CHAPTER – VI

QUEUE LENGTH ANALYSIS OF UNRELIABLE $M^X/(G_1, G_2)/1$ FEEDBACK QUEUE WITH TWO PHASE SERVICES, SETUP, OPTIONAL SERVER VACATION, DELAYED REPAIR, UNDER MULTIPLE ADAPTED VACATION POLICY DURING IDLE PERIOD

INTRODUCTION

In the following Chapters VI to VIII the classical 1-policy is considered.

The present chapter analyses the steady-state behaviour of an $M^X/G/1$ vacation queueing model, with two successive phases of service subject to breakdowns, occurring randomly at any instant while serving the customers and delayed repair. After completing two phases of service, the dissatisfied customers can demand for immediate feedback. A random setup time is also considered at the beginning of each busy period. The queueing system considered in the present chapter is more general since many of the queueing features are combined in a single model and the results of the model generalise many of the results exist in the literature.

The present work also considers both vacation policies (exhaustive and non-exhaustive) together in a single model.

It is assumed that whenever the system becomes idle, the server adopts multiple adapted vacation (Mytalas and Zazanis, 2015) policy controlled by a sequence of probabilities $\{\gamma_k\}$, k = 0, 1, ... This policy provides additional flexibility and allows us to deduce the corresponding results of other vacation (single vacation, multiple vacation, randomized (p, J) vacation) models including non-vacation models, with the suitable selection of the probabilities $\{\gamma_k\}$, k = 0, 1, 2, ...

Each busy period starts with a setup operation and at the end of the setup operation, the server begins to serve the waiting customers in two-successive phases of service. At the end of the second phase service, any customer who is dissatisfied with the service, may demand for re-service immediately to any number of times and leaves the system after successfully completing the service. The server before starting service to the next customer in the queue may opt for a single vacation with probability **p** or continue to serve the next customer with probability **(1 – p)**.

There are limited number of research work on vacation queueing problems with an unreliable server and delayed repair. The earlier papers assume that as soon as a service channel fails, the repair work will start instantaneously. But recently some of the authors (Ke et al. (2010) and Choudhury et al. (2009)) studied the breakdown situations, taking into account a possible practical situation namely, the delay time of a repair. In the present chapter, it is assumed that the server may breakdown at any time while providing service and the service of the customer being served is then interrupted and cannot resume service until the server channel is repaired. Once the system breaks down, its repair does not start immediately and there is a delay time. The system is studied under steady-state and the distributions of some important system characteristics, such as the system size distributions at random epoch and at departure epoch, the mean system size when the system is in different states are derived. Numerical examples are presented for illustrative purposes.

6.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

6.1.1 Model Description

Arrival Pattern

The present chapter considers an $M^X/(G_1,G_2)/1$ queueing system in which the arrivals occur in batches in accordance with a time-homogeneous Poisson process with random batch size X, group arrival rate λ and probability distribution g_k = Pr (X=k), k = 1, 2, 3… . The customers are served one by one according to the order in the queue.

Multiple Adapted Vacation (MAV) Policy

A cycle starts whenever the system becomes empty and the server is deactivated. The deactivated server either leaves the system for a vacation (first vacation) of random length (VI) with probability γ_0 or remains in the system and stays idle with probability $(1 - \gamma_0)$. Upon returning from the first vacation, if the server finds at least one customer waiting in the system, then the server immediately starts a setup operation of random length D. Otherwise, if there are no customers found waiting in the queue, the server either joins the system and remains idle in the system with probability $(1 - \gamma_1)$ or takes a new vacation (second vacation) with probability γ_1. This pattern continues until atleast one customer is found in the system. Thus the vacation policy is determined by the sequence of probabilities (γ_i), i = 0, 1, 2, . . . The vacations have independent duration with common distribution function VI(t) and density function vI(t) of finite moments. Thus the setup operation starts either at the end of a vacation or as soon as a customer arrives when the server is waiting in the system. At the end of setup operation, the server begins to serve the customers one by one exhaustively.

Busy Period and Breakdown Period

During busy period, the server provides two phases of heterogeneous service in succession to each customer. Every customer undergoes both the stages of service one after the other to complete a service. The service times of the two stages of service follow different general (arbitrary) distributions with distribution functions $S_i(t)$ and the density functions $s_i(t)$ of finite moments $E(S_i^k)$, k = 1, 2 and i = 1, 2.

If a customer after completing both the stages of service, is unsatisfied with the service then he may demand for a re-service with probability f (or) leave the system with probability (1 – f).

The server may breakdown at anytime, while working with regular service or re-service, and the service channels will not function for a short interval of time. The breakdowns occur according to the Poisson process with rate $\mathbf{a_1}$ in the first phase of service and $\mathbf{a_2}$ in the second phase service.

As soon as the server fails, the service is stopped for the customer until the channel is repaired. The customer whose service is interrupted (in any phases of service) joins the head of the queue and waits for the server to return from the repair facility to start a new service from phase 1. It is also assumed that the repair does not start immediately. The interval between the instant at which the breakdown occurs and the repair begins is defined as delay time. The successive delay times Δ_i in each case form a sequence of independent and identically distributed random variables having continuous probability distribution functions $\Delta_i(x)$, density functions $\delta_i(x)$ with finite moments $E(\Delta_i^k)$, i = 1, 2 and k = 1, 2. The repair times (denoted by R_i for the i^{th} phase) of the server in both the phases of service are assumed to be arbitrarily distributed with distribution functions $R_i(t)$, density function $r_i(t)$ for $t \geq 0$, i = 1, 2 with the finite moments. The sum of delayed time and repair time is defined as breakdown period.

The customers continue to arrive according to the compound Poisson process independent of the state of the system and wait in the queue during the busy period, breakdown period and vacation period. The service completion period of a customer consists of the service time in both the phases along with the re-service time and the repair time of the server including a possible repair delay.

At the end of each service (i.e., when the customer leaves the system after completing the service), the server may take a vacation of random duration VB with probability p or continue to serve the next customer if any with probability (1 – p). The vacation time VB (different from VI) follows general (arbitrary) distribution. The distribution function and density function of VB are respectively denoted by VB(t) and vB(t) and its LST is denoted by $VB^*(\theta)$. Thus a cycle is made up of idle vacation period, setup period, completion period and the vacation period between the services.

We denote the model by M^X / (G_1, G_2) / 1 / BSV / MAV / Delayed repair / feedback, where BSV denotes the Bernoulli schedule vacation between services and MAV denotes the multiple adapted vacation policy. Various stochastic processes involved in the queueing system are assumed to be independent of each other. Using supplementary variable technique the steady state system equations under the steady-state condition are analysed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it.

The notations $N_S(t)$, λ, X, $g_n^{(i)}$ ($i \geq 1$, $n \geq 1$), X(z) are same as in Table 2.0 of Chapter II. Let Y(t) denote the states of the system at time t.

Y(t)	=	0	if the server is in idle-vacation state
		1	if the server is doing setup operation
		2	if the server is busy with first phase of service.
		3	If the server is busy with second phase of service
		4	If the server is in VB type vacation
		5	If the server is waiting for the first type of repair
		6	If the server is waiting for the second type of repair
		7	If the server is under first type of repair
		8	If the server is under second type of repair
		9	The system is empty.

The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RV_S are listed below.

