# The Rings of Continuous Functions

BY

Vijayalakshmi. S.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043,
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS

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CERTIFIED AS BONAFIDE RESEARCH WORK

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INTRODUCTION

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# Introduction

# INTRODUCTION.

In this thesis we attempt to give a brief survey of the fundamental results of "Rings of continuous functions on a topological space".

In the first chapter, we define lattice operations and Ring operations in C(X), the set of all continuous functions from a topological space X into a topological space R and study the interrelations between them. The zero-sets of continuous functions play an important role in the study of C(X). The important results proved in the first chapter are as follows :

(1) Every (ring) homomorphism from C(Y) or C*(Y) into C(X) is a lattice homomorphism.

(2) Every (ring) homomorphism from C(Y) or C*(Y) into C(X) takes bounded functions to bounded functions.

Denoting by Z(f), the zero-set of the function f in C(X), by Z [X], the collection of all Z (f); f in C(X), we find that Z[X] is closed under finite intersection. Completely separated sets are characterized in terms of zero-sets also as follows :

"Two sets are completely separated iff they are contained in disjoint zero-sets. Moreover, completely separated sets have disjoint zero-set-neighborhoods".

A subspace S of X is C - embedded in X if every function in C(S) can be extended to a function in C(X). Likewise, S is C* - embedded in X if every function in C*(S) can be extended to a function C*(X). The basic result about C* - embedding (Urysohn's theorem) and the relation between C* - embedding and C - embedding are as follows :

"A subspace S of X is C* - embedded in X iff any two completely separated sets in S are completely separated in X".

"A C* - embedded subset is C - embedded iff it is completely separated from every zero-set disjoint from it".

In the second chapter, we study the ideals and z - filters in C(X). The important results proved here are as follows :

(1) If I is an ideal in C(X), then the family $Z[I] = \{ Z(f) : f \in I \}$ is a z - filter on X.

(2) If $\mathscr{F}$ is a z - filter on X, then the family $Z^{\leftarrow}[\mathscr{F}] = \{f : Z(f) \in \mathscr{F}\}$ is an ideal in C.

Connecting maximal ideal and z - ultrafilter on C(X) we have the following results :

(1) If M is a maximal ideal in C(X), then Z(M) is a z - ultrafilter on X.

(2) If A is a z - ultrafilter on X, then $Z^{\leftarrow}[A]$ is a maximal ideal in C.

The mapping Z is one - one from the set of all maximal ideals in C onto the set of all z - ultrafilters.

Connecting z - ideals and prime ideals we have the following theorem :

"For any z - ideal I in C, the following are equivalent".

(1) I is prime
(2) I contains a prime ideal
(3) For all $g, h \in C$, if $gh = 0$ then $g \in I$ or $h \in I$
(4) For every $f \in C$, there is a zero-set in Z[I] on which f does not change sign.

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# Chapter I

# CHAPTER I

## RING OPERATIONS AND LATTICE OPERATIONS IN C (X)

### Section 1

**Definition : 1.1.1**

S is denumerable (countably infinite) iff there is a 1 - 1 function f ; $I^{+} \xrightarrow{\text{onto}} S$ where $I^{+}$ is a set of positive integers.

**Definition : 1.1.2**

A set S is said to be countable if it is finite or denumerably infinite.

**Definition : 1.1.3**

Let  be a nonempty family of sets. $\mathcal{S}$ is said to have the finite (respectively countable) intersection property provided that the intersection of any finite (respectively countable) number of members of $\mathcal{S}$ is nonempty.

**Definition : 1.1.4**

Let S be a nonempty set. A partial order relation in S is a relation which is symbolized by $\leqq$ and assumed to have the following properties:

1) $a \leq a$ for every $a$ (reflexivity)

2) $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetry)

3) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)

A nonempty set $S$ is which there is defined a partial order relation is called a partially ordered set.

Definition : 1.1.5

A partial ordering $<$ on $S$ is a total ordering iff for every $a, b \in S$ either $a \leq b$ or $b < a$

Definition : 1.1.6

A mapping $\emptyset$ from a partially ordered set $A$ into a partially ordered set $E$ is said to preserve order if $a \leq b$ in $A$ implies $\emptyset a \leq \emptyset b$ in $E$.

Definition : 1.1.7

A maximal element of $A$ is an element $a$ such that $X \geq a$ implies $X = a$. The largest element of $A$ necessarily unique. If it exists - is the element $C$ such that $c \geq x$ for all $X \in A$. Minimal and smallest are defined similarly.

Definition : 1.1.8

In a partially ordered set, the symbol $a v b$ denotes sup $\{a, b\}$, that is, the smallest element $C$ - if one exists - such that $c \geq a$ and $c \geq b$ Likewise, $a \wedge b$ stands for $\inf\{a, b\}$.

When both $a \vee b$ and $a \wedge b$ exist for all $a, b \in A$, then A in called a lattice. A subset S is a sublattice of A provided that, for all $x, y \in S$ the elements $x \vee y$ and $x \wedge y$ of A belongs to S.

