. Second thing is every point can be approached very closely from other points and if a sequence of points are taken generally they will get to cluster around a point. Topological spaces become interesting precisely because they are good in above both features. So far any conditions which guarantee the existence of sufficient number of open sets are not yet studied. Indeed the only open sets whose existence was absolutely required so far are the empty set and whole set. In case of the indiscrete space, they are the only open sets. But most of the theorem reduced to trivialities when specialized to indiscrete space. Since every concept in topology is defined in terms of open sets in order to make non trivial and interesting statements about a space it is necessary that the space possesses a fairly rich collection of open sets.

In this Chapter various degrees of such richness are studied. Separation axioms are one among the most common, important and interesting concepts. They can be used to define more restricted class of topological spaces. Replacing the sets being separated or doing separation in the separation axioms by different types of sets, several extensions of separation axioms have been introduced by mathematicians from time to time. In this chapter, four new types of spaces are defined and its properties are analysed. Also, δP_S-convergence, δP_S-accumulation, δP_S-open cover are defined and using these results δP_S-Compactness is defined. Some of the existing results were analysed using the newly defined spaces.

4.2 Separation Axioms

4.2.1 $\delta P_S T_0$ Spaces

In this section, we introduce a new type of separation axioms called $\delta P_S T_0$ space, this space lies strictly between δT_0 and pre-T_0 space, and we give some properties of $\delta P_S T_0$ space.

Definition 4.2.1.1: A topological space is **$\delta P_S T_0$-space** if to each pair of distinct points x,y of X, there exists a δP_S-open set containing one, but not the other.

It is evident that strongly δT_0- space is $P_S T_0$-space is $\delta P_S T_0$, and $\delta P_S T_0$ is δ-pre-T_0. But the converses may not be true as shown in the following examples:

Example 4.2.1.2: Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ be a topology on X, then X is $\delta P_S T_0$ but not δT_0, since $c \neq d$, there is no δ-open set containing one of them, but not the other.

Example 4.2.1.3: Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\}$ be a topology on X, then X is $\delta P_S T_0$ but not $P_S T_0$, since $P_S O(\tau) = \{X, \emptyset\}$.

Example 4.2.1.4: Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a, b\}\}$ be a topology on X. Then X is δ-pre-T_0, since $\delta PO(\tau) = \mathcal{P}(X)$ but not $\delta P_S T_0$, since for $a \neq b$ in X, there is no δP_S-open set containing one of them, but not the other.

Proposition 4.2.1.5: A topological space X is $\delta P_S T_0$ space if and only if $\delta P_S Cl\{x\} \neq \delta P_S Cl\{y\}$, for every pair of distinct points x, y of X.

Proof: Let X be a $\delta P_S T_0$-space and x, y be any two distinct points of X. There exists a δP_S-open set U containing x or y, say x, but not y. Then $X \backslash U$ is a δP_S-closed set, which does not contain x, but contains y. Since $\delta P_S\, Cl\{y\}$ is the smallest δP_S-closed set containing y, $\delta P_S\, Cl\{y\} \subseteq X \backslash U$, and so $x \notin \delta P_S\, Cl\{y\}$. Consequently $\delta P_S\, Cl\{x\} \neq \delta P_S\, Cl\{y\}$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, $\delta P_S\, Cl\{x\} \neq \delta P_S Cl\{y\}$. Now, let $z \in X$ such that $z \in \delta P_S\, Cl\{x\}$, but $z \notin \delta P_S\, Cl\{y\}$. Now we claim that $x \in \delta P_S\, Cl\{y\}$. For if $x \in \delta P_S\, Cl\{y\}$, then $\{x\} \subseteq \delta P_S Cl\{y\}$, which implies that $\delta P_S\, Cl\{x\} \subseteq \delta P_S Cl\{y\}$. This is contradiction to the fact that $z \notin \delta P_S\, Cl\{y\}$.Consequently x belongs to the δP_S-open set $X \backslash \delta P_S\, Cl\{y\}$ to which y does not belong. It gives that X is $\delta P_S T_0$ space.

Proposition 4.2.1.6: Every semi-regular subspace of a $\delta P_S T_0$ space is $\delta P_S T_0$ space.

Proof: Let Y be an semi-regular) subspace of a $\delta P_S T_0$ space X and x, y be two distinct points of Y. Then there exists a δP_S-open set U containing x or y, say, x but not y. Now by Proposition 2.3.5, $U \cap Y$ is a δP_S-open set in Y containing x but not y. Hence Y is $\delta P_S T_0$-space.

Definition 4.2.1.7: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be a **point $\boldsymbol{\delta P_S}$-closure 1-1** if $x, y \in X$ such that $\delta P_S Cl\{x\} \neq \delta P_S Cl\{y\}$, then $\delta P_S Cl\{f(x)\} \neq \delta P_S Cl\{f(y)\}$.

Proposition 4.2.1.8: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a point δP_S-closure 1-1 function and X is $\delta P_S T_0$ space, then f is 1-1.

Proof: Let $x, y \in X$ with $x \neq y$. Since X is a $\delta P_S T_0$ space, then by Proposition 4.2.1.4, $\delta P_S\, Cl\{x\} \neq \delta P_S\, Cl\{y\}$. But f is point δP_S closure 1-1 implies that $\delta P_S Cl\{f(x)\} \neq \delta P_S Cl\{f(y)\}$. Hence $f(x) \neq f(y)$. Thus, f is 1-1.

Proposition 4.2.1.9: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a mapping from $\delta P_S T_0$ space X into $\delta P_S T_0$ space Y. Then f is point δP_S closure 1-1 if and only if f is 1-1.

Proof: Follows from Proposition 4.2.1.8 and retracing the steps for converse.

Proposition 4.2.1.10: If a topological space X is $\delta P_S T_0$ then it is semi- T_0.

Proof: Let U be a $\delta P_S T_0$ space and x, y be any two distinct points of X. There exists a δP_S-open set U containing x or y, say, x but not y. Then there exists a semi-closed set F such that $x \in F \subseteq U$, so $X \backslash F$ is a semi-open set containing y, and it is obvious that $x \notin X \backslash F$. Therefore X is semi- T_0 space.

4.2.2 $\delta P_S T_1$ Space

In this section, we introduce a new type of separation axioms called $\delta P_S T_1$ space, this space lies strictly between δT_1 and δ-pre-T_1 space, and we give some properties of $\delta P_S T_1$-space.

Definition 4.2.2.1: A topological space X is **$\delta P_S T_1$-space** if to each pair of distinct points x, y of X, there exists a pair of δP_S-open sets containing x but not y, and the other containing y but not x.

Remark 4.2.2.2: Obviously, every δT_1 space is $P_S T_1$-space, $P_S T_1$-space is $\delta P_S T_1$-space and $\delta P_S T_1$-space is $\delta P T_1$-space but the converse is not always true, as shown in the following examples.

Example 4.2.2.3: Let X be an infinite set with the cofinite topology. Simplify the space X is T_1 and hence it is semi- T_1 and pre- T_1. Then by Proposition 2.2.23, X is $P_S T_1$ but not δT_1, since for x and y in X, there is no δ-open set containing one of them, but not the other.

Remark 4.2.2.4: Every $\delta P_S T_1$ space is δ-pre- T_1, but the following example shows that the converse is not always true.