	RV	CDF	PDF	LST	k^{th} moments k = 1, 2
Setup time	D	D(x)	d(x)	$D^*(\theta)$	$E(D^k)$
Service time in first and second stage	S_i	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S^k)$, i = 1, 2
Vacation time during idle period	VI	VI(x)	vI(x)	$VI^*(\theta)$	$E(VI^k)$
Vacation time during busy period	VB	VB(x)	vB(x)	$VB^*(\theta)$	$E(VB^k)$
Delayed repair time in first and second stage of service	Δ_i	$\Delta_i(x)$	$\delta_i(x)$	$\Delta_i^*(\theta)$	$E(\Delta_i^k)$, i = 1, 2
Repair time in first and second stage of service	R_i	$R_i(y)$	$r_i(y)$	$R_i^*(\theta)$	$E(R_i^k)$, i = 1, 2

Let $D^o(t)$, $S_i^o(t)$, $VI^o(t)$, $VB^o(t)$, $R_i^o(t)$, $\Delta_i^o(t)$ denote the remaining times of the random variables ; setup time, service time in i^{th} phase, vacation time during idle period, vacation between services, repair mode in i^{th} phase, delayed repair mode in i^{th} phase at time t. Then the state space is $\{N_S(t), \delta(t)\}$ where $\delta(t) = (VI^o(t), D^o(t), S_1^o(t), S_2^o(t), VB^o(t), \Delta_1^o(t), \Delta_2^o(t), R_1^o(t), R_2^o(t), 0)$ according as Y(t) = 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 respectively defines a Markov process. Further the following system size probabilities t time t are defined.

Let Z(t) = j, j ≥ 1 denote that the server is in j^{th} vacation at time t.

$$QI_{n,j}(x, t)\,dt = Pr\{N_S(t) = n, x < VI^o(t) \le x + dt, Y(t) = 0, Z(t) = j, j \ge 1\}, n \ge 0$$
$$SE_n(x, t)\,dt = Pr\{N_S(t) = n, x < D^o(t) \le x + dt, Y(t) = 1\}$$
$$P_{i,n}(x, t)\,dt = Pr\{N_S(t) = n, x < S_i^o(t) \le x + dt, Y(t) = 2 \text{ or } 3\}, n \ge 1$$
$$QB_n(x, t)\,dt = Pr\{N_S(t) = n, x < VB^o(t) \le x + dt, Y(t) = 4\}, n \ge 1$$
$$DE_{i,n}(x, t)\,dt = Pr\{N_S(t) = n, x < \Delta_i^o(t) \le x + dt, Y(t) = 5 \text{ or } 6\}, n \ge 1$$
$$BR_{i,n}(x, t)\,dt = Pr\{N_S(t) = n, x < R_1^o(t) \le x + dt, Y(t) = 7 \text{ or } 8\}, n \ge 1$$
$$PI(t) = Pr\{N_S(t) = 0, Y(t) = 9\}$$

6.1.2 The Steady State System Size Equations

Assuming that, the system size probabilities are independent of time t in the steady state, the Kolmogorov equations that govern the system under steady state conditions can be written as follows :

Vacation During Idle Period

$$-\frac{d}{dx}QI_{0,1}(x) = -\lambda\, QI_{0,1}(x) + (1 - f)\, P_{2,1}(0)\, \gamma_0\, vI(x)$$
$$-\frac{d}{dx}QI_{0,j}(x) = -\lambda\, QI_{0,j}(x) + QI_{0,j-1}(0)\, \gamma_{j-1}\, vI(x) j \ge 2$$
$$-\frac{d}{dx}QI_{n,j}(x) = -\lambda\, QI_{n,j}(x) + \lambda \sum_{k=1}^{n} QI_{n-k}(x)\, g_k, \qquad n \ge 1 \; ; j \ge 1$$

Server Idle in Empty System

$$\lambda\, PI = \sum_{j=1}^{\infty} (1-\gamma_j)\, QI_{0,j}(0) + P_{2,1}(0)\,(1 - f)\,(1 - \gamma_0)$$

Busy with First Stage of Service

$$-\frac{d}{dx}P_{1,n}(x) = -(\lambda + a_1)\,P_{1,n}(x) + QB_n(0)\,s_1(x) + \sum_{i=1}^{2} BR_{i,n}(0)\,s_1(x)$$
$$+ SE_n(0)\,s_1(x) + P_{2,n}(0)\,f\,(1-p)\,s_1(x)$$
$$+ P_{2,n+1}(0)\,(1-f)\,(1-p)\,s_1(x) + (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{1,n-k}(x)\,g_k \qquad n \geq 1$$

Busy with Second Stage of Service

$$-\frac{d}{dx}P_{2,n}(x) = -(\lambda + a_2)\,P_{2,n}(x) + P_{1,n}(0)\,s_2(x) + (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{2,n-k}(x)\,g_k, \qquad n \geq 1$$

Vacation During Busy Period

$$-\frac{d}{dx}QB_n(x) = -\lambda\,QB_n(x) + P_{2,n+1}(0)\,(1-f)\,p\,vB(x) + P_{2,n}(0)\,f\,vB(x)\,p$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} QB_{n-k}(x)\,g_k, \qquad n \geq 1$$

Setup State

$$-\frac{d}{dx}SE_n(x) = -\lambda\,SE_n(x) + \sum_{j=1}^{\infty} QI_{n,j}(0)\,d(x) + \lambda\,PI\,g_n\,d(x)$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} SE_{n-k}(x)\,g_k, \qquad n \geq 1$$

Delayed Repair States

$$-\frac{d}{dx}DE_{i,n}(x) = -\lambda\,DE_{i,n}(x) + a_i\left(\int_0^{\infty} P_{i,n}(w)\,dw\right)\delta_i(x)$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} DE_{i,n-k}(x)\,g_k \qquad n \geq 1, \quad i = 1, 2$$

i.e.,

$$-\frac{d}{dx}DE_{i,n}(x) = -\lambda\,DE_{i,n}(x) + a_i\,P^*_{i,n}(0)\,\delta_i(x)$$
$$+ (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} DE_{i,n-k}(x)\,g_k \qquad n \geq 1, \quad i = 1, 2$$

Breakdown States

$$-\frac{d}{dx}BR_{i,n}(x) = -\lambda\,BR_{i,n}(x) + DE_{i,n}(0)\,r_i(x) + (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR_{i,n-k}(x)\,g_k$$
$$i = 1, 2\ ; \quad n \geq 1$$

The LST of the steady-state equations are,

$$\theta\ QI^*_{0,1}(\theta) - QI_{0,1}(0) = \lambda\ QI^*_{0,1}(\theta) - (1-f)\,P_{2,1}(0)\,\gamma_0\ VI^*(\theta) \qquad (6.1)$$

$$\theta\ QI^*_{0,j}(\theta) - QI_{0,j}(0) = \lambda\ QI^*_{0,j}(\theta) - \gamma_{j-1}\,QI_{0,j-1}(0)\ VI^*(\theta), \qquad j \geq 2 \qquad (6.2)$$

$$\theta\ QI^*_{n,j}(\theta) - QI_{n,j}(0) = \lambda\ QI^*_{n,j}(\theta) - \lambda \sum_{k=1}^{n} QI^*_{n-k,j}(\theta)g_k, \qquad n \geq 1,\ j \geq 1 \qquad (6.3)$$

$$\begin{aligned}\theta\ P^*_{1,n}(\theta) - P_{1,n}(0) = {} & (\lambda + a_1)\ P^*_{1,n}(\theta) - [QB_n(0) + SE_n(0) + \sum_{i=1}^{2} BR_{i,n}(0)]\ S^*_1(\theta) \\ & - [P_{2,n+1}(0)\,(1-f)\,(1-p) + P_{2,n}(0)\,f\,(1-p)]\ S^*_1(\theta) \\ & - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P^*_{1,n-k}(\theta)g_k, \qquad n \geq 1 \qquad (6.4)\end{aligned}$$