Definition : 1.1.9

A mapping $\phi$ from a lattice A into a lattice E is a lattice homomorphism into E provided that

$$\emptyset(a \vee b) = \emptyset a \vee \emptyset b \text{ and } \emptyset(a \wedge b) = \emptyset a \wedge \emptyset b.$$

Note :

$\emptyset\,[A]$ is a sublattice of E.

Definition : 1.1.10

A partially ordered set in which every nonempty subset has both a supremum and an infimum is said to be lattice - complete.

Definition : 1.1.11

A subset S of a totally ordered set A is said to be confinal (respectively coinitial) if for every $x \in A$ there exists $s \in S$ such that $s \geq x$ (respectively $s \leq x$)

Proposition : 1.1.12

Hausdorff's maximal principle.

Every partially ordered set contains a maximal chain. (that is maximal in the class of all chains as partially ordered by set inclusion).

Definition : 1.1.13

A discrete subspace means a subspace that is discrete in its relative topology but not necessarily closed in the space.

**Example** : $\{1/n\}_{n \in N}$ is a discrete subspace of R.

Definition : 1.1.14

A will denote a commutative ring having a unity element, that is, an element 1, necessarily unique such that 1.a = a for all a. A unit of A is an element a that has a multiplicative inverse $a^{-1}$ that is an element such that $aa^{-1} = 1$.

Definition : 1.1.15

Ideal will always mean proper ideal that is a subring I $\neq$ A such that a $\in$ I implies xa $\in$ I for all X $\in$ A. Thus an ideal cannot contain a unit.

Definition : 1.1.16

The intersection of any nonempty family of ideals is an ideal. The smallest ideal containing an ideal I and an element a is denoted by (I,a); it consists of all elements of the form $i + xa$ where $i \in I$ and $x \in A$.

Definition : 1.1.17

An ideal P in A is prime if $ab \in P$ implies $a \in P$ or $b \in P$, that is A/P is an integral domain.

Proposition : 1.1.18

Every maximal ideal is prime.

**Proof :**

If M is a maximal ideal, with respect to set inclusion then $a \notin M$ implies $1 \in (M, a)$, so that $1 \equiv xa$ (Mod M) for some $X \in A$. Conversely $1 \equiv xa \pmod{M}$ implies $1 \in (M, a)$. Thus, an ideal M is maximal if and only if A/M is afield, that is, every maximal ideal is prime.

Result : 1.1.19

The maximal principle every ideal is contained in a maximal ideal and hence that every non-unit of A belongs to some maximal ideal.

Definition : 1.1.20

Let a partial ordering relation be defined on the ring A. Then A is called a partially ordered ring provided that

1) $a \geqq b$ implies $a + x \geqq b + x$ for all x
2) $a \geqq b$ and $b \geqq 0$ implies $ab \geqq 0$.

Remark : 1.1.21

$a \geqq b$ if and only if $a - b \geqq 0$

$a \geqq 0$ if and only if $-a \leqq 0$

if $a \leqq r$ and $b \leqq s$ then $a + b \leqq r + s$.

To define such a partial ordering relation, it is enough to spacify the elements $\geqq 0$ subject to :

$a \geqq 0$ and $-a \geqq 0$ iff $a = 0$ and

$a \geqq 0$ and $b \geqq 0$ implies $a + b \geqq 0$ and $ab \geqq 0$ and then to define $a \geqq b$ to mean $a - b \geqq 0$.

To establish that a homomorphism $\phi$ from A into a partially ordered ring is order-preserving, it suffices to show that $a \geqq 0$ implies $\phi a \geqq 0$.

Definition : 1.1.22

If $a \vee b$ exists, for all a and b then $a \wedge b$ exists and $a \wedge b = -(-a \vee -b)$. A is called a lattice in which case it is called a lattice - ordered ring if $a \vee b$ exists for each a and b.

In a lattice - ordered ring $|a|$ denotes the element $a \vee -a$; it satisfies $|a| \geqq 0$

A is called totally ordered if every element is comparable with 0.

Definition : 1.1.23

Let X be any set . A collection $\mathcal{A}$ of nonempty subsets of X is said to be a filter on x if

1) $\mathcal{A} \neq \emptyset$

2) if A and B are in $\mathcal{A}$ then $A \cap B$ is also in $\mathcal{A}$

3) If $A \in \mathcal{A}$ and $A \subset B$ then $B \in \mathcal{A}$

Note :

An ultrafilter on X is a maximal filter on X.

## Section 2

Notation :

The collection of all functions from the topological space X into the set R of real numbers is denoted by $R^X$

Preposition : 1.2.1

$R^X$ is a commutative ring with unity element provided that X is not empty.