Example 4.2.2.5: Let $X = \{a, b, c\}$ with $\tau = \{X, \emptyset, \{a\}\}$. Then X is δ-pre-T_1, since $\delta PO(\tau) = \mathcal{P}(X)$ but not $\delta P_S T_1$.

Remark 4.2.2.6: Every $\delta P_S T_1$ space is $\delta P_S T_0$, but the converse is not always true as seen in the Example 4.2.1.2, X is $\delta P_S T_0$ but not $\delta P_S T_1$.

Theorem 4.2.2.7: For a topological space X, the following statements are equivalent:

a) X is $\delta P_S T_1$-space

b) Each singleton set {x} is δP_S closed.

c) Each subset of X is the intersection of all δP_S containing it.

d) The intersection of all δP_S-open sets containing the point $x \in X$ is the set $\{x\}$.

Proof: $(a) \Rightarrow (b)$ Suppose (a). Let $x \in X$. Then for any $y \in X, x \neq y$, there exists δP_S-open set U containing y but not x. Hence $y \in U \subseteq X \backslash \{x\}\}$. This implies that $X \backslash \{x\} = \cup \{U: y \in X\{x\}\}$. So, $X \backslash \{x\}$ being a union of δP_S-open sets. Hence, {x} is δP_S-closed.

$(b) \Rightarrow (c)$ Suppose (b). If $A \subseteq X$, then for each point $y \notin A$, there exists a δP_S-open. Hence $A = \cap \{X \backslash \{y\}: y \in X \backslash A\}$ so that the intersection of all δP_S-open sets containing A is the set A itself.

$(c) \Rightarrow (d)$ Follows Directly.

$(d) \Rightarrow (a)$ Suppose (d). Let $x, y \in X$ and $x \neq y$. Hence there exists a δP_S-open set U_x such that $x \in U_x$ and $y \notin U_x$. Similarly, there exists a δP_S-open set U_y such that $y \in U_y$ and $x \notin U_y$. Hence X is $\delta P_S T_1$.

Proposition 4.2.2.8: Every open (or semi-regular) subspace of a $\delta P_S T_1$ space is $\delta P_S T_1$-space.

Proof: Let A be a open (or semi-regular) subspace of a $\delta P_S T_1$ space X. Let $x \in A$. Since X is $\delta P_S T_1$, then by Theorem 4.2.2.7, {x} is δP_S-closed set in X and hence X\{x} is δP_S-open in A, by Proposition 2.3.5.

Consequently, $\{x\}$ is δP_S-closed in A. Hence by Theorem 4.2.2.7, A is $\delta P_S T_1$ space.

4.2.3 $\boldsymbol{\delta P_S T_2}$ Spaces

In this section, we introduce a new type of separation axioms called $\delta P_S T_2$ space, this space lies strictly between δT_2 and δ-pre- T_2 space, and we give some properties of δT_2-space.

Definition 4.2.3.1: A topological space X is $\boldsymbol{\delta P_S T_2}$**-space** if to each pair of distinct points x, y of X, there exists pair of disjoint δP_S-open sets, one containing x and the other containing y.

Remark 4.2.3.2: Since every δ-open set is δP_S-open, then every δT_2 space is $\delta P_S T_2$, but the converse is not always true as shown in the following example.

Example 4.2.3.3: Consider the Prime Integer topology, then the space is both semi- T_1 and pre-T_2 and consequently by Corollary 4.2.3.10, it is $\delta P_S T_2$ but not δT_2, since for x=2 and y=any other number with $x \neq y$. There exists no δ-open sets U_x and V_y such that $U_x \cap V_y = \emptyset$, the only δ-open set containing the prime number p=2 is X.

Remark 4.2.3.4: Clearly every $\delta P_S T_2$ is pre- T_2, but the converse need not be true in general, as in Example 4.2.2.5, the space X is δ-pre- T_2 but not $\delta P_S T_2$.

Remark 4.2.3.5: Clearly every $\delta P_S T_2$ space is $\delta P_S T_1$, but the converse is not always true as shown in the following example.

Example 4.2.3.6: Consider the space (X, τ) given in Example 4.2.2.3. Then the space X is $\delta P_S T_1$ but not $\delta P_S T_2$, since for x and y in X, there is no pair of disjoint δP_S-open sets, one containing x and the other containing y.

Theorem 4.2.3.7: For a topological space X, the following statements are equivalent:

a) X is $\delta P_S T_2$

b) If $x \in X$, then there exists a δP_S-neighborhood N(x) of x such that $y \notin \delta P_S\, Cl(N(x))$

c) For each $x \in X$, $\cap \{(\delta P_S\, Cl(N): N \text{ is a } \delta P_S \text{neighborhood of } x\} = \{x\}$

Proof: $(a) \Rightarrow (b)$ Suppose (a). Let $x \in X$. Then, there exists disjoint δP_S-open sets U, V such that $x \in U, x \notin V$. Then $x \in U \subseteq X \backslash V$, so that X\V is a δP_S-neighborhood of x. We write $N(x) = X \backslash V$. Then N(x) is δP_S-closed and $y \notin \delta P_S\, Cl\big(N(x)\big)$. Hence $y \notin \delta P_S\, Cl\big(N(x)\big)$.

$(b) \Rightarrow (c)$ Follows directly

$(c) \Rightarrow (a)$ Suppose (c). Let $x, y \in X$ and $x \neq y$. Then, by hypothesis there exists a δP_S-closed δP_S neighborhood N of x such that $y \notin N$. Now there is a δP_S-open set U such that $x \in U \subseteq N$.Thus U and $X \backslash N$ are disjoint δP_S-open sets containing x and y respectively. Hence, X is $\delta P_S T_2$.

Proposition 4.2.3.8: Every open (or semi-regular) subspace of a $\delta P_S T_2$ space is $\delta P_S T_2$-space.

Proof: Follows from Proposition 4.2.1.6.

Corollary 4.2.3.9: Let X be a semi-T_1 space. Then X is $\delta P_S T_i$ if and only if is pre-T_i, for $i = 0,1,2$.

Proof: Follows from Proposition 2.2.23.

Proposition 4.2.3.11: Every s-regular T_i space is $\delta P_S T_i$ for $i = 0,1,2$.

Proof: Follows from Proposition 2.2.35.

Remark 4.2.3.12: The following diagram represents the implication between the separation axioms that we have defined in this paper and examples show that no other implications hold between them.