$$\begin{aligned}\theta\ P^*_{2,n}(\theta) - P_{2,n}(0) = {} & (\lambda + a_2)\ P^*_{2,n}(\theta) - P_{1,n}(0)\ S^*_2(\theta) \\ & - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P^*_{2,n-k}(\theta)g_k, \qquad n \geq 1 \qquad (6.5)\end{aligned}$$

$$\begin{aligned}\theta\ QB^*_n(\theta) - QB_n(0) = {} & \lambda\ QB^*_n(\theta) - [P_{2,n+1}(0)\,(1-f) + P_{2,n}(0)\,f]\,VB^*(\theta) \\ & - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} QB^*_{n-k}(\theta)g_k, \qquad n \geq 1 \qquad (6.6)\end{aligned}$$

$$\begin{aligned}\theta\ SE^*_n(\theta) - SE_n(0) = {} & \lambda\ SE^*_n(\theta) - \sum_{j=1}^{\infty} QI_{n,j}(0)\,D^*(\theta) - \lambda\,PI\,g_n\ D^*(\theta) \\ & - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} SE^*_{n-k}(\theta)g_k, \qquad n \geq 1 \qquad (6.7)\end{aligned}$$

$$\theta\ DE^*_{i,n}(\theta) - DE_{i,n}(0) = \lambda\ DE^*_{i,n}(\theta) - a_i\ P^*_{i,n}(0)\ \Delta^*_i(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} DE^*_{i,n-k}(\theta)g_k, \qquad (6.8)$$

$$\theta\ BR^*_{i,n}(\theta) - BR_{i,n}(0) = \lambda\ BR^*_{i,n}(\theta) - DE_{i,n}(0)\ R^*_i(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR^*_{i,n-k}(\theta)g_k, \qquad i = 1, 2,\ n \geq 1 \qquad (6.9)$$

$$\lambda\,PI = \sum_{j=1}^{\infty} (1-\gamma_j)\ QI_{0,j}(0) + P_{2,1}(0)\,(1-f)\,(1-\gamma_0) \qquad (6.10)$$

6.1.3 Probability Generating Functions

To analyse the system, the following partial PGFs are defined

$$QI^*_j(z,\theta) = \sum_{n=0}^{\infty} QI^*_{n,j}(\theta)z^n, \qquad QI_j(z,0) = \sum_{n=0}^{\infty} QI_{n,j}(0)z^n \qquad j \geq 1$$

$$QB^*(z,\theta) = \sum_{n=0}^{\infty} QB^*_n(\theta)z^n, \qquad QB(z,0) = \sum_{n=0}^{\infty} QB_n(0)z^n$$

$$SE^*(z,\theta) = \sum_{n=1}^{\infty} SE^*_n(\theta)z^n, \qquad SE(z,0) = \sum_{n=1}^{\infty} SE_n(0)z^n$$

For i = 1, 2,

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{i,n}^*(\theta) z^n, \qquad P_i(z, 0) = \sum_{n=1}^{\infty} P_{i,n}(0) z^n$$

$$DE_i^*(z,\theta) = \sum_{n=1}^{\infty} DE_{i,n}^*(\theta) z^n, \qquad DE_i(z, 0) = \sum_{n=1}^{\infty} DE_{i,n}(0) z^n$$

$$BR_i^*(z,\theta) = \sum_{n=1}^{\infty} BR_{i,n}^*(\theta) z^n, \qquad BR_i(z, 0) = \sum_{n=1}^{\infty} BR_{i,n}(0) z^n,$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, partial generating functions are derived, through some algebraic manipulations.

Equations (6.1) and (6.3) at j = 1 imply,

$$(\theta - w_X(z))\ QI_1^*(z,\theta) = QI_1(z, 0) - (1 - f)\,\gamma_0\, P_{2,1}(0)\ VI^*(\theta) \tag{6.11}$$

where $w_X(z) = \lambda\,(1 - X(z))$

At $\theta = w_X(z)$, $QI_1(z, 0) = (1 - f)\,\gamma_0\, P_{2,1}(0)\ \ VI^*(w_X(z))$ (6.12)

$$\text{and hence } QI_1^*(z,\theta) = \frac{(1-f)\ \gamma_0\, P_{2,1}(0)\,(VI^*(w_X(z)) - VI^*(\theta))}{\theta - w_X(z)} \tag{6.13}$$

Similarly equations (6.2) and (6.3) for $j \geq 2$ lead to

$$QI_j(z, 0) = \gamma_{j-1}\, QI_{0,j-1}(0)\ VI^*(w_X(z)), \qquad j \geq 2 \tag{6.14}$$

$$\text{and} \quad QI_j^*(z,\theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} QI_{0,j-1}(0)\,\gamma_{j-1}, \quad j \geq 2 \tag{6.15}$$

Thus the sum of the equations ((6.12) and (6.14)) and ((6.13) and (6.15)) over j = 1 to ∞ respectively give,

$$QI(z, \quad 0) = \sum_{j=1}^{J} QI_j(z,0) = VI^*(w_X(z)) \left(\sum_{j=1}^{\infty} QI_{0,j}(0)\,\gamma_j + \gamma_0\,(1-f)\,P_{2,1}(0) \right)$$

(6.16)

$$\sum_{j=1}^{\infty} QI_j^*(z,\theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} \left(\sum_{j=1}^{\infty} QI_{0,j}(0)\,\gamma_j + \gamma_0\,(1-f)\,P_{2,1}(0) \right) \tag{6.17}$$

if αI_n denotes the probability that n-customers arrive during a vacation time (VI) then the equations (6.12) and (6.14) also imply

$$\sum_{n=0}^{\infty} QI_{n,1}(0)\, z^n = (1 - f)\,\gamma_0\, P_{2,1}(0) \sum_{n=0}^{\infty} \alpha I_n\, z^n \text{ and} \tag{6.17.1}$$

$$\sum_{n=0}^{\infty} QI_{n,j}(0)\, z^n = \gamma_{j-1}\, QI_{0,j-1}(0) \sum_{n=0}^{\infty} \alpha I_n\, z^n, \qquad j \geq 2 \tag{6.17.2}$$

Thus collecting the constant terms for j ≥ 1 on both sides of 6.17.1 and 6.17.2 and use them recursively, it is found that,

$$QI_{0,j}(0) = (1-f)\, P_{2,1}(0)\, (\alpha I_0)^j \prod_{i=0}^{j-1} \gamma_i, \ \forall\, j \geq 1 \tag{6.18}$$

Substituting for $QI_{0,j}(0)$ ($j \geq 1$) in (6.16) and (6.17),

$$QI(z, 0) = VI^*(w_X(z))\,(1-f)\, P_{2,1}(0) \left(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i \right) \tag{6.19}$$

and

$$QI^*(z,\theta) = \frac{(VI^*(w_X(z)) - VI^*(\theta))}{(\theta - w_X(z))} (1-f) P_{2,1}(0) \sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i \tag{6.20}$$

Next multiplying the equation (6.7) by z^n and adding over n = 1 to ∞,

$$(\theta - w_X(z))\, SE^*(z,\theta) = SE(z, 0) - \left[\sum_{j=1}^{\infty} [QI_j(0) - QI_{0,j}(0)] + \lambda\, PI\, X(z)\right] D^*(\theta) \tag{6.20.1}$$