**Proof :**

Define the addition and multiplication of two functions f, g in $R^X$ as follows :

$$(f+g)(x) = f(x)+g(x) \text{ and } (fg)(x) = f(x)\,g(x)$$

Then $R^X$ is a commutative ring with unity element. In which we have the following: The zero element is the constant function 0 and the unity element is the constant function 1. The additive inverse $-f$ of f is characterized by the formula $(-f)(x) = -f(x)$. The multiplicative inverse $f^{-1}$ of f is characterized by the formula $f^{-1}(x) = \frac{1}{f(x)}$

Definition : 1.2.2

The partial ordering on $R^X$ is defined by $f \geq g$ iff $f(x) \geq g(x)$ for all $x \in X$.

**Note :** This is a partial ordering relation, since R is ordered.

Theorem : 1.2.3

$R^X$ is a lattice - ordered ring.

**Proof :**

Let $f, g \in R^X$

Then the partial ordering relation on $R^X$ is defined by

$f \geqq g$ iff $f(x) \geqq g(x)$ for all $x \in X$.

Therefore, for every h in $R^X$

$f+g \geqq g+h$ iff $f \geqq g$.

That is, the ordering relation is invariant under translation.

Also we have $f \geqq 0$ and $g \geqq 0 \Rightarrow fg \geqq 0$ Thus, $R^X$ is a partially ordered ring. For any f and g in $R^X$, if we define the function k as $k(x) = f(x) \vee g(x)$ then k satisfies $k \geqq f$ and $k \geqq g$ Also, for all h such that $h \geqq f$ and $h \geqq g$ we have $h \geqq k$. Therefore, $f \vee g$ exists and it is equal to k. Similarly $(f \wedge g)(x) = f(x) \wedge g(x)$. Hence, $R^X$ is a lattice ordered ring.

Result : 1.2.4

The function $|f|$ defined as $f \vee -f$ satisfies $|f|(x) = |f(x)|$

**Result : 1.2.5**

R is totally ordered but $R^X$ is not if X contains at least two points.

Definition : 1.2.6

The set of all continuous functions from the topological space X into the topological space R is denoted by C(x) or C.

Theorem : 1.2.7

C(x) is a commutative ring, a subring of $R^X$

**Proof :**

We know that the sum and product of two continuous functions are continuous. And if $f \in C$ then $-f \in C$. The constant function $1 \in C$ and it is the unity element. Therefore C(x) is a commutative ring, a subring of $R^X$ .

Theoram : 1.2.8

C is a sublattice of $R^X$

**Proof :**

Since f is continuous, $|f|$ is also continuous. For $f,g \in C$, we have $f \vee g = 2^{-1}(f+g+|f-g|)$

$\Rightarrow f \vee g \quad C$

Similarly $f \vee g \in C$

$\therefore$ C is a sublattice of $R^X$

Result : 1.2.9

If $f \geq 0$ , then f has a unique nonnegative $r^{th}$ power ($r \in R$, $r>0$) dentoted by $f^r$ and defined by $f^r(x) = f(x)^r$ $(x \in X)$

and if f is continuous, then $f^r$ is also continuous. If n is odd ($n \in N$) then $f^{1/n}$ may be defined as a function in C for any $f \in C$.

Proposition : 1.2.10

If the space X is discrete, then every function on X is continuous, so that $R^X$ is the same as C(x). Conversely if $R^X = C(x)$, then the characteristic function of every set in X is continuous, which shows that the space is discrete.

Definition : 1.2.11

The set of all bounded function in C(x) is defined by $C^* = C^*(X)$

Result : 1.2.12

$C^*$ is a subring and sublattice of C

Definition : 1.2.13

If the subring $C^*(X)$ is all of C(X) that is, every function in C(X) is bounded then X is said to be pseudocompact.

**Example : 1.2.14**

Every compact space is pseudocompact.

Proposition : 1.2.15

Every countably compact space is pseudocompact.

**Proof :**

By definition, X is countably compact provided that every family of closed sets with the finite intersection property has the countable intersection property, that is, every countable open cover has a finite subcover. Suppose, now, that X is countably compact, and consider any function f in C(X). The sets $\{x : |f(x)|<n\}$ for $n \in N$ constitute a countable open cover of X. Hence a finite subfamily covers X, that is, f is bounded.

### Theorem : 1.2.16

Every (ring) homomorphism t from C(Y) or C*(Y) into C(X) is a lattice homomorphism.

**Proof :**

Let t be a ring homomorphism from C(Y) or C* (Y) into C(X)

Since $g = 1^2$ implise $tg = (t1)^2$ , t sends nonnegative functions into nonnegative functions, that is, t is order - preserving.

$$(t|g|)^2 = t(|g|)^2 = t(g^2) = (tg)^2$$

and since $t|g| \geq 0$, we have $t|g| = |tg|$.

combaining this with the formula

$(g \vee h) + (g \vee h) = g+h+|g-h|$

We get

$$t(gvh)+t(gvh) = tg+th+|tg-th| = (tgvth) + (tgvth)$$

But $t(gvh)$ and $tgvth$ are real - valued functions (defined on X) and therefore $t(gvh) = tgvth$.

Theorem : 1.2.17

Every (ring) homomorphism t from C(Y) or C* (Y) into C(X) takes bounded functions to bounded functions.

