# Avinashilingam Institute for Home Science and Higher Education for Women

**(Deemed to be University under Category 'A' by MHRD, Estd. u/s 3 of UGC Act 1956)**
**Re-accredited with 'A+' Grade by NAAC. Recognised by UGC Under Section 12B**
**Coimbatore - 641 043, Tamil Nadu, India**

**Bachelor's Degree Examination – June 2021**
**VI Semester**

**Class : III UG** | **Time : 3 Hours**
**Major : Mathematics / Special Education and Mathematics** | **Max. Marks: 100**

## 18BMAC21/18BSMC13 Abstract Algebra II

**Part A** **10 x 1 = 10**
**Choose the Correct Answer**

1. If $a, b \in R$ then $d \in R$ is said to be a greatest common divisor of $a$ and $b$ if CO1 K2
   a. $d|a$ b. $d|b$ c. $d|a$ and $d|b$ d. $d|a$ or $d|b$

2. Let $R$ be a commutative ring with unit element. An element $a \in R$ is a unit in $R$ if there exists an element $b \in R$ such that CO1 K4
   a. $a = 1$ b. $ab = 1$ c. $a = 0$ d. $ab = 0$

3. $L(L(S))$= _____, where $L(S)$ is the linear span of $S$. CO2 K3
   a. $L(S)$ b. $S$ c. T d. $L(T)$

4. If $n > m$ then there is a homomorphism of $F^{(n)}$ onto $F^{(m)}$ with a kernel $W$ which is isomorphic to ___ CO3 K1
   a. $F^{(m)}$ b. $F^{(n)}$ c. $F^{(n+m)}$ d. $F^{(n-m)}$

5. If $V$ is a finite-dimensional vector space and $W$ is a subspace of $V$ such that $\dim V < \dim W$, then CO2 K2
   a. $W \subset V$ b. $V \subset W$ c. $V = W$ d. $V \neq W$

6. If $\dim_F V = 10$ then $\dim_F(Hom(V, V)$ is CO2 K1
   a. 10 b. 100 c. 20 d. 1

7. If $\dim_F V = m$ then $\dim_F Hom(V, F) =$ CO2 K2
   a. $m$ b. $m^2$ c. 0 d. $-m$

8. If $V$ is a vector space over $F$ then its dual space is _____ CO3 K1
   a. $V$ b. $F$ c. $Hom(V, F)$ d. $Hom(V, V)$

9. If $u, v \in V$ then $u$ is said to be orthogonal to $v$ if _________ CO5 K2
   a. $(u, v) = 0$ b. $(u, u) = 0$ c. $(v, v) = 0$ d. $(u, v) = 1$

10. If $V$ is a finite-dimensional inner product space and $W$ is subspace of $V$, then $(W^{\perp})^{\perp} =$ ___. CO5 K1
    a. $W$ b. $W^{\perp}$ c. $V$ d. $F$

**Part B** 5 x 6 = 30

**Answer ALL questions**

**Each answer should not exceed 400 words or two pages**

11.a. Let $R$ be an Euclidean ring. Then prove that any two elements $a$ and $b$ in $R$ have a greatest common divisor $d$ and also show that $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$ CO1 K2

(or)

11.b. If $p$ is a prime number of the form $4n + 1$, then prove that $p = a^2 + b^2$ for some integers$a, b$. CO1 K4

12.a. If $V$ is the internal direct sum of $U_1, \dots, U_n$, then prove that $V$ is isomorphic to the external direct sum of $U_1, \dots, U_n$. CO1 K1

(or)

12.b. If $v_1, \dots, v_n \in V$ are linearly independent, then prove that every element in their linear span has a unique representation in the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with the $\lambda_i \in F$. CO2 K2

13.a. If $A$ and $B$ are finite-dimensional subspaces of a vector space $V$, prove that $A + B$ is finite-dimensional and $\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$. CO2 K3

(or)

13.b. Prove that if $v_1, v_2, \dots v_n$ is a basis of $V$ over $F$ and if $w_1, \dots, w_m$ in $V$ are linearly independent over $F$, then $m \leq n$. CO2 K1

14.a. Show that $A(A(W)) = W$ CO3 K2

(or)

14.b. If $n > m$, that is, if the number of unknowns exceeds the number of equations, prove that there is a solution $(x_1, \dots, x_n)$ where not all of $x_1, \dots, x_n$ are 0. CO1 K3

15.a. If $u, v \in V$ prove that $|u, v| \leq ||u||||v||$. CO4 K1

(or)

15.b.If $\{v_i\}$ is an orthonormal set, prove that the vectors in $\{v_i\}$ are linearly independent. If $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$ then prove that $\alpha_i = (w, v_i)$ for $i = 1,2, \dots, n$. CO5 K2

**Part C** 5 x 12 = 60

**Answer ALL questions**

**Each answer should not exceed 800 words or four pages**

16.a. State and prove Unique factorization theorem. CO4 K1

(or)

16.b. Prove that $J[i]$ is a Euclidean ring. CO1 K2

17.a. If $T$ is a homomorphism of $U$ onto $V$ with kernel $W$, then prove that $V$ is isomorphic to $U|W$. Conversely, if $U$ is a vector space and $W$ a subspace of $U$, then prove that there is a homomorphism of $U$ onto $U|W$. CO2 K2

(or)

17.b.(i) Prove that $L(S)$ is a subspace of $V$. CO2 K4

ii. If $v_1, \dots, v_n$ are in $V$ then prove that either they are linearly independent or some $v_k$ is a linear combination of the preceding ones, $v_1, \dots, v_{k-1}$. CO3 K3

18.a. If $V$ is a finite dimensional vector space and if $W$ is a subspace of $V$, then prove that $W$ is finite-dimensionaland $\dim W \leq \dim V$and $\dim V/W = \dim V - \dim W$. CO2 K1

(or)

18.b.i. If $V$ is finite-dimensional over $F$, prove that any two bases of $V$ have the same number of elements. CO3 K3

(ii)Show that $F^{(n)}$ is isomorphic $F^{(m)}$ if and only if $n = m$.

19.a. Prove that if $V$ and $W$ are of dimensions $m$ and $n$, respectively, over $F$, then $Hom(V, W)$ is of dimension $mn$ over $F$. CO5K4

(or)

19.b. Show that if $V$ is finite-dimensional and $v \neq 0 \in V$, then there is an element $f \in \hat{V}$ such that $f(v) \neq 0$. CO2K2

20.a. Prove that every finite dimensional inner product space has an orthonormal set as a basis. CO5K2

20.b. If $V$ is a finite-dimensional inner product space and if $W$ is a subspace of $V$, prove that $V = W + W^{\perp}$ and $V$ is the direct sum of $W$ and $W^{\perp}$. CO3K1

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