Adding equation (6.20.1) with, equation (6.10) multiplied by $(-D^*(\theta))$,

$$\begin{aligned}(\theta - w_X(z))\, SE^*(z,\theta) = {} & SE(z, 0) - D^*(\theta)[QI(z, 0) - \sum_{j=1}^{\infty} \gamma_j\, QI_{0,j}(0) - PI\, w_X(z) \\ & + P_{2,1}(0)\,(1-f)\,(1-\gamma_0)]\end{aligned} \tag{6.20.2}$$

Using (6.18) in equation (6.10),

$$\lambda\, PI = \sum_{j=1}^{\infty} (1-\gamma_j)\, QI_{0,j}(0) + P_{2,1}(0)\,(1-f)\,(1-\gamma_0) = \phi_\gamma\,(1-f)\, P_{2,1}(0) \tag{6.20.3}$$

where

$$\phi_\gamma = \sum_{j=1}^{\infty} (1-\gamma_j)(\alpha I_0)^j \prod_{i=0}^{j-1} \gamma_i + (1-\gamma_0) \tag{6.20.4}$$

Therefore, the PGF of the system size at setup termination epoch and arbitrary epoch are obtained by substituting for QI(z, 0), $QI_{0,j}(0)$ and PI from equations (6.19), (6.18) and (6.20.3) in (6.20.1) and (6.20.2). They are

$$SE(z, 0) = D^*(w_X(z))(1-f)P_{2,1}(0)\left(\left(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i\right)(VI^*(w_X(z)) - 1) + 1 - \frac{\phi_\gamma}{\lambda} w_X(z)\right) \tag{6.21}$$

and $SE^*(z,\theta) = (1-f)\, P_{2,1}(0)\, ((\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i)\,(VI^*(w_X(z))-1)$

$$+ 1 - \frac{\phi_\gamma}{\lambda} w_X(z))\ \frac{(D^*(w_X(z)) - D^*(\theta))}{(\theta - w_X(z))} \qquad (6.22)$$

Equation (6.5) gives the generating functions of the system size when the server is busy with second stage service, at service completion epoch and at arbitrary epoch, namely,

$$P_2(z,0) = P_1(z,0)\ S_2^*(g_{a_2}(w_X(z))) \qquad (6.23)$$

$$P_2^*(z,\theta) = P_1(z,0)\ \frac{(S_2^*(g_{a_2}(w_X(z))) - S_2^*(\theta))}{(\theta - g_{a_2}(w_X(z)))} \qquad (6.24)$$

where $g_{a_2}(w_X(z)) = a_2 + \lambda\,(1 - X(z))$ (6.24.1)

The partial probability generating functions of the system size, when the server is in vacation period VB(t) is obtained by using equation (6.6) and it is found that

$(\theta - w_X(z))\ QB^*(z,\theta)$

$$= QB(z,0) - p\ VB^*(\theta) P_2(z,0)\ \frac{(1-f+fz)}{z} + (1-f)\ p\ VB^*(\theta) P_{2,1}(0)$$

At $\theta = w_X(z)$

$$QB(z,0) = p\ VB^*(w_X(z))(P_2(z,0)\ \frac{(1-f+fz)}{z} - (1-f)\ P_{2,1}(0)) \qquad (6.25)$$

Substituting the value of QB(z, 0)

$$QB^*(z,\theta) = \frac{p\,(VB^*(w_X(z)) - VB^*(\theta))}{(\theta - w_X(z))} (P_2(z,0) \frac{(1-f+fz)}{z} - (1-f)\ P_{2,1}(0)) \qquad (6.26)$$

Similarly the equation (6.8) is used to calculate the generating functions of the system size when the server is in delayed repair period in i^{th} phase (i = 1, 2), at service completion epoch and arbitrary epoch. They are given by

$$DE_i(z,0) = a_i\ \Delta_i^*(w_X(z))\, P_i^*(z,0) \qquad (6.27)$$

$$DE_i^*(z,\theta) = a_i\ P_i^*(z,0)\, \frac{(\Delta_i^*(w_X(z)) - \Delta_i^*(\theta))}{(\theta - w_X(z))}, \quad i = 1, 2 \qquad (6.28)$$

The partial probability generating functions of the system size, when the server is in breakdown state, during first phase and second phase of service are obtained by using the equation (6.9) and are given by,

$$BR_i(z, 0) = DE_i(z, 0)\, R_i^*(w_X(z)) \tag{6.29}$$

$$BR_i^*(z,\theta) = DE_i(z, 0)\, \frac{(R_i^*(w_X(z)) - R_i^*(\theta))}{(\theta - w_X(z))}, \qquad i = 1, 2 \tag{6.30}$$

Next to calculate the partial generating functions corresponding to the first stage service, the equation (6.4) is used. It gives,

$$\theta\, P_1^*(z,\theta) - P_1(z, 0)$$

$$= (\lambda + a_1)\, P_1^*(z,\theta) - QB(z, 0)\, S_1^*(\theta) - \sum_{i=1}^{2} BR_i(z,0)\, S_1^*(\theta) - SE(z, 0)\, S_1^*(\theta)$$

$$- P_2(z, 0)\, f\,(1 - p)\, S_1^*(\theta) - (1-p)\, S_1^*(\theta)\, \frac{(1-f)}{z}\, [P_2(z, 0) - P_{2,1}(0)\, z]$$

$$- \lambda\, X(z)\, P_1^*(z,\theta) \tag{6.31}$$

By using the equations (6.21), (6.23), (6.25), (6.29) in (6.31) and simplifying it, we have

$$[\theta - g_{a_1}(w_X(z))]\, P_1^*(z,\theta)$$

$$= P_1(z, 0)\, \{1 - S_1^*(\theta)[\frac{(fz + 1 - f)}{z}(1-p+p\, VB^*(w_X(z)))\, S_2^*(g_{a_2}(w_X(z))) + Y_2(z)]\}$$

$$- S_1^*(\theta)[a_1\, R_1^*(w_X(z))\, \Delta_1^*(w_X(z))\, P_1^*(z,0)]$$

$$+ S_1^*(\theta)(1 - f)\, P_{2,1}(0)\, [(1 - p) + p\, VB^*(w_X(z))$$

$$- D^*(w_X(z))(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i\, (VI^*(w_X(z)) - 1) + 1 - \frac{\phi_\gamma}{\lambda} w_X(z))] \tag{6.31.1}$$

where ϕ_γ is given by the equation (6.20.4), $g_{a_1}(w_X(z)) = a_1 + \lambda(1 - X(z))$.