**Proof :**

Let t be a ring homomorphism for C(Y) or C*(Y) into C(X). As with any homomorphism $t1 = t(1.1) = (t1)(t1)$, so that the function $t1$ in C(X) is an idempotent. Therefore it can assume no values on X other than 0 or 1. Hence for each n N , the function $tn = t1+t1+..............+t1$ assumes no values other than 0 or n, consider, now, any function g in C*(Y). Since $|g| \leq n$, for suitable $n \in N$, we have $|tg| \leq tn \leq n$

Corollary : 1.2.18

If X is not pseudocompact then C(X) is not a homomorphic image of C*(Y), for any Y

**Note :**

C(X) and C*(X) are isomorphic only if they are identical.

Corollary : 1.2.19

An isomorphism from C(Y) onto C(X) carries C*(Y) onto C*(X).

The above result is also a corollary of the next theorem.

Theorem : 1.2.20

Let $t$ be a homomorphism from $C(Y)$ into $C(X)$ whose image contains $C^*(X)$. Then $t$ carries $C^*(Y)$ into $C^*(X)$.

**Proof :**

To prove $t1 = 1$

Let $k \in C(Y)$ such that $tk = 1$

Then $t1 = (tk)(t1) = t(k1) = tk = 1$

For each $n \in N$ we have $tn = n$

Claim :

To find $g \in C^*(Y)$ such that $tg = f$ where given $f \in C^*(X)$

Choose $h \in C(Y)$ for which $th = f$ and Choose $n \in N$ such that $|f| \leq n$ .

If we define $g = (-n \vee h) \wedge n$, then $g \in C^*(Y)$ and $tg = (-n \vee f) \wedge n = f$.

## Section 3

**Definition : 1.3.1**

The set $f^{\leftarrow}(0) = \{x \in X | f(x)=0\}$ Where $f \in C(X)$ is called the Zero - set of f. It is denoted by $Z(f)$ or $Z_X(f)$

**Note :**

Any set that is a zero-set of some function in C(X) is called a zero-set in X. Thus, Z is a mapping from the ring C into the set of all zero-sets in X.

**Results : 1.3.2**

(1) $Z(f) = Z(|f|) = Z(f^n)$ (for all $n \in N$), $Z(0) = X$ and $Z(1) = \emptyset$.

(2) $Z(fg) = Z(f) \cup Z(g)$

(3) $Z(f^2+g^2) = Z(|f|+|g|) = Z(f) \cap Z(g)$

(4) If $f \in C$ and $g = |f| \wedge 1$, then $g \in C^*$ and $Z(g) = Z(f)$. Hence C and C* yield the same zero-sets.

**Definition : 1.3.3**

Every set of the form $\{x : f(x) \geq 0\}$ is a zero-set.

$\{x : f(x) \geq 0\} = Z(f \wedge 0) = Z(f - |f|)$

Likewise,

$\{x : f(x) \leq 0\} = Z(f \vee 0) = Z(f + |f|)$

Thus, the open sets

$\text{Pos } f = \{x : f(x) > 0\}$

and

$\text{neg } f = \{x : f(x) < 0\} = \text{pos }(-f)$ are cozero-sets, that is, complements of zero-sets. Conversely, every cozero-set is of the form $X - Z(f) = \text{pos } |f|$

Definition : 1.3.4

For a function f in C(X), $f^{-1}$ exists if and only if f vanishes no where on X. In other words, f is a unit of C if and only if $Z(f) = \emptyset$

Result : 1.3.5

If f is a unit of C*, then $Z(f) = \emptyset$ . The converse need not hold, however, as the multiplicative inverse $f^{-1}$ of f in C may not be a bounded function. In-fact, the condition for C* is clearly the following a function f in C* is a unit of C* if and only if it is bounded away, from zero, that is, $|f| \geq r$ for some $r>0$.

Definition : 1.3.6

$Z[C']$ where $C' \subset C(X)$ is defined as the family of zero-sets $\{Z(f) \mid f \in C'\}$. The family $Z[C(X)]$ of all zero-sets in X will be denoted by $Z(X)$.

Note :

$Z[C^*(X)]$ is same as $Z(X)$ and $Z(X)$ is closed under the formation of finite unions and finite intersections.

Proposition : 1.3.7

Z(X) is closed under countable interesection.

**Proof :**

Given $f_n \in C$, define $g_n = |f_n| \wedge 2^{-n}$ , and let $g(x) = \sum_{n \in N} g_n(x)$ $(x \in X)$ since $|g_n| \leq 2^{-n}$, the series converges uniformly, and therefore g is a continuous function. Clearly,

$$Z(g) = \sum_{n \in N} Z(g_n) = \sum_{n \in N} Z(f_n)$$

Proposition : 1.3.8

Z(X) need not be closed under infinite union. For example, every one-element set in R is a zero-set in R, so that an infinite union of zero-sets need not even be closed. In a general space, even a closed, countable union of zero-sets need not be a zero-set. Nor need Z(X) be closed under arbitrary intersection.