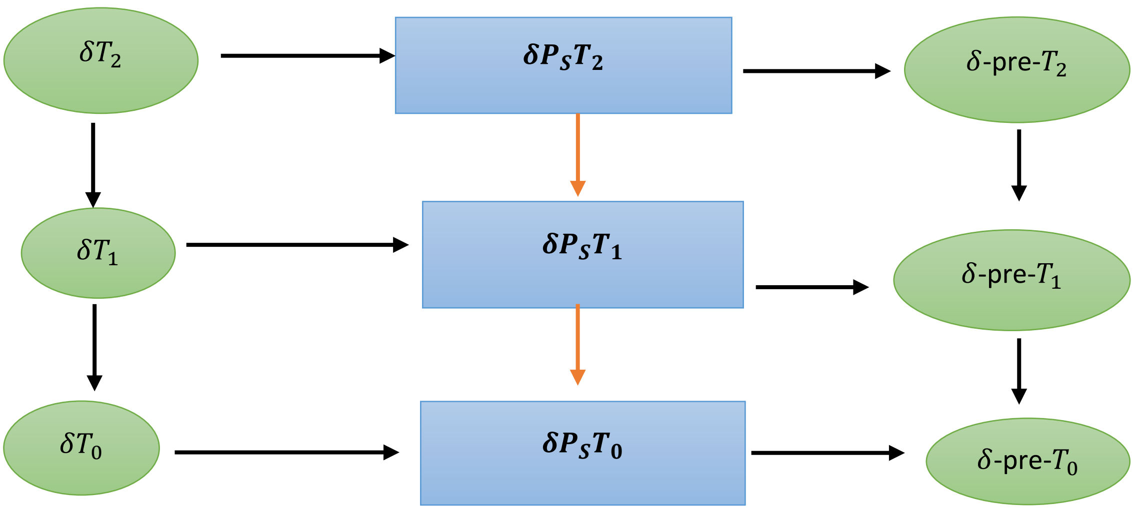

Figure 4.1

4.3 Associated Spaces with Separation Axioms and its relation

Definition 4.3.1. A topological space (X, τ) is said to be a

a) $_{\delta Ps}T_{\delta}$**-space** if every δP_S-open subset of (X, τ) is δ-open in (X, τ).

b) $_{\delta Ps}T_{\theta}$**-space** if every δP_S-open subset of (X, τ) is θ-open in (X, τ).

c) $_{\delta Ps}T_{Ps}$**-space** if every δP_S-open subset of (X, τ) is P_S-open in (X, τ).

d) $_{\delta P}T_{\delta Ps}$**-space** if every δ-pre-open subset of (X, τ) is δP_S-open in (X, τ).

Theorem 4.3.2. If a topological space (X, τ) is $_{\delta Ps}T_{\delta}$ with $A \subseteq X$, then the following are equivalent.

a) A is δP_S-open;

b) A is δ-open.

Proof: Follows from Proposition 2.2.14 and by Definition 4.3.1 (a).

Theorem 4.3.3. If a topological space (X, τ) is $_{\delta Ps}T_{\theta}$ with $A \subseteq X$, then the following are equivalent.

a) A is δP_S-open

b) A is θ -open.

Proof. Follows from Proposition 2.2.18 and Definition 4.3.1 (b)

Theorem 4.3.4. If a topological space (X, τ) is $_{\delta Ps}T_{Ps}$ with $A \subseteq X$, then the following are equivalent.

a) A is δP_S-open

b) A is P_S-open.

Proof: Follows from Proposition 2.2.13 and Definition 4.3.1(c).

Proposition 4.3.5. Every $_{\delta Ps}T_{\theta}$ -space is a $_{\delta Ps}T_{Ps}$ -space but not conversely.

Proof. Let (X, τ) be a $_{\delta Ps}T_{\theta}$ -space and A be a P_S-open set in (X, τ). Since every P_S-open set is δP_S-open, A is δP_S-open. Since (X, τ) is $_{\delta Ps}T_{\theta}$, A is δP_S-open in (X, τ). Hence (X, τ) is a $_{\delta Ps}T_{Ps}$-space.

Proposition 4.3.6. Every $_{\delta Ps}T_{\delta}$-space is also $_{\delta Ps}T_{Ps}$ -space but not conversely.

Proof: Follows from Remark 2.2.19.

Proposition 4.3.7. Every $_{\delta Ps}T_{\theta}$ space is also a $_{\delta Ps}T_{\delta}$-space but not conversely.

Proof: Follows from Remark 2.2.19.

Therefore, from above Definitions we have:

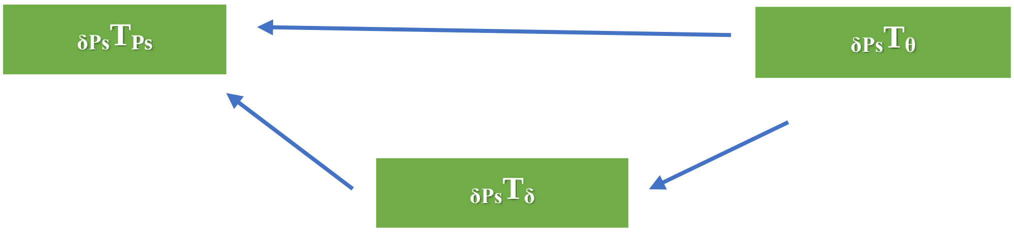

Figure 4.2

In the sequel, none of the above implications is irreversible from the above diagram which is shown in the following examples:

Example 4.3.8. Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\delta O(X) = \delta P_S O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Therefore, it is $_{\delta Ps}T_{\delta}$-space but not $_{\delta Ps}T_{\theta}$ space as $\theta O(X) = \{X, \emptyset\} \neq \delta P_S O(X)$.

Example 4.3.9. Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\delta P_S O(X) = P_S O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Therefore, it is $_{\delta Ps}T_{Ps}$ -space but not $_{\delta Ps}T_{\delta}$-space. Since $\delta O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \neq \delta P_S O(X)$.

Example 4.3.10. Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Thus $\delta P_S O(X) = P_S O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$.Therefore, (X, τ) is $_{\delta Ps}T_{Ps}$-space but not $_{\delta Ps}T_{\delta}$ space since $\delta O(\tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \} \neq \delta P_S O(X, \tau)$.

Proposition 4.3.11. If X is semi-T_1, then X is $_{\delta P}T_{\delta Ps}$-space.

Proof: Follows from Proposition 2.2.23.

Example 4.3.12. Consider X as in Example 4.3.8, is semi-T_1-space. In which $\delta P_S O(X) = \delta PO(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Proposition 4.3.13. If X is discrete, then X is $_{\delta P}T_{\delta Ps}$-space.

Proof: Follows from Theorem 2.2.36(b).

Proposition 4.3.14. Every $_{\delta Ps}T_{\delta}$-space is a hyperconnected space.

Proof: Follows from Lemma 2.2.26(a) and Proposition 2.2.28.

Proposition 4.3.15. In a space if $\theta O(X) = \delta PO(X)$, then (X, τ) is $_{\delta Ps}T_{\theta}$, $_{\delta Ps}T_{\delta}$, $_{\delta Ps}T_{Ps}$, and $_{\delta P}T_{\delta Ps}$-spaces.

Proof: We know from Remark 2.2.19.