And for i = 1, 2,

$$Y_i(z) = a_i\, \Delta_i^*(w_X(z))\, R_i^*(w_X(z))\, \frac{(1 - S_i^*(g_{a_i}(w_X(z))))}{g_{a_i}(w_X(z))} \tag{6.31.2}$$

At θ = 0, $P_1^*(z,0)$ can be obtained as,

$$P_1^*(z,0)=\frac{\{\frac{P_1(z,0)}{z}[z-((fz+1-f)(1-p+p\,VB^*(w_X(z)))S_2^*(g_{a_2}(w_X(z)))+Y_2(z)] + (1-f)P_{2,1}(0)\,w_X(z)\,I_{MAV}^{BV}(z)\}}{(R_1^*(w_X(z))-a_1\,\Delta_1^*(w_X(z))-g_{a_1}(w_X(z)))} \tag{6.32}$$

where

$$I_{MAV}^{BV}(z) = D^*(w_X(z))\left[\left(\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\gamma_i\right)\left(\frac{1-VI^*(w_X(z))}{w_X(z)}\right)+\frac{\phi_\gamma}{\lambda}\right] + \frac{1-D^*(w_X(z))}{w_X(z)} - p\,\frac{(1-VB^*(w_X(z)))}{w_X(z)} \tag{6.32.1}$$

substituting for $P_1^*(z,0)$ in (6.31.1), and evaluating $P_1(z, 0)$ at $\theta = g_{a_1}(w_X(z))$, we have,

$$P_1(z,0) = \frac{z\,S_1^*(g_{a_1}(w_X(z)))(1-f)P_{2,1}(0)(-w_X(z))\,I_{MAV}^{BV}(z)}{D_{DE}^{BV}(z)} \tag{6.33}$$

where

$$D_{DE}^{BV}(z) = Y(z) + p\,(1-VB^*(w_X(z)))\,(fz+1-f)\,S_1^*(g_{a_1}(w_X(z)))\;S_2^*(g_{a_2}(w_X(z))) \tag{6.33.1}$$

and

$$Y(z) = z\,[1-(Y_1(z)+Y_2(z)\;S_1^*(g_{a_1}(w_X(z))))] - (fz+1-f)\;S_1^*(g_{a_1}(w_X(z)))\;S_2^*(g_{a_2}(w_X(z))) \tag{6.33.2}$$

substituting for $P_1(z, 0)$ in equation (6.32) and simplifying it we have

$$P_1^*(z,0) = \frac{z\,(S_1^*(g_{a_1}(w_X(z)))-1)\,(1-f)P_{2,1}(0)\,(w_X(z))\,I_{MAV}^{BV}(z)}{D_{DE}^{BV}(z)\,g_{a_1}(w_X(z))} \tag{6.34}$$

Thus the partial generating functions corresponding to different states at arbitrary epochs are given by

$$QI^*(z,0) = (1-f)\,P_{2,1}(0)\,\frac{(1-VI^*(w_X(z)))}{w_X(z)}\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\gamma_i \tag{6.35.1}$$

$$SE^*(z,0) = \frac{(1-D^*(w_X(z)))}{w_X(z)}(1-f)\,P_{2,1}(0)\,[(\sum_{j=0}^{\infty}(\alpha I_0)^j\prod_{i=0}^{j}\gamma_i)(VI^*(w_X(z))-1) + 1 - \frac{\phi_\gamma}{\lambda}w_X(z)] \quad (6.35.2)$$

where ϕ_γ is given by the equation (6.20.4)

$$P_1^*(z,0) = \frac{z\,(S_1^*(g_{a_1}(w_X(z)))-1)(1-f)P_{2,1}(0)(w_X(z))\,I_{MAV}^{BV}(z)}{g_{a_1}(w_X(z))\,D_{DE}^{BV}(z)} \quad (6.35.3)$$

$$P_2^*(z,0) = \frac{z\,(1-f)P_{2,1}(0)\,S_1^*(g_{a_1}(w_X(z)))(S_2^*(g_{a_2}(w_X(z)))-1)\,w_X(z)\,I_{MAV}^{BV}(z)}{g_{a_2}(w_X(z))\,D_{DE}^{BV}(z)} \quad (6.35.4)$$

$$QB^*(z,0) = (1-f)\,P_{2,1}(0)\,p\,\frac{(VB^*(w_X(z))-1)}{w_X(z)}\left[\frac{(fz+1-f)\,S_1^*(g_{a_1}(w_X(z)))\,S_2^*(g_{a_2}(w_X(z)))\,w_X(z)\,I_{MAV}^{BV}(z)}{D_{DE}^{BV}(z)}+1\right] \quad (6.35.5)$$

$$DE_i^*(z,0) = \frac{a_i\,(1-\Delta_i^*(w_X(z)))}{w_X(z)}P_i^*(z,0), \qquad i = 1, 2 \quad (6.35.6)$$

$$BR_i^*(z,0) = a_i\,\Delta_i^*(w_X(z))\,P_i^*(z,0)\,\frac{(1-R_i^*(w_X(z)))}{w_X(z)}, \qquad i = 1, 2 \quad (6.35.7)$$

$$PI = \frac{\phi_\gamma}{\lambda}(1-f)\,P_{2,1}(0) \quad (6.35.8)$$

To derive the total PGF of the system size distribution, the following generating functions are considered.

$P_{Idle}(z)$ = Probability generating function of the system size when the server is idle

$$= PI + SE^*(z,0) + QI^*(z,0) + QB^*(z,0)$$

$$= \frac{I_{MAV}^{BV}(z)\,Y(z)}{D_{DE}^{BV}(z)}(1-f)\,P_{2,1}(0) \quad (6.36)$$

where $I_{MAV}^{BV}(z)$, Y(z) and $D_{DE}^{BV}(z)$ are given by the equations (6.32.1), (6.33.2) and (6.33.1) respectively.

$P_{Comp}(z)$ = The PGF of the system size when the server is busy or breakdown state or delayed repair state

$$= \sum_{i=1}^{2} P_i^*(z,0) + BR_i^*(z,0) + DE_i^*(z,0)$$

$$= \frac{I_{MAV}^{BV}(z)\,P_{2,1}(0)\,(1-f)}{D_{DE}^{BV}(z)}[\,S_1^*(g_{a_1}(w_X(z)))\;S_2^*(g_{a_2}(w_X(z)))\,(1-f)(z-1)-Y(z)] \quad (6.37)$$

Thus the total PGF of the system size distribution is given by

$$P_{MAV}^{BV}(z) = P_{Idle}(z) + P_{Comp}(z)$$

$$= \frac{(1-f)^2\,P_{2,1}(0)\,I_{MAV}^{BV}(z)\,(z-1)\,S_1^*(g_{a_1}(w_X(z)))\,S_2^*(g_{a_2}(w_X(z)))}{D_{DE}^{BV}(z)} \quad (6.38)$$

$P_{2,1}(0)$ can be calculated by using the normalizing condition $P_{MAV}^{BV}(1) = 1$ and found to be $P_{2,1}(0) = \dfrac{(1-\rho_{DE}^{BV})}{(1-f)\,I_{MAV}^{BV}(1)}$ (6.39)

where

$$\rho_{DE}^{BV} = \frac{\lambda\,E(X)}{(1-f)\,S_1^*(a_1)\,S_2^*(a_2)}[F_C + pE(VB)\;S_1^*(a_1)\,S_2^*(a_2)] \quad (6.39.1)$$

where $F_C = F(S_1, \Delta_1, R_1, a_1) + S_1^*(a_1)\,F(S_2, \Delta_2, R_2, a_2)$, (6.39.2)

$F(S_i, \Delta_i, R_i, a_i) = (1 - S_i^*(a_i))(E(\Delta_i) + E(R_i) + \frac{1}{a_i})$, i = 1, 2 and

$$I_{MAV}^{BV}(1) = \frac{\phi_\gamma}{\lambda} + \left(\sum_{j=0}^{\infty}(\alpha I_0)^j \prod_{i=0}^{j}\gamma_i\,E(VI) - p\,E(VB) + E(D)\right) \quad (6.39.3)$$

Hence, substituting for $P_{2,1}(0)$ in equation (6.38),

$$P_{MAV}^{BV}(z) = \frac{(1-f)(1-\rho_{DE}^{BV})(z-1)\,S_1^*(g_{a_1}(w_X(z)))\,S_2^*(g_{a_2}(w_X(z)))}{D_{DE}^{BV}(z)}\;\frac{I_{MAV}^{BV}(z)}{I_{MAV}^{BV}(1)} \quad (6.40)$$