Definite : 1.3.9

Two subsets A and B of X are said to be completely separated (from one another) in X if there exists a function f in C*(X) such that $0 \leq f \leq 1$, $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

Clearly, it is enough to find a function g in C(X) satisfying $g(x) \leq 0$ for all $x \in A$ and $g(x) \geq 1$ for all $x \in B$;

for then $(0 \vee g) \wedge 1$ has the required properties. And, the numbers 0 and 1 may be replaced in the definition by any real numbers r and s. (with $r < s$)

**Result : 1.3.10**

Two sets contained (respectively) in completely separated sets are completely separated, and that two sets are completely separated if and only if their closures are.

**Note :**

When a zero-set **Z** is a neighborhood of a set A, we refer to **Z** as a zero-set-neighborhood of A.

**Theorem : 1.3.11**

Two sets are compltely separated if and only if they are contained in disjoint zero-sets. Moreover, completely separated sets have disjoint zero-set-neighborhoods.

**Proof :**

We begin with the sufficiency. If $Z(f) \cap Z(g) = \emptyset$, Then $|f| + |g|$ has no zeros, and we may define

$$h(x) = \frac{|f(x)|}{|f(x)| + |g(x)|} \qquad (x \in X)$$

-in brief, $h = |f| \cdot [|f|+|g|]^{-1}$. Then $h \in C(X)$, and h is equal to 0 on $Z(f)$ and to 1 on $Z(g)$

Coversely, if A and A' are completely separated, there exists $f \in C(X)$ equal to 0 on A and to 1 on A'. The disjoint sets $F = \{x : f(x) \leqq 1/3\}$, $F' = \{x : f(x) \geqq 2/3\}$ are zero-set-neighborhoods of A and A', respectively.

Corollary : 1.3.12

If A and A' are completely separated, then there exist zero-sets F and **Z** such that $A \subset X - Z \subset F \subset X - A'$

In the above theorem if we take $Z = \{x : f(x) \geq 1/3\}$ We get the result.

Definition : 1.3.13

A subspace S of X is C-embedded in X if every function in C(S) can be extended to a function in C(X). Likewise, we say that S in C* -embedded in X if every function in C* (S) can be extended to a function in C*(X).

Result : 1.3.14

If a function f in C*(S) has an extension g in C(X), then f also has a bounded extension: if n is a bound for $|f|$, then $(-n \vee g) \wedge n$ belongs to C*(X) and agrees with f on S. Thus, S is C* -embedded in X if and only if every function in C*(S) can be extended to function in C(X).

If $S \subset X \subset Y$, and X is C-embedded in Y, then S is C-embedded in Y if and only if it is C-embedded in X. The corresponding transitivity is valid for C*.

Theorem : 1.3.15

Urysohn's Extension Theorem.

A subspace S of X is C* - embedded in X if and only if any two completely separated sets in S are completely separated in X.

**Proof** :

Necessity : If A and B are completely separated sets in S, there exists a function f in $C^*(S)$ that is equal to 0 on A and 1 on B. By hypothesis, f has an extension to a function g in $C^*(X)$. Since g is 0 on A and 1 on B, these sets are completely separated in X.

Sufficiency : Let $f_1$ be a given function in $C^*(S)$. Then $|f_1| \leqq m$ for some $m \in N$. For convenience of notation, define

$$r_n = \frac{m}{2}\left(\frac{2}{3}\right)^n \qquad (n \in N)$$

Then $|f_1| \leqq m = 3r_1$. Inductively, given $f_n \in C^*(S)$ with $|f_n| \leqq 3r_n$ define.

$$A_n = \{s \in S : f_n(s) \leqq -r_n\} \text{ and}$$
$$B_n = \{s \in S : f_n(s) \geqq r_n\}$$

Then $A_n$ and $B_n$ are completely separated in S, and so, by hypothesis, they are completely separated in X. Accordingly, there exists a function $g_n$ in $C^*(X)$, equal to $-r_n$ on $A_n$, and to $r_n$ on $B_n$, and with $|g_n| \leqq r_n$.

The values of $f_n$ and $g_n$ on $A_n$ lie between $-3r_n$ and $-r_n$ ; on $B_n$, they lie between $r_n$ and $3r_n$; and else where on S, they are between $-r_n$ and $r_n$. We now define

$$f_{n+1} = f_n - g_n|S.$$

and we have $|f_{n+1}| \leqq 2r_n$

that is, $|f_{n+1}| \leqq 3r_{n+1}$

This completes the induction step. Now put $g(x) = \sum_{n \in N} g_n(x)$

Because the series converges uniformly, this defines g as a continuous function on X. Next, we observe that

$$(g_1 + g_2 + \ldots\ldots\ldots + g_n)|S = (f_1 - f_2) + \ldots\ldots + (f_n - f_{n+1})$$

$$= f_1 - f_{n+1}.$$

Since the sequence $(f_{n+1}(s))$ approaches 0 at every point s of S, this shows that $g(s) = f_1(s)$. Thus, g is an extension of $f_1$. This completes the proof.