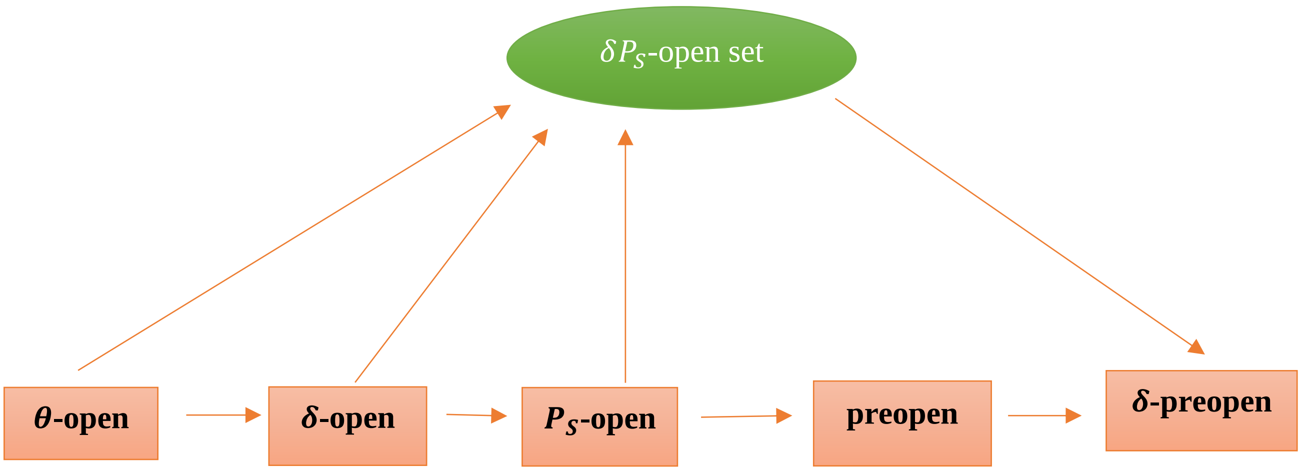

Figure 4.3

From the diagram, all these sets $O(X) = \delta O(X) = P_S O(X) = \delta P_S O(X) = PO(X) = \delta PO(X)$ are equal and hence the result follows.

Proposition 4.3.16. In a space if $\delta O(X) = \delta PO(X)$, then X is $_{\delta P_s}T_{\delta}$, $_{\delta P_s}T_{P_s}$, and $_{\delta P}T_{\delta P_s}$-spaces.

Proof: Follows from the above diagram.

Example 4.3.17: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, $\delta O(X) = \delta PO(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Here we can see it is also $\delta O(X) = PO(X) = P_S O(X) = \delta PO(X)$. Hence (X, τ) is $_{\delta P_s}T_{\delta}$, $_{\delta P}T_{\delta P_s}$, $_{\delta P_s}T_{\theta}$ and $_{\delta P_s}T_{P_s}$ -space.

4.4. δP_S-Compact Spaces

In this section, we introduce a new class of topological spaces called δP_S-compact. This class of spaces lies strictly between the classes of strongly compact space and nearly compact space, but it is not comparable with compact space.

Definition 4.4.1: A filter base $\mathfrak{I}$ in a topological space (X, τ) **δP_S-converges** (resp., **δP_S-θ-converges**) to a point $x \in X$ if for every δP_S-open set V containing x, there exists an F ∈ $\mathfrak{I}$ such that $F \subseteq V$ (resp., $F \subseteq \delta P_S Cl(V)$).

Definition 4.4.2: A filter base $\mathfrak{I}$ in a topological space (X, τ) **δP_S-accumulates** (resp., **δP_S-θ-accumulates**) to a point $x \in X$ if $F \cap V \neq \emptyset$ (resp., $F \cap \delta P_S Cl(V) \neq \emptyset$), for every δP_S-open set V containing and every $F \in \mathfrak{I}$.

It is clear from the above definitions that δP_S-converges (resp., δP_S-accumulates) of filter bases in topological spaces implies δP_S-θ-converges (resp., δP_S-θ-accumulates), but the converses are not true in general as shown in the following example.

Example 4.4.3: Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathfrak{J} = \{X, \emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then $\delta P_S O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Thus, $\mathfrak{J}$ δP_S-θ-converges to a, but $\mathfrak{J}$ does not δP_S-converges to a, because the set {a} is δP_S-open containing a, there is no $F \in \mathfrak{J}$ such that $F \subseteq \{a\}$. Also $\mathfrak{J}$ δP_S- θ-accumulates to b, but $\mathfrak{J}$ does not δP_S-accumulates to b, because the set $\{b\}$ is δP_S-open containing b, there exists an $F \in \mathfrak{J}$ such that $F \cap \{b\} = \emptyset$.

The following propositions are the consequences of the above definitions.

Proposition 4.4.4: If $\mathfrak{J}$ is a maximal filter base in a topological space (X, τ), then $\mathfrak{J}$ δP_S-converges (resp., δP_S-θ-converges) to a point $x \in X$ if and only if $\mathfrak{J}$ δP_S-accumulates (resp., δP_S-θ- accumulates) to a point x.

Lemma 4.4.5: Let $\mathfrak{J}$ be a filter base in a topological space (X, τ). If $\mathfrak{J}$ δ-p-converges (resp., δ-pre-θ-converges) to a point $x \in X$, then $\mathfrak{J}$ δP_S-converges (resp., δP_S-θ-converges) to a point x.

Proof: Suppose that $\mathfrak{J}$ δ-p-converges (resp., δ-pre-θ-converges) to a point $x \in X$,

Let V be any δP_S-open set containing x, then V is δ-preopen set containing x. Since $\mathfrak{J}$ δ-p-converges (resp., δ-pre-θ-converges) to a point $x \in X$, there exists an $F \in \mathfrak{J}$ such that $F \subseteq V$ (resp., $F \subseteq \delta\, pCl(V) \subseteq \delta P_S Cl(V)$) This shows that $\mathfrak{J}$ δP_S-converges (resp., δP_S-θ-converges) to a point x.

The following example shows that the converse of Lemma 4.4.5 is not true in general.

Example 4.4.6: Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}$ and $\mathfrak{J} = \{X, \emptyset, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$. Then $\delta P_S O(X) = \{X, \emptyset, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. Thus, $\mathfrak{J}$ δP_S-converges (resp., δP_S-θ-converges) to a, but $\mathfrak{J}$ does not pre-θ-converges to a and hence does not p-converges to a, because the set $\{a\}$ is preopen containing a, there is no $F \in \mathfrak{J}$ such that $F \subseteq \{a\}$ (resp., $F \subseteq \delta\, pCl(\{a\}) = \{a\}$).

Lemma 4.4.7: Let $\mathfrak{J}$ be a filter base in a topological space (X, τ). If $\mathfrak{J}$ δ-p-accumulates (resp., pre-θ-accumulates) to a point $x \in X$, then $\mathfrak{J}$ δP_S-accumulates (resp., δP_S-θ-accumulates) at a point x.

Proof: The proof is similar to Lemma 4.4.5.

The converse of Lemma 4.4.7 is not true in general as shown by the following example.

Example 4.4.8: Consider the space (X,τ) given in Example 4.4.6. Let $\Im = \{X, \emptyset, \{a,c\}, \{a,c,d\}, \{a,b,c\}\}$. Then $\Im$ δP_S-accumulates (resp., δP_S- θ-accumulates) to b, but $\Im$ does not pre-θ- accumulates to b and hence does not p -accumulates to b, because the set $\{b,d\}$ is preopen containing b, there exists an $F \in \Im$ such that $F \cap pCl(\{b,d\}) = \emptyset$ and hence $F \cap \{b,d\} = \emptyset$.

Lemma 4.4.9: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ δP_S-converges to a point $x \in X$, then $\Im$ δ-converges to a point x.

Proof: Suppose that $\Im$ δP_S-converges to a point $x \in X$. Let V be any open set containing x, then $Int\ Cl(V) \in RO(X)$. Since $RO(X) \subseteq \delta P_S O(X)$ in general, so $Int\ Cl(V) \in \delta P_S O(X)$. Since $\Im$ δP_S- converges to a point $x \in X$, there exists an $F \in \Im$ such that $F \subseteq Int\ Cl(V)$. This shows that $\Im$ δ-converges at a point x.