6.1.4 Decomposition Property

Since $D_{DE}^{BV}(1) = Y(1) = 0$ and $D_{DE}^{BV\,\prime}(1) = (1 - \rho_{DE}^{BV})\,(1-f)\;S_1^*(a_1)\,S_2^*(a_2)$, equation (6.40) can be re-written as

$$P_{MAV}^{BV}(z) = \left(\frac{(z-1)\,S_1^*(g_{a_1}(w_X(z)))\,S_2^*(g_{a_2}(w_X(z)))\,Y'(1)}{S_1^*(a_1)\;S_2^*(a_2)\,Y(z)}\right)\left(\frac{P_{Idle}(z)}{P_{Idle}(1)}\right) \quad (6.40.1),$$

where Y(z) is given by (6.33.2). The equation (6.40.1) shows that, the PGF of the system size of the model under consideration is decomposed into the product of two probability generating functions, one of which is the PGF of the single server two phase service queueing model with server breakdown, delayed repair and Bernoulli feedback (without vacation) and the other is the PGF of the conditional system size distribution $\frac{P_{Idle}(z)}{P_{Idle}(1)}$ during the server idle period.

By calculation $Y'(1) = (1 - f)\left(1 - \frac{\lambda E(X) F_C}{(1-f) S_1^*(a_1) S_2^*(a_2)}\right)$. The stability condition $\rho_{DE}^{BV} < 1$ implies $\frac{\lambda E(X) F_C}{(1-f) S_1^*(a_1) S_2^*(a_2)}$ is also < 1. (from 6.39.1)

6.1.5 Queue Size Distribution at Departure Epoch

The PGF $\pi^+(z)$ of the queue size distribution $\{\pi_n^+ ;\ n \geq 0\}$ at departure epoch can be obtained as,

$\pi^+(z) = \sum_{n=0}^{\infty} \pi_n^+ z^n = \sum_{n=0}^{\infty} D_1[(1-f) P_{2,n+1}(0)] z^n$ where D_1 is the normalizing constant.

i.e., $\pi^+(z) = \frac{D_1}{z}[(1 - f) P_2(z, 0)]$

using equations (6.23) and (6.33) and evaluating the normalizing constant D,

$\pi^+(z) = \frac{(X(z) - 1)}{E(X)(z - 1)} P_{MAV}^{BV}(z)$ where $P_{MAV}^{BV}(z)$ is the PGF of the system size of the present model at arbitrary point of time.

6.1.6 Performance Measures

In this section, the steady-state system size probabilities and the mean number of customers in the system corresponding to various system states are calculated.

The Server in Idle State

Let P_{VI}, P_{Set} and P_{VB} denote the steady-state system size probabilities and L_{VI}, L_{Set} and L_{VB} denote the mean number of customers, when the system is on vacation (during idle period), in setup state and on vacation during busy period respectively. These measures can be calculated using equations (6.35.1), (6.35.2) and (6.35.5).

$$P_{VI} = \lim_{z\to 1} QI^*(z,0) = \left(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i\right) E(VI)\, \frac{(1-\rho_{DE}^{BV})}{I_{MAV}^{BV}(1)} \quad (6.41)$$

$$L_{VI} = \left[\frac{d}{dz} QI^*(z,0)\right]_{z=1} = \frac{(1-\rho_{DE}^{BV})}{I_{MAV}^{BV}(1)} (\lambda E(X)) \left(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i\right) \quad (6.41.1)$$

$$P_{Set} = \lim_{z\to 1} SE^*(z,0) = E(D)\, \frac{(1-\rho_{DE}^{BV})}{I_{MAV}^{BV}(1)} \quad (6.42)$$

$$L_{Set} = \left[\frac{d}{dz} SE^*(z,0)\right]_{z=1}$$

$$= \frac{(1-\rho_{DE}^{BV})}{I_{MAV}^{BV}(1)} (\lambda E(X)) \left(\frac{E(D^2)}{2} + E(D)\left(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i\, E(VI) + \frac{\phi_\gamma}{\lambda}\right)\right) \quad (6.42.1)$$

$$P_{VB} = \lim_{z\to 1} QB^*(z,0) = p\, E(VB) \left(\frac{\lambda E(X)}{1-f} - \frac{(1-\rho_{DE}^{BV})}{I_{MAV}^{BV}(1)}\right) \quad (6.43)$$

where ρ_{DE}^{BV} and $I_{MAV}^{BV}(1)$ is given by the equations (6.39.1) and (6.39.2) respectively.

$$L_{VB} = \left[\frac{d}{dz} QB^*(z,0)\right]_{z=1}$$

$$= \frac{p\,E(VB)}{(1-f)} \Big[\frac{\lambda E(X(X-1))}{2} + \lambda E(X) \Big(\frac{(-D_{DE}^{BV''}(1))}{2 D_{DE}^{BV'}(1)} + \frac{I_{MAV}^{BV'}(1)}{I_{MAV}^{BV}(1)} \Big)$$

$$-(\lambda E(X))^2 \left(\frac{S_1^{*^1}(a_1)}{S_1^*(a_1)} + \frac{S_2^{*^1}(a_2)}{S_2^*(a_2)}\right) \Big] + \frac{p\,E(VB^2)}{2} (\lambda E(X)) \Big(\frac{\lambda E(X)}{1-f} - \frac{(1-\rho_{DE}^{BV})}{I_{MAV}^{BV}(1)} \Big) \quad (6.43.1)$$

where $$I_{MAV}^{BV'}(1) = \left[\frac{d}{dz} I_{MAV}^{BV}(z)\right]_{z=1}$$

$$= \lambda E(X) \left(\sum_{j=0}^{\infty} (\alpha I_0)^j \prod_{i=0}^{j} \gamma_i \left(\frac{E(VI^2)}{2} + E(VI) + E(D)\right) - \frac{p\,E(VB^2)}{2} + \frac{E(D^2)}{2} + E(D)\frac{\phi_\gamma}{\lambda}\right) \quad (6.43.2)$$

and $$(-D_{DE}^{BV''}(1)) = \lambda E(X(X-1)) [F_C + p\, S_1^*(a_1)\, S_2^*(a_2) E(VB)]$$

$$+ (\lambda E(X))^2 \{G_C - 2\, S_1^{*'}(a_1)\, F(S_2, \Delta_2, R_2, a_2) \quad (6.43.3)$$

$$+ p\, S_1^*(a_1)\, S_2^*(a_2) E(VB^2) - 2p\, E(VB)\, [S_1^{*'}(a_1)$$

$$+ S_2^{*'}(a_2)\, S_1^*(a_1)]\} + 2\lambda E(X) \{p\, S_1^*(a_1)\, S_2^*(a_2) E(VB)\, f$$

$$+ (1-f)\, (S_1^{*'}(a_1)\, S_2^*(a_2) + S_2^{*'}(a_2)\, S_1^*(a_1)) + F_C\}$$

where $G_C = G(S_1, \Delta_1, R_1, a_1) + S_1^*(a_1)\, G(S_2, \Delta_2, R_2, a_2)$,

$$G(S_i, \Delta_i, R_i, a_i) = (1 - S_i^*(a_i))\,(E(\Delta_i^2) + E(R_i^2) + 2\,E(\Delta_i)\,E(R_i)) + \frac{2}{a_i}(F(S_i, \Delta_i, R_i, a_i)) + 2\,S_i^{*'}(a_i)\,(E(\Delta_i) + E(R_i) + \frac{1}{a_i})$$

for i = 1, 2 and F_C is given by equation (6.39.2).