Theorem : 1.3.16

A C*- embedded subset is C-embedded if and only if it is completely separated from every zero-set disjoint from it.

**Proof :**

Let S be C* - embedded in X.

Necessity : Given a zero-set $Z(h)$ in X, disjoint from S, put $f(s) = \frac{1}{h(s)}$ for $s \in S$. This defines f as a continuous function on S. Let g be continuous extension of f to all of X. Then gh belongs to C(X), and is equal to 1 on S and to 0 on **Z**(h)

Sufficiency : Consider any function f in C(S). Then arctan $\circ$ f belongs to C*(S), and has an extension to a function g in C(X). The set $\mathbf{Z} = \{x \in X : |g(x)| \geqq \pi/2\}$ belongs to **Z** (X), and is disjoint from S. By hypothesis, there is a function h in C(X) equal to 1 on S and to 0 on **Z** and with $|h| \leqq 1$. The function gh then agrees with arctan $\circ$ f on S, and satisfies $|(gh)(x)| < \frac{\pi}{2}$ for every x. Hence tan $\circ$ (gh) is a real - valued, continuous extension of f to all of X.

Theorem : 1.3.17

If there exists a function in C(X) that carries S homeomorphically onto a closed set in R, then S is C-embedded in X.

**Proof :**

Let h denote the postulated function in C(X). Then $\theta = \overset{\leftarrow}{(h/s)}$ is a continuous mapping from H = h[S] onto S, with $\theta(h(s)) = s$ (for $s \in S$), consider, now, an arbitrary function f in C(S). The composition $f \circ \theta$ belongs to C(H). Since H is closed in R, by hypothesis, it is C-embedded and so there is a function g in C(R) that agrees with $f \circ \theta$ on H. Then $g \circ h$ is in C(X), and for all $s \in S$, We have

$(g \circ h)(s) = f(\theta(h(s))) = f(s)$,

that is, $g \circ h$ is an extension of $f$.

Corollary : 1.3.18

Let $E \subset X$, and suppose that some function h in $C(X)$ is unbounded on E. Then E contains a copy of N, C-embedded in X, on which h approaches infinity. Also, X is pseudocompact if and only if it contains no C-embedded copy of N.

# Chapter II

# CHAPTER 2

## PROPERTIES OF Z - FILTERS

### Section 1

Definition : 2.1.1

A proper subset I of C is an ideal in C provided that I is a subring such that $gf \in I$ whenever $f \in I$, for arbitrary $g \in C$.

Result : 2.1.2

Since a proper ideal contains no unit, The ring C itself is an improper ideal.

**Note :**

Throughout the chapter the word ideal will always mean proper ideal.

Proposition : 2.1.3

The intersection of any nonempty family of ideal is an ideal. Every ideal is embeddable in a maximal ideal. Every maximal ideal M is prime, that is, if $fg \in M$, then $f \in M$ or $g \in M$.

Definition : 2.1.4

The smallest ideal (perhaps improper) containing a given collection of ideals I, .......and elements f..... is denoted by (I,...f,....)

It consists of all elements of C expressible as (finite) sums $i$ +.......+sf+.....where $i \in I$, .....and where s,..... are arbitrary function in C.

Remark : 2.1.5

If I is an ideal in C, then $I \cap C^*$ is an ideal in $C^*$

Definition : 2.1.6

A nonempty subfamily $\mathcal{F}$ of Z(X) is called a z-filter on X provided that

(1) $\emptyset \notin \mathcal{F}$

(2) if $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$ and

(3) if $Z \in \mathcal{F}$, $Z' \in Z(X)$ and $Z' \supset Z$ then $Z' \in \mathcal{F}$

Result : 2.1.7

By (3), X belongs to every z - filter. Because of (3), (2) may be replaced in the above list by (2') if $Z_1, Z_2 \in \mathcal{F}$ then, $Z_1 \cap Z_2$ contains a member of $\mathcal{F}$

Definition : 2.1.8

Every family of $\mathcal{B}$ of zero-sets that has the finite intersection property is contained in a z-filter; the smallest such is the family $\mathcal{F}$ of all zero-sets containing finite intersections of mebers of $\mathcal{B}$. We say that $\mathcal{B}$ generates the z-filter $\mathcal{F}$. When $\mathcal{B}$ itself is closed under finite intersection, it is called a base for $\mathcal{F}$.

Remark : 2.1.9

The definition of z-filter is an analogue of the definition of filter. A z-filter is a topological object, While a filter is a purely set-theoretic one. In a discrete space, every set is a zero-set, so that filters and z-filters are the same in discrete spaces.

Proposition : 2.1.10

In any space X, the intersection with $Z(X)$ of any filter is a z-filter. Conversely, if $\mathcal{F}'$ is the smallest filter containing a given z-filter $\mathcal{F}$ (that is, $\mathcal{F}$ is a base for $\mathcal{F}'$), then $\mathcal{F}' \cap Z(X) = \mathcal{F}$

Theorem : 2.1.11

(a) If I is an ideal in $C(X)$, then the family $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X.