Lemma 4.4.10: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ δP_S-accumulates to a point $x \in X$, then $\Im$ δ-accumulates to x.

Proof: The proof is similar to Lemma 4.4.9.

Proposition 4.4.11: Let $\Im$ be a filter base in a topological space (X,τ) and E is any semi-closed set containing x. If there exists an $F \in \Im$ such that $F \subseteq E$ (resp., $F \subseteq \delta P_S Cl(E)$), then $\Im$ δP_S- converges (resp., δP_S-θ-converges) to a point $x \in X$.

Proof: Let V be any δP_S-open set containing x, then for each $x \in V$, there exists a semi-closed set E such that $x \in E \subseteq V$. By hypothesis, there exists an $F \in \Im$ such that $F \subseteq E \subseteq V$ (resp., $F \subseteq \delta P_S Cl(E) \subseteq \delta P_S Cl(V)$) which implies that $F \subseteq V$ (resp., $F \subseteq F \subseteq \delta P_S Cl(V)$). Hence $\Im$ δP_S-converges (resp., δP_S-θ-converges) to a point $x \in X$..

Proposition 4.4.12: Let $\Im$ be a filter base in a topological space (X,τ) and E is any semi-closed set containing x. If there exists an $F \in \Im$ such that $F \cap E \neq \emptyset$ (resp., $F \cap \delta P_S Cl(E) \neq \emptyset$), then $\Im$ is δP_S-accumulation (resp., δP_S-θ-accumulation) to a point $x \in X$.

Proof: The proof is similar to Proposition 4.4.11.

Theorem 4.4.13: If a function $f: (X,\tau) \to (Y,\sigma)$ is δP_S-continuous (resp., almost δP_S-continuous), then for each point $x \in X$ and each filter base $\Im$ in X δP_S-converging to x, the filter base $f(\Im)$ is convergent (resp., δ-convergent) to $f(x)$.

Proof: Suppose that x belongs to X and $\Im$ is any filter base in X which δP_S-converges to x. By the δP_S-continuity (resp., almost δP_S-continuity) of f, for any open set V in Y containing $f(x)$, there exists $U \in \delta P_S O(X)$ containing x such that $f(U) \subseteq V$ (resp.,

$f(U) \subseteq IntCl(V)$). But $\mathfrak{J}$ is δP_S-convergent to x in X, then there exists an $F \in \mathfrak{J}$ such that $F \subseteq U$. It follows that $f(F) \subseteq V$ (resp., $f(F) \subseteq IntCl(V)$). This means that $f(\mathfrak{J})$ is convergent (resp., δ-convergent) to $f(x)$.

Definition 4.4.14: Let $\{Q_\alpha : \alpha \epsilon J\}$ be a collection of δP_S-Open subsets of X whose union is X. Then $\{Q_\alpha : \alpha \epsilon J\}$ is called a ***δP_S*-open cover**, if for every δP_S-open cover $\{Q_\alpha : \alpha \epsilon J\}$ of X, there exists a finite subset J_0 of J such that $X = \cup \{Q_\alpha : \alpha \epsilon J_0\}$

Definition 4.4.15: A topological space (X, τ) is called ***δP_S*- Compact** if for every δP_S-open cover of X has a finite subcover.

Proposition 4.4.16: If every semi-closed cover of a space X has a finite subcover, then X is δP_S-compact.

Proof: Let $\mathcal{F} = \{Q_\alpha : \alpha \in J\}$ be any δP_S-open cover of X, then for each $x \in X$ there exists $Q_\alpha(x) \in \mathcal{F}$ such that $x \in Q_\alpha(x)$. Since $Q_\alpha(x)$ is a δP_S-open set there exists a semi-closed set $G_\alpha(x)$ such that $X \in G_\alpha(x) \subseteq Q_\alpha(x)$

Consider the family $\mathcal{G} = \{G_\alpha(x) : x \epsilon X\}$ of semi-closed sets. Then X is covered by $\mathcal{G}$

By hypothesis, this semi-closed cover of X has a finite sub-cover such that

$$X = \cup \{G_\alpha(x_i)/i = 1,2 \dots n\} \subseteq \cup \{Q_\alpha(x_i)/i = 1,2 \dots n\}$$

$\therefore X$ is covered by a finite collection of $\mathcal{F}$

Hence X is δP_S-compact.

Proposition 4.4.17: For a topological space δP_S-Compact space is a P_S-Compact Space.

Proof: Let (X, τ) be a δP_S-Compact space

Consider $\mathcal{F} = \{Q_\alpha : \alpha \in J\}$ be any P_S-open cover of X. Since every P_S-Open set is δP_S-open set by Proposition 2.2.13.

$\mathcal{F}$ is also a δP_S-open cover of X. since X is δP_S-compact. $\mathcal{F}$ has a finite sub-cover $\{Q_i/i = 1,2, \dots n\}$ of X

Hence X is δP_S-Compact.

Lemma 4.4.18: If a topological space (X,τ) is strongly compact, then it is δP_S-compact.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any δP_S-open cover of X. Then $\{V_\alpha : \alpha \in \Delta\}$ is a preopen cover of X. Since X is strongly compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{V_\alpha : \alpha \in \Delta_0\}$. Hence X is δP_S-compact.

The converse of Lemma 4.4.18 is not true as shown by the next example.

Example 4.4.19: Let $X = \mathfrak{R}$ with the topology $\tau = \{X, \emptyset, \{0\}\}$. Then (X,τ) is not strongly compact [Example 1.2.5(b)]. Since the space X is hyperconnected, then by Proposition 2.2.28

$\delta P_S O(X) = \{X, \emptyset\}$. Then (X,τ) is δP_S-compact.

Proposition 4.4.20: Every semi-T_1 and δP_S-compact space is strongly compact.

Proof: Suppose that X is semi-T_1 and δP_S-compact space. Let $\{V_\alpha: \alpha \in \Delta\}$ be any preopen cover of X. Then for every x ∈ X, there exists α(x) ∈ Δ such that $x \in V_{\alpha(x)}$. Since X is semi-T_1, by Proposition 2.2.23, the family $\{V_\alpha: \alpha \in \Delta\}$ is a δP_S-open cover of X. Since X is δP_S-compact, there exists a finite subset Δ_0 of Δ in X such that $X = \cup\{V_\alpha: \alpha \in \Delta_0\}$. Hence X is strongly compact.

Corollary 4.4.21: Let a space X be semi-T_1. Then X is δP_S-compact if and only if X is strongly compact.

Proof: Follows from Lemma 4.4.18 and Proposition 4.4.20.

Proposition 4.4.22: If a topological space (X, τ) is δP_S-compact then it is nearly compact.

Proof: By Proposition 4.4.17, if X is δP_S-compact it is also Ps-compact. Then by Lemma 1.2.4, (X, τ) is nearly compact.

The following example shows that the converse of the above lemma is not true.

Example 4.4.23: The unit interval [0,1] with the usual topology is compact [Example 1.2.5(a)] and hence it is nearly compact, but not δP_S-compact.