The Server in Busy State

The probability that the server is busy and the expected number of customers in the system when the server is in first and second stage of services are calculated using the equations (6.35.3) and (6.35.4) respectively.

$$P_{busy} = \lim_{z \to 1} [P_1^*(z, 0) + P_2^*(z, 0)] = \frac{\lambda E(X)}{(1-f)} \left[\frac{(1 - S_1^*(a_1))}{a_1\, S_1^*(a_1)\, S_2^*(a_2)} + \frac{(1 - S_2^*(a_2))}{a_2\, S_2^*(a_2)} \right] \quad (6.44)$$

$$L_{busy} = \left[\frac{d}{dz} P_1^*(z, 0) + P_2^*(z, 0) \right]_{z=1}$$

$$= \frac{(S_1^*(a_1))}{S_2^*(a_2)} + (S_2^*(a_2)) + \frac{(\lambda E(X))^2\, S_1^{*'}(a_1)\,(S_2^*(a_2) - 1)}{a_2\,(1-f)\, S_1^*(a_1)\, S_2^*(a_2)} \quad (6.45)$$

where

$$(S_i^*(a_i)) = \frac{1}{(1-f)\, S_i^*(a_i)} \left[\frac{(1 - S_i^*(a_i))}{a_i} \left(\frac{\lambda E(X(X-1))}{2} + \lambda E(X) \left(\frac{(-D_{DE}^{BV''}(1))}{2\, D_{DE}^{BV'}(1)} + 1 + \frac{I_{MAV}^{BV'}(1)}{I_{MAV}^{BV}(1)} + \frac{\lambda E(X)}{a_i} \right) \right) + (\lambda\, E(X))^2\, \frac{S_i^{*'}(a_i))}{a_i} \right], \; i=1,2 \quad (6.45.1)$$

The Server in Breakdown State

The probability that the server in breakdown states (P_{BR_1} and P_{BR_2}) and the corresponding expected number of customers in the system (L_{BR_1} and L_{BR_2}) are calculated by using the equations (6.9).

Thus, for i = 1, 2

P_{BR_i} = Probability that the server in breakdown state due to i^{th} stage of service

$$= \lim_{z \to 1} BR_i^*(z, 0) = a_i\, E(R_i)\, P_{busy_i} \quad (6.46)$$

$$L_{BR_i} = \left[\frac{d}{dz} BR_i^*(z,0)\right]_{z=1}$$

$$= a_i\,[E(R_i)\,L_{busy_i} + (\lambda\,E(X))\,P_{busy_i}\,(\frac{E(R_i^2)}{2} + E(R_i)\,E(\Delta_i))] \qquad (6.46.1)$$

Delayed Repair State

The probability that the server in delayed repair period (P_{DE_i}) and the expected number of customers in the system (L_{DE_i}) are calculated by using the equation (6.35.6).

Thus, for i = 1, 2,

$$P_{DE_i} = a_i\,E(\Delta_i)\,P_{busy_i}, \qquad (6.47)$$

$$L_{DE_i} = a_i\,(E(\Delta_i)\,L_{busy_i} + \lambda\,E(X)\,P_{busy_i}\,\frac{E(\Delta_i^2)}{2}) \qquad (6.47.1)$$

Mean System Size

The expected system size L_{MAV}^{BV} of the model is given by

$$L_{MAV}^{BV} = L_I + L_{busy}$$

$$L_{MAV}^{BV} = (-\lambda\,E(X))\left(\frac{S_1^{*'}(a_1))}{S_1^*(a_1)} + \frac{S_2^{*'}(a_2))}{S_2^*(a_2)}\right)$$

$$+ \frac{(-D_{DE}^{BV''}(1))}{2\,(1-f)\,(1-\rho_{DE}^{BV})\,S_1^*(a_1)\,S_2^*(a_2)} + \frac{I_{MAV}^{BV'}(1)}{I_{MAV}^{BV}(1)} \qquad (6.48)$$

where $(-D_{DE}^{BV''}(1))$, $I_{MAV}^{BV}(1)$, $I_{MAV}^{BV'}(1)$ and ρ_{DE}^{BV} are given by the equations (6.43.3), (6.39.3), (6.43.2) and (6.39.1) respectively.

6.2 PARTICULAR CASES

In this section, some special cases of the model of the present chapter are analysed. With the suitable selection of the vacation controlled probabilities γ_j ($j \geq 0$), the corresponding results of single, multiple, non vacation, BSV and randomized J vacation models are deduced. For this $I_{MAV}^{BV}(z)$ and $I_{MAV}^{BV}(1)$ are calculated for different possible values of γ_is′ and given in the following Table 6.0.

Table 6.0

	Types of vacation during idle period	Probabilities (γ_j)	ϕ_γ	$\sum_{j=0}^{\infty} \alpha I_0^j \prod_{i=0}^{j} \gamma_i$	$I^{BV}_{MAV}(z) = I_z$; $I^{BV}_{MAV}(1) = I_1$
1.	Single vacation	$\gamma_0 = 1$, $\gamma_j = 0 \ \forall\ j \geq 1$	αI_0	1	$I_z = D^*(w_X(z)) \left[\frac{1-VI^*(w_X(z))}{w_X(z)} + \frac{\alpha I_0}{\lambda}\right] + \frac{1-D^*(w_X(z))}{w_X(z)} - \frac{p(1-VB^*(w_X(z)))}{w_X(z)}$ $I_1 = \frac{\alpha I_0}{\lambda} + E(VI) - p\,E(VB) + E(D)$
2.	Multiple vacation	$\gamma_j = 1, \forall_j$	0	$\frac{1}{1-\alpha I_0}$	$I_z = D^*(w_X(z)) \left[\frac{1-VI^*(w_X(z))}{w_X(z)} \frac{1}{1-\alpha I_0}\right] + \frac{1-D^*(w_X(z))}{w_X(z)} - \frac{p(1-VB^*(w_X(z)))}{w_X(z)}$ $I_1 = \frac{E(VI)}{1-\alpha I_0} - p\,E(VB) + E(D)$
3.	Non vacation	$\gamma_j = 0, \forall\ j, p = 0$	1	0	$I_z = \frac{D^*(w_X(z))}{\lambda} + \frac{1-D^*(w_X(z))}{w_X(z)}$; $I_1 = E(D) + \frac{1}{\lambda}$
4.	< p, J > vacation	$\gamma_0 = 1$, $\gamma_j = \bar{p}$, $1 \leq j \leq J-1$, $\gamma_j = 0 \ \forall\ j \geq J$	$\frac{[\alpha I_0 (1-\bar{p})(1-(\alpha I_0 \bar{p})^{J-1})]}{1-\alpha I_0 \bar{p}} + \alpha I_0^J\ \bar{p}^{\,J-1}$	$\frac{1-(\alpha I_0 \bar{p})^J}{1-\alpha I_0 \bar{p}}$	$I_z = D^*(w_X(z)) \left[\frac{1-(\alpha I_0 \bar{p})^J}{1-\alpha I_0 \bar{p}} \frac{1-VI^*(w_X(z))}{w_X(z)} + \frac{1}{\lambda}\left[\alpha I_0 (1-\bar{p}) \frac{(1-(\alpha I_0 \bar{p})^{J-1})}{1-\alpha I_0 \bar{p}} + (\alpha I_0)^J\ \bar{p}^{\,J-1}\right]\right] + \frac{1-D^*(w_X(z))}{w_X(z)} - \frac{p(1-VB^*(w_X(z)))}{w_X(z)}$ $I_1 = \frac{1}{\lambda}\left[\alpha I_0 (1-\bar{p}) \frac{1-(\alpha I_0 \bar{p})^{J-1}}{1-\alpha I_0 \bar{p}} + (\alpha I_0)^J\ \bar{p}^{\,J-1}\right] + \frac{1-(\alpha I_0 \bar{p})^J}{1-\alpha I_0 \bar{p}} E(VI) - pE(VB) + E(D)$
5.1	Optional Bernoulli Schedule Vacation (BSV)	$\gamma_0 = p$, $\gamma_j = 0 \ \forall\ j$, VI = VB	$(1-p) + p\,\alpha I_0$	p	$I_z = \frac{D^*(w_X(z)) - 1}{w_X(z)} [p(1-VI^*(w_X(z))) - 1] + D^*(w_X(z)) \frac{[1-p+p\alpha I_0]}{\lambda}$ $I_1 = \frac{1-p+p\alpha I_0}{\lambda} + E(D)$
5.2	BSV without setup	Pr (D=0) = 1			$\frac{I_z}{I_1} = 1$