(b) If $\mathcal{F}$ is a z-filter on X, then the family $Z^{\leftarrow}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$ is an ideal in C.

**Proof :**

(a) (i) since I contains no unit, $\emptyset \notin Z[I]$

(ii) Let $Z_1, Z_2 \in Z[I]$. Let $f_1, f_2 \in I$ satisfy $Z_1 = Z(f_1)$, $Z_2 = Z(f_2)$. Since I is an ideal $f_1^2 + f_2^2 \in I$. Hence $Z_1 \cap Z_2 = Z(f_1^2 + f_2^2) \in Z[I]$

(iii) Let $Z \in Z[I]$, and $Z' \in Z(X)$. Let $f \in I$, $f' \in C$ satisfy $Z = Z(f)$, $Z' = Z(f')$. Since I is an ideal, we have $ff' \in I$. Hence if $Z' \supset Z$ then $Z' = Z \cup Z' = Z(ff') \in Z(I)$

(b) Let $J = Z^{\leftarrow}[\mathcal{F}]$. By (1) of definition 2.1.6 J contains no unit. Let $f, g \in J$ and let $h \in C$. Then $Z(f-g) \supset Z(f) \cap Z(g) \in \mathcal{F}$, by (2) of definition 2.1.6 $Z(hf) \supset Z(f)$ Hence $Z(f-g) \in \mathcal{F}$ and $Z(hf) \in \mathcal{F}$ J, by (3) of definition 2.1.6 Therefore $f-g \in J$ and $hf \in J$. Thus, J is an ideal in C.

**Remark : 2.1.12**

$(f\ g) \neq C$ if and only if $Z(f)$ meets $Z(g)$, hence if and only if $f^2 + g^2$ or $|f| + |g|$ is not a unit of C.

Any mapping, Z satisfies (for $\mathcal{F} \subset Z(X)$) and $Z[Z^{\leftarrow}(\mathcal{F})] = \mathcal{F}$ and $Z[Z[I]] \supset I$.

The first relation implies that every z-filter is of the form $Z[J]$ for some ideal J in C. In the second relation, the inclusion may be proper.

**Definition : 2.1.13**

By a z-tltrafilter on X is meant a maximal z-filter, that is, one not contained in any other z-filter.

Remark : 2.1.14

A z-ultrafilter is a maximal subfamily of $\mathbf{Z}(X)$ with the finite intersection property. It follows from the maximal principle that every subfamily of $\mathbf{Z}(X)$ with the finite intersection property is contained in some z-ultrafilter.

In a discrete space, z-ultrafilters are the same as ultrafilters, that is, maximal filters.

Theorem : 2.1.15

(a) If $M$ is a maximal ideal in $C(X)$, then $\mathbf{Z}[M]$ is a z-ultrafilter on $X$.

(b) If $A$ is a z-ultrafilter on $X$, then $\mathbf{Z}^{\leftarrow}[A]$ is a maximal ideal in $C$.

The mapping $\mathbf{Z}$ is one - one from the set of all maximal ideals in $C$ onto the set of all z-ultrafilters.

**Proof :**

Since $\mathbf{Z}$ and $\mathbf{Z}^{\leftarrow}$ preserve inclusion, the result follows at once from 2.1.11

Theorem : 2.1.16

(a) Let $M$ be a maximal ideal in $C(X)$; if $\mathbf{Z}(f)$ meets every member of $\mathbf{Z}[M]$, then $f \in M$.

(b) Let $\mathbf{A}$ be a z-ultrafilter on X; if a zero-set Z meets every member of $\mathbf{A}$, then $Z \in \mathbf{A}$.

**Proof :**

By theorem 2.1.15, the two statements are equivalent. In (b) $\mathbf{A} \cup \{Z\}$ generates a z-filter. As this contains the maximal z-filter $\mathbf{A}$, it must be $\mathbf{A}$.

The properties stated in the theorem are, infact, charateristic of maximal ideals and z-filters; if a z-filter $\mathbf{A}$ contains every zero-set that meets all members of $\mathbf{A}$, then, clearly $\mathbf{A}$ is a z-ultrafilter.

## Section 2.

**Definition : 2.2.1**

An ideal I in C(X) is called a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$ - that is to say, if $I = Z^{\leftarrow}[Z[I]]$

**Result : 2.2.2**

If $\mathcal{F}$ is a z-filter, then $Z^{\leftarrow}[\mathcal{F}]$ is a z-ideal (since $\mathcal{F} = Z[Z^{\leftarrow}[\mathcal{F}]]$). Hence if J is any ideal in C, then $I = Z^{\leftarrow}[Z[J]]$ is a z-ideal, which is the smallest z-ideal containing J. Also every maximal ideal is a z-ideal.

The intersection of an arbitrary (nonempty) family of z-ideals is a z-ideal.

The mapping Z is one - one from the set of all z-ideals onto the set of all z-filters.