Corollary 4.4.24: If a topological space (X, τ) is δP_S-compact, then it is quasi-H-closed.

Proof: From Proposition 4.4.22, a δP_S-compact space is nearly compact space. It is known that every nearly compact space is quasi H-Closed Space.

In general, δP_S-compact spaces and compact spaces are not comparable as shown by the following two examples:

Example 4.4.25: Let $X = (0,1)$ with the topology τ consisting of $X, \emptyset$ and all subsets of X of the form $(0,1 - {}^1\!/_n)$, where $n = 2,3, \ldots$. Then (X, τ) is not compact. Since the space X is hyperconnected, then by Proposition 2.2.28, the family of δP_S-open sets are only $\emptyset$ and X. Therefore, X is δP_S-compact.

Example 4.4.26: Any closed interval $[a, b]$, where $a, b \in \mathcal{R}$ with the relative usual topology is compact, but it is not P_S-compact which implies it is not δP_S-compact from Proposition 4.4.17.

Proposition 4.4.27: If (X, τ) is locally indiscrete, then X is compact if and only if X is δP_S-compact.

Proof: From Proposition 2.2.31 it follows that in an indiscrete space $\delta P_S O(X) = \tau$.

Hence Compactness and δP_S-Compactness are equivalent.

Proposition 4.4.28: If (X,τ) be s-regular and X is δP_S-compact, then it is compact.

Proof: Follows from Proposition 2.2.35.

Theorem 4.4.29: For any topological space (X,τ). The following statements are equivalent:

a) (X,τ) is δP_S-compact,
b) Every maximal filter base $\mathfrak{I}$ in X δP_S-converges to some point $x \in X$.
c) Every filter base $\mathfrak{I}$ in X δP_S-accumulates to some point $x \in X$.
d) For every family $\{F_\alpha : \alpha \in \Delta\}$ of δP_S-closed subsets of X such that $\cap \{F_\alpha : \alpha \in \Delta\} = \emptyset$, there exists a finite subset Δ_0 of Δ such that $\cap \{F_\alpha : \alpha \in \Delta_0\} = \emptyset$.

Proof: (a) ⇒ (b): Suppose that X is δP_S-compact space and let $\mathfrak{I} = \{F_\alpha : \alpha \in \Delta\}$ be a maximal filter base. Suppose that $\mathfrak{I}$ does not δP_S-converges to any point of X. Since $\mathfrak{I}$ is maximal, by Proposition 4.4.4, $\mathfrak{I}$ does not δP_S-accumulates to any point of X. This implies that for every $x \in X$, there exists a δP_S-open set V_x and an $F_\alpha(x) \in \mathfrak{I}$ such that $F_\alpha(x) \cap V_x = \emptyset$. The family $\{V_x : x \in X\}$ is a δP_S-open cover of X and by hypothesis, there exists a finite number of points $x_1, x_2, \dots, x_n$ of X such that $X = \cup \{V(x_i) : i = 1,2,\dots,n\}$. Since $\mathfrak{I}$ is a filter base on X, there exists an $F_0 \in \mathfrak{I}$ such that $F_0 \subseteq \cap \{F_{\alpha(x_i)} : i = 1,2,\dots,n\}$. Hence $F_0 \cap V_{(x_i)} = \emptyset$ for $i = 1,2,\dots,n$. Which implies that $F_0 \cap \left\{\cup V_{(x_i)} : i = 1,2,\dots,n\right\} = F_0 \cap X = \emptyset$. Therefore, we obtain $F_0 = \emptyset$. Contradicting the fact that $F_0 \neq \emptyset$.

(b) ⇒ (c): Let $\mathfrak{I}$ be any filter base on X. Then, there exists a maximal filter base $\mathfrak{I}_0$ such that $\mathfrak{I} \subseteq \mathfrak{I}_0$. By hypothesis, $\mathfrak{I}_0$ δP_S converges to some point x ∈ X. For every F ∈ $\mathfrak{I}$ and every δP_S open set V containing x, there exists an $F_0 \in \mathfrak{I}_0$ such that $F_0 \subseteq V$, hence $\emptyset \neq F_0 \cap F \subseteq V \cap F$. This shows that $\mathfrak{I}$ δP_S-accumulates at x.

(c) ⇒ (d): Let $\{F_\alpha : \alpha \in \Delta\}$be a family of δP_S-closed subsets of X such that $\cap \{F_\alpha : \alpha \in \Delta\} = \emptyset$. Suppose that every finite subfamily $\cap \{F_{\alpha_i} : i = 1,2,\dots n\} \neq \emptyset$. Therefore $\mathfrak{I} = \{\cap F_{\alpha_i} : i = 1,2,\dots n, F_{\alpha_i} \in \{F_\alpha : \alpha \in \Delta\}\}$ form a filter base on X. By hypothesis, $\mathfrak{I}$ δP_S-accumulates to some point $x \in X$. This implies that for every δP_S-open set V containing x, $F_\alpha \cup V \neq \emptyset$ for every $F_\alpha \in \mathfrak{I}$ and every $\alpha \in \Delta$. Since $x \notin \cap F_\alpha$ there exists $\alpha_0 \in \Delta$ such that $x \notin F_{\alpha_0}$. Hence, x is contained in the δP_S-open set $X \backslash F_{\alpha_0}$ and $F_{\alpha_0} \cap X \backslash F_{\alpha_0} = \emptyset$. Contradicting the fact that $\mathfrak{I}$ δP_S-accumulates to x.

(d) ⇒ (a): Let $\{V_\alpha : \alpha \in \Delta\}$ be a δP_S-open cover of X. Then $\{X \backslash V_\alpha : \alpha \in \Delta\}$ is a family of δP_S-closed subsets of X such that $\cap \{X \backslash V_\alpha : \alpha \in \Delta\} = \emptyset$. By hypothesis, there exists a finite subset Δ_0 of Δ such that $\cap \{X \backslash V_\alpha : \alpha \in \Delta_0\} = \emptyset$. Hence $X = \cup \{V_\alpha : \alpha \in \Delta_0\}$. This shows that

X is δP_S-compact.

4.5. $\boldsymbol{\delta P_S}$-Sets and $\boldsymbol{\delta P_S}$-Compact Subspaces

In this section, we introduce a new class of topological space called δP_S-set and δP_S-compact subspace.

Definition 4.5.1: A subset A of a topological space (X,τ) is said to be **$\boldsymbol{\delta P_S}$-set** (resp., **$\boldsymbol{\delta P_S}$-compact subspace**) if for every cover $\{V_\alpha: \alpha \in \Delta\}$ of A by δP_S-open subsets of (X,τ) (resp., by δP_S-open subsets of A), there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup \{V_\alpha: \alpha \in \Delta_0\}$ (resp., $A = \cup \{V_\alpha: \alpha \in \Delta_0\}$).

Lemma 4.5.2: A subset A of a space X is δP_S-set (resp., δP_S-compact subspace) if and only if for every cover of A by δP_S-open sets of X (resp., by δP_S-open sets of A) has a finite subcover.

Proof: The proof follows directly from Definition 4.5.1.

Now we will give several characterizations to δP_S-sets (resp., δP_S-compact subspaces) of topological spaces and also we give some other conditions each of which makes a given topological space a δP_S-compact space.