Ke and Huang (2010) have analysed an unreliable single service $M^X/G/1$ system under randomized vacation policy and delayed repair (without setup operation or BSV between services). They have considered the case, that the service interrupted customers resume service from where the service got interrupted when the server is fixed. It is found that $\frac{I_z}{I_1}$ of case 4 with $\Pr(D = 0) = 1$ coincides with the PGF of the conditional system size distribution during server idle period obtained by Ke and Huang (2010).

Khalaf et al. (2011a) investigates an $M^X/G/1$ queue with BSV, and general delayed repair time, with no setup operation. They have considered that the service interrupted customers repeat their service from the beginning when the server is fixed. They have obtained the PGF $S_q(z)$ of the queue size, irrespective of the state of the system.

The expression $(QI^*(z,0) + QB^*(z,0) + \frac{P_1^*(z,0)}{z} + DE_1^*(z,0) + BR_1^*(z,0))$ with $VI = VB$, $f = 0$, $a_2 = 0$ and $\Pr(D = 0) = 1$ is calculated from the equations (6.35.1) to (6.35.7) for the single service model and found to be

$$\frac{\begin{array}{l}(1-\rho)[p\, g_a(w_X(z))\, S^*(g_a(w_X(z)))(VB^*(w_X(z))-1)\\ +(S^*(g_a(w_X(z)))-1)(w_X(z)+a\,z\,(1-\Delta^*(w_X(z))R^*(w_X(z))))]\end{array}}{\begin{array}{l}g_a(w_X(z))[z-S^*(g_a(w_X(z)))(1-p+p\,VB^*(w_X(z)))]\\ -a\,z\,(1-S^*(g_a(w_X(z))))\Delta^*(w_X(z))R^*(w_X(z))\end{array}} \tag{6.49}$$

It is verified that equation (6.49) gives $S_q(z)$ of the model of Khalaf et al. (2011a).

6.3 NUMERICAL ANALYSIS

In this section numerical results are obtained to study the effects of the parameters namely (i) mean vacation time ($E(VI)$), (ii) mean service time $E(S_i)$, (iii) breakdown rates (a_i), (iv) delayed repair time $E(\Delta_i)$, (v) feedback probability (f) and (vi) arrival rate (λ), on mean system size (L) and on the system size probabilities (P_{busy}, P_{idle}) when the server is in different states. For the computation purpose the following distributions are assumed for different random variables.

Random variables (Y)	Distribution F(Y)		
First service (S_1)	Two-stage hyper-exponential	$S_1^*(a_1) = \frac{b_1\mu_1}{\mu_1+a_1}+\frac{(1-b_1)\mu_2}{\mu_2+a_1}$	$S_1^{*'}(a_1) = -\left(\frac{b_1\mu_1}{(\mu_1+a_1)^2}+\frac{(1-b_1)\mu_2}{(\mu_2+a_1)^2}\right)$
Second service (S_2)	Gamm-2 type	$S_2^*(a_2) = \left(\frac{\mu_{21}}{\mu_{21}+a_2}\right)^2$	$S_2^{*'}(a_2) = -\frac{2}{\mu_{21}}\left(\frac{\mu_{21}}{\mu_{21}+a_2}\right)^3$
		Mean E(Y)	**Second order moments E(Y^2)**
Setup time (D)	Erlang-3 type	$\frac{1}{\gamma}$	$\frac{4}{3\gamma^2}$
Vacation times (VI, VB)	Erlang -3 type (Gamma-3 type)	$(\frac{1}{\eta_1}, \frac{3}{\eta_2})$	$(\frac{4}{3\eta_1^2}, \frac{12}{\eta_2^2})$
Repair time (R_1, R_2)	Gamma-4 type (Exponential)	$(\frac{4}{rp_1}, \frac{1}{rp_2})$	$(\frac{20}{rp_1^2}, \frac{2}{rp_2^2})$
Delay time Δ_i, i = 1, 2	Exponential	$(1 / \delta_i)$	$2 / \delta_i^2$
Batch size (X)	Geometric (Geo(p_1))	$(\frac{1}{1-p_1})$	$\frac{p_1+1}{(1-p_1)^2}$
	Binomial (B(4, p_1))	$4p_1$	$4p_1(1 + 3p_1)$

Tables 6.1 and 6.2 analyse the sensitivity of the service rate μ_i, the feedback probability f, arrival rate λ, failure rate a_i and mean repair time $E(R_i)$ (i = 1, 2). Table 6.1 shows that the system size L increases with feedback probability and decreases as the service rates increases. The values of Table 6.2 justify that L increases with arrival rate (λ) or the breakdown rates (a_i) and decreases as the mean repair time ($E(R_i)$, i = 1, 2) decreases. The system size (L) increases with the feedback probability f is depicted in Figure 6.1 for different values of μ_1 when μ_{21} = 5 (given in Table 6.1).

Table 6.1 Expected System Size (L) Vs Service rates (μ_1, μ_{21}) for Different Values of Feedback Probability (f)

$(\delta_1, \delta_2) = (0.5, 0.7)$ $(p_1, \lambda, p, \gamma_0) = (0.5, 0.03, 0.44, 0.7)$, $\gamma_j = \frac{1}{2^j}, j \geq 1$

$(\mu_2, b_1, a_1, a_2, \gamma, \eta_1, \eta_2, rp_1, rp_2) = (2, 0.3, 1, 2, 0.7, 0.5, 0.6, 5, 3)$

	μ_1	f = 0.45		f = 0.50		f = 0.54	
		L_{busy}	L	L_{busy}	L	L_{busy}	L
μ_{21} = 5	4	0.927	5.329	1.819	10.482	7.797	44.972
	6	0.767	4.490	1.342	7.870	3.325	19.523
	8	0.698	4.124	1.165	6.900	2.495	14.797
	$\mu_{2,1}$						
μ_1 = 0.01	50	0.736	4.077	1.208	6.709	2.472	13.746
	60	0.694	3.861	1.110	6.193	2.128	11.888
	70	0.666	3.718	1.048	5.866	1.931	10.825

Figure 6.1 L Vs f for Different Values of μ_1 with μ_{21} = 5