**Result : 2.2.3**

In C(N), every ideal I is a z-ideal

**Proof :**

Suppose that $Z(f) = Z(g)$, where $g \in I$. Define h as follows; $h(n) = 0$ for $n \in Z(g)$ and $h(n) = \frac{f(n)}{g(n)}$ for $n \notin Z(g)$. Since N is discrete, h is continuous. Evidently, $f = hg$. Therefore $f \in I$ which implies I is a z-ideal.

**Theorem : 2.2.4**

Every z-ideal in C(X) is an intersection of prime ideals.

**Proof :**

$Z(f^n) = Z(f)$ for every $n \in N$. Therefore if I is any z-ideal, then $f^n \in I$ implies $f \in I$. But this property characterizes I as the intersection of all the prime ideals containing it, Hence the proof.

The next theorem clarifies to some extend the relation between prime ideals and z-ideals.

**Theorem : 2.2.5**

For any z - ideal I in C, the following are equivalent.

(1) I is prime

(2) I contains a prime ideal.

(3) For all $g, h \in C$, if $gh = 0$, then $g \in I$ or $h \in I$

(4) For every $f \in C$, there is a zero-set in Z[I] on which f does not change sign.

**Proof :**

(1) Implies (2)

Trivial

(2)implies (3).

If I contains a prime ideal P, and $gh = 0$, then $gh \in P$ whence either g or h is in P and hence in I.

(3) implies (4)

If suffices to observe that $(f \vee 0)(f \wedge 0) = 0$ for every $f \in C$.

(4) implies (1)

Given $gh$ I, consider the function $|g| - |h|$. By hypothesis, there is a zero-set Z of I on which $|g| - |h|$ is nonnegative, say. Then every zero of g on Z is a zero of h. Hence $Z(h) \supset Z \cap Z(h) = Z \cap Z(gh) \in Z[I]$, so that $Z(h) \in Z[I]$. Since I is a z-ideal, $h \in I$. Thus, I is prime.

Proposition : 2.2.6

If J and J' are ideals, neither containing the other, the $J \cap J'$ is not prime.

Inpact, this holds in any commutative ring. For, when $a \in J-J'$ and $a' \in J'-J$ then neither a nor a' belongs to JnJ', but $aa' \in J \cap J'$.

Theorem : 2.2.7

Every prime ideal in C(X) is contained in a unique maximal ideal.

**Proof :**

We know that every ideal is contained in at least one maximal ideal. If M and M' are distinct maximal ideals, their intersection is a z - ideal (since M and M' are z - ideals), but it is not prime. Therefore by theorem 2.2.5 $M \cap M'$ contains no prime ideal.

**Note :**

The corresponding theorem is valid for C*.

Definition : 2.2.8

By a prime z - filter, we shall mean z - filter $\mathcal{F}$ with the following property: whenever the union of two zero-sets belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

Theorem : 2.2.9

(a) If P is a prime ideal in C(X), then Z[P] is a prime z - filter.

(b) If $\mathcal{F}$ is a prime z - filter, then $\overset{\leftarrow}{Z}[\mathcal{F}]$ is a prime z - ideal.

**Proof :**

(a) Let $Q = Z^{\leftarrow}[Z[P]]$ Then $Z[Q] = Z[P]$, and Q is a z - ideal containing the prime ideal P. By theorem 2.2.5 Q is prime. Suppose, now, that $Z(f) \cup Z(g) \in Z[P]$ This implies that $Z(fg) \in Z[Q]$;

therefore fg belongs to the z - ideal Q. Since Q is prime, it contains f, say. Then $\mathbf{Z}(f) \in \mathbf{Z}[Q] = \mathbf{Z}[P]$

(b) We know that the ideal $P = \mathbf{Z}^{\leftarrow}[\mathcal{F}]$ is a z -ideal. Suppose that $fg \in p$. Then $\mathbf{Z}(fg) = \mathbf{Z}(f) \cup \mathbf{Z}(g) \in \mathbf{Z}[P] = \mathcal{F}$ By hypothesis, $\mathbf{Z}(f)$, say, belongs to $\mathbf{Z}[P]$. Then f belongs to the z - ideal P.

Result : 2.2.10

A prime z - filter is contained is a unique z - ultrafilter.

Since every maximal in C is prime, every z - ultrafilter is a prime z - filter. If zero-sets $\mathbf{Z}$ and $\mathbf{Z}'$ do not belong to a z - ultrafilter $\mathbf{A}$, then, by theorem 2.1.16 (b), there exist $A, A' \in \mathbf{A}$ such that $\mathbf{Z} \cap A = \mathbf{Z}' \cap A' = \emptyset$. Then $\mathbf{Z} \cup \mathbf{Z}'$ does not meet the member $A \cap A'$ of $\mathbf{A}$, and hence does not belong to $\mathbf{A}$.

Result : 2.2.11

In a discrete space X, there is no difference between prime and maximal, that is, every prime filter. $\mathbf{U}$ is an ultrafilter. For if $A \notin \mathbf{U}$, then $X - A \in \mathbf{U}$; hence A cannot be adjoined to $\mathbf{U}$.

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