Proposition 4.5.3: Let A be a subset of a topological space (X,τ). If every cover of A by semi-closed subsets of X (resp., by semiclosed subsets of A) has a finite subcover, then A is δP_S-set (resp., δP_S-compact subspace).

Proof: Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of A by δP_S-open subsets of X (resp., by δP_S-open subsets of A), then for each x ∈ X, there exists $\alpha \in \Delta_0, x \in V_{\alpha(x)}$, there exists a semi-closed set $F_{\alpha(x)}$ such that $x \in F_{\alpha(x)} \subseteq V_{\alpha(x)}$. So, the family $\{F_{\alpha(x)}: x \in X\}$ is a cover of A by semi-closed subsets of X (resp., by semi-closed subsets of A), then by hypothesis, this family has a finite subcover such that $A \subseteq \cup \left\{F_{\alpha(x_i)}: i = 1,2, \dots, n\right\} \subseteq \cup \{V_{\alpha(x_i)}: i = 1,2, \dots, n\}$ (resp., $A = \cup \left\{F_{\alpha(x_i)}: i = 1,2, \dots, n\right\} \subseteq \cup \{V_{\alpha(x_i)}: i = 1,2, \dots, n\}$. Therefore, $A = \cup \{V_{\alpha(x_i)}: i = 1,2, \dots, n\}$. Hence A is a δP_S-set (resp., δP_S-compact subspace).

Theorem 4.5.4: For any topological space (X,τ). The following statements are equivalent:

a) A is δP_S-set (resp., δP_S-compact subspace),
b) Every maximal filter base $\Im$ on X which meets A δP_S-converges to some point of A,
c) Every filter base $\Im$ on X which meets A δP_S-accumulates to some point x ∈ X.
d) For every family $\{F_\alpha: \alpha \in \Delta\}$ of δP_S-closed subsets of (X,τ) such that $[\cap \{F_\alpha: \alpha \in \Delta\}] \cap A = \emptyset$, there exists a finite subset Δ_0 of Δ such that [∩

$\{F_\alpha : \alpha \in \Delta_0\}] \cap A = \emptyset$.

Proof: Similar to Theorem 4.4.29.

Proposition 4.5.5: A space X is δP_S-compact if and only if every proper δP_S-closed set of X is δP_S-set.

Proof: Necessity: Let F be any proper δP_S-closed set of X. Let {Vα: α ∈ Δ} be a cover of F and $V_\alpha \in \delta P_S O(X)$ for every $\alpha \in \Delta$. Since F is δP_S-closed set, then $X \backslash F$ is δP_S-open set. So, the family $\{V_\alpha : \alpha \in \Delta\} \cup X \backslash F$ is a δP_S-open cover of X. Since X is δP_S- compact, there exists a finite subset Δ_0 of Δ such that $X = \cup \{V_\alpha : \alpha \in \Delta_0\} \cup (X \backslash F)$. Therefore, we obtain $F \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$. Hence F is δP_S-set.

Sufficiency: Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of X and $V_\alpha \in \delta P_S$O(X) for every α ∈ Δ. Suppose that $X \neq V_{\alpha_0} \neq \emptyset$ for every $\alpha_0 \in \Delta$. Then $X \backslash V_{\alpha_0}$ is a proper δP_S-closed subset of X. Therefore, by hypothesis, there exists a finite subset Δ_0 of Δ such that $X \backslash V_{\alpha_0} \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$. Therefore, we obtain $X = \cup \{V_\alpha : \alpha \in \Delta_0 \cup \{\alpha_0\}\}$. Which shows that X is δP_S-compact.

Proposition 4.5.6: If a space X is δP_S-compact and A is both regular open and δP_S-closed subset of X, then A is δP_S-compact subspace.

Proof: Let $\{A_\alpha : \alpha \in \Delta\}$ be any cover of A by δP_S-open set of A. Since A ∈ RO(X), by Proposition 2.3.2, $A_\alpha \in \delta P_S O(X)$ for each $\alpha \in \Delta$. Since A is a δP_S-closed subset of X, then $X \backslash A \in \delta P_S$O(X) and $\{A_\alpha : \alpha \in \Delta\} \cup X \backslash A = X$ and $\{A_\alpha : \alpha \in \Delta\} \cup X \backslash A$ forms a δP_S-open cover of X. Since X is δP_S-compact, there exists a finite subset Δ_0 of Δ such that $X = \cup \{A_\alpha : \alpha \in \Delta_0\} \cup X \backslash A$, hence $A = \cup \{A_\alpha : \alpha \in \Delta_0\}$. Therefore, A is δP_S-compact subspace.

Proposition 4.5.7: If there exists either a proper regular semi-open or a δ-open subset A of a topological space (X, τ) such that A and $X \backslash A$ are δP_S-compact subspace, then X is also δP_S-compact.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any δP_S-cover of X. Since A is either regular semi-open or a δ-open subset of X, then for every $\alpha \in \Delta$, by Corollary 2.3.7, we have $A \cap V_\alpha \in \delta P_S O(A)$. Therefore, $\{A \cap V_\alpha : \alpha \in \Delta\}$ is a δP_S-open cover of A. Since A is δP_S-compact subspace, there exists a finite subset Δ_0 of Δ such that $A = \cup \{A \cap V_\alpha : \alpha \in \Delta_0\}$. Therefore, we have $A \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$. Since A is either regular semi-open or a δ-open subset of X, then $X \backslash A$ is also either regular semi-open or a δ-open. By the same way we can find a finite subset Δ_1 of Δ such that $X \backslash A \subseteq \cup \{V_\alpha : \alpha \in \Delta_1\}$. Hence $X = \cup \{V_\alpha : \alpha \in \Delta_0 \cup \Delta_1\}$. This shows that X is δP_S-compact.

Proposition 4.5.8: If there exists a proper clopen subset A of a topological space

(X, τ) such that A and X\A are δP_S-sets, then X is also δP_S-compact.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any δP_S-cover cover of X. Since A is clopen subset of X, then A is regular open subset of X. Therefore, for every $\alpha \in \Delta$, by Proposition 2.3.9, we have $A \cap V_\alpha \in \delta P_S O(X)$. Therefore, $\{A \cap V_\alpha : \alpha \in \Delta\}$ is a cover of A by δP_S-open sets of X. Since A is δP_S-set, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\} \cap A \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$.Since A is clopen subset of X, then X\A is also clopen. By the same way we can find a finite subset Δ_1 of Δ such that $X \backslash A \subseteq \cup \{V_\alpha : \alpha \in \Delta_1\}$. Hence $X = \cup \{V_\alpha : \alpha \in \Delta_0 \cup \Delta_1\}$This shows that X is δP_S- compact.

Proposition 4.5.9: If a regular open set G of a space X is δP_S-set, then G is δP_S-compact subspace.

Proof: Suppose that G ∈ RO(X) and G is δP_S-set. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of G and $V_\alpha \in \delta P_S$O(G) for every $\alpha \in \Delta$. Since $G \in RO(X)$, then by Proposition 2.3.2, we have $V_\alpha \in \delta P_S$O(X) for every $\alpha \in \Delta$. Since G is δP_S-set, there exists a finite subset Δ_0 of Δ such that $G \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$, which implies that G is δP_S-compact subspace.

Proposition 4.5.10: If either G ∈ RSO(X) or G ∈ τ or G is a δ-open set G of a space X is δP_S-compact subspace, then G is δP_S-set.

Proof: Suppose that either $G \in RSO(X)$ or $G \in \tau$ or G is a δ-open set, and $\{V_\alpha : \alpha \in \Delta\}$ be a cover of G and $V_\alpha \in \delta P_S O(X)$ for every $\alpha \in \Delta$. Since either $G \in RSO(X)$ or $G \in \tau$ or G is a δ-open, then for every $\alpha \in \Delta$, by Corollary 2.3.7, we have $G \cap V_\alpha \in \delta P_S O(G)$. Therefore, the family $\{G \cap V_\alpha : \alpha \in \Delta\}$ is a δP_S-open cover of G. Since G is δP_S-compact subspace, there exists a finite subset Δ_0 of Δ such that $G = \cap \{G \cap V_\alpha : \alpha \in \Delta_0\}$. Therefore, $G \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$, which implies that G is δP_S-set.

Corollary 4.5.11: A regular open set G of a space X is δP_S-set if and only if G is δP_S-compact subspace.

Proof: This is an immediate consequence of Proposition 4.5.9 and Proposition 4.5.10.

Proposition 4.5.12: Let A and B be subsets of a space X. If A is δP_S- closed set and B is δP_S-set, then $A \cap B$ is δP_S-set.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of A ∩ B by δP_S-open subsets of X. Since A is δP_S-closed set, then X\A is δP_S-open. So, $B \subseteq \cup \{V_\alpha : \alpha \in \Delta\} \cup X \backslash A$ and the family $\{V_\alpha : \alpha \in \Delta\} \cup X \backslash A$ is a δP_S-open cover of B. Since B is δP_S-set, then there exists a finite subset Δ_0 of Δ such that $B \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\} \cup (X \backslash A)$. Therefore, we obtain A∩ $B \subseteq \cup \{V_\alpha : \alpha \in \Delta_0\}$. Hence $A \cap B$ is δP_S-set.

Proposition 4.5.13: Let Y be any regular open subspace of a space X and A be any subset of Y. Then A is δP_S-set of X if and only if A is δP_S-set of Y.

Proof: Necessity: Suppose that A is δP_S-set of X and $Y \in RO(X)$.

Let $\{V_\alpha: \alpha \in \Delta\}$ be a cover of A and $V_\alpha \in \delta P_S O(Y)$ for every $\alpha \in \Delta$. Since Y ∈ RO(X). Then by Proposition 2.3.4, there exists a δP_S-open set U_α of X such that $V_\alpha = U_\alpha \cap Y$ for every $\alpha \in \Delta$. So, $A \subseteq \cup \{V_\alpha: \alpha \in \Delta\} = \cup\{U_\alpha \cap Y: \alpha \in \Delta\} \subseteq \cup\{U_\alpha: \alpha \in \Delta\}$. Then the family $\{U_\alpha: \alpha \in \Delta\}$ is a cover of A and $U_\alpha \in \delta P_S O(X)$. Since A is δP_S-set of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{U_\alpha: \alpha \in \Delta_0\}$. Since $A \subseteq Y$. Hence $A \subseteq \cup\{U_\alpha \cap Y: \alpha \in \Delta_0\} = \cup\{V_\alpha: \alpha \in \Delta_0\}$. Therefore, A is δP_S-set of Y.

Sufficiency: Suppose that A is δP_S-set of Y and $Y \in RO(X)$. Let $\{U\alpha: \alpha \in \Delta\}$ be a cover of A and $U_\alpha \in \delta P_S O(X)$ for every $\alpha \in \Delta$. Since $A \subseteq Y$. Then $A \subseteq \cup\{U_\alpha: \alpha \in \Delta\} \cap Y = \cup\{U_\alpha \cap Y: \alpha \in \Delta\}$. Since Y ∈ RO(X). Then by Proposition 2.3.4, there exists a δP_S-open set Vα of Y such that $V_\alpha = U_\alpha \cap Y$ for every $\alpha \in \Delta$. Then the family $\{V_\alpha: \alpha \in \Delta\}$ is a cover of A and $V_\alpha \in \delta P_S O(Y)$. Since A is δP_S-set of Y, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{V_\alpha: \alpha \in \Delta_0\} = \cup\{U_\alpha \cap Y: \alpha \in \Delta_0\} \subseteq \cup\{U_\alpha: \alpha \in \Delta_0\}$. Therefore, A is δP_S-set of X.

4.6. Results on Images of δP_S-Compactness

Proposition 4.6.1: If a function $f: (X, \tau) \to (Y, \sigma)$ is δP_S-continuous (resp., almost δP_S-continuous) and A is δP_S-set, then f (A) is compact (resp., N-closed) relative to Y.

Proof: Let $\{G_\alpha: \alpha \in \Delta\}$ be any cover of f (A) by open sets of Y. For each x ∈ A, there exists an α(x) ∈ Δ such that $f(x) \in G_\alpha(x)$. Since f is δP_S-continuous (resp., almost δP_S-continuous), there exists a δP_S-open set U_x of X containing x such that $f(U_x) \subseteq G_\alpha(x)$ (resp., $f(U_x) \subseteq Int(Cl(G_\alpha(x)))$). Then the family $\{U_\alpha: x \in A\}$ is a δP_S-open cover of A. For some finite subset A_0 of A, we have $A \subseteq \cup\{U_x: x \in A_0\}$. Therefore, $f(A) \subseteq \cup\{G_\alpha(x): x \in A_0\}$ (resp., $f(A) \subseteq \cup\{Int(Cl(G_\alpha(x))): x \in A_0\}$). This shows that f (A) is compact (resp., N-closed) relative to Y.

Corollary 4.6.2: If $f: (X, \tau) \to (Y, \sigma)$ is δP_S-continuous (resp., almost δP_S- continuous) surjection function and X is δP_S-compact, then Y is compact (resp., nearly compact).

Proposition 4.6.3: If a function $f: (X, \tau) \to (Y, \sigma)$ is δP_S-continuous (resp., almost δP_S-continuous), A is δP_S-set and F is δP_S-closed subset of X, then $f(A \cap F)$ is compact (resp., N-closed) relative to Y.

Proof: Follows from Proposition 4.6.1 and Proposition 4.7.12.

Proposition 4.6.4: If $f:(X,\tau) \to (Y,\sigma)$is a precontinuous (resp., almost precontinuous) surjection function and X is semi-T_1 and δP_S-compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Proposition 4.6.1 and Corollary 4.4.21.

Proposition 4.6.5: If $f:(X,\tau) \to (Y,\sigma)$ is a δP_S-continuous (resp., almost δP_S-continuous) surjection function and X is locally indiscrete and compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Proposition 4.6.1 and Proposition 4.4.27.

Proposition 4.6.6: If $f:(X,\tau) \to (Y,\sigma)$ is a continuous (resp., almost continuous) surjection function and X is locally indiscrete and δP_S- compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Proposition 4.6.1 and Proposition 4.4.27.

Proposition 4.6.7: If $f:(X,\tau) \to (Y,\sigma)$ is a continuous (resp., almost continuous) surjection function and X is s-regular and δP_S-compact space, then Y is compact (resp., nearly compact).

Proof: Follows from Proposition 4.6.1 and Lemma 4.4.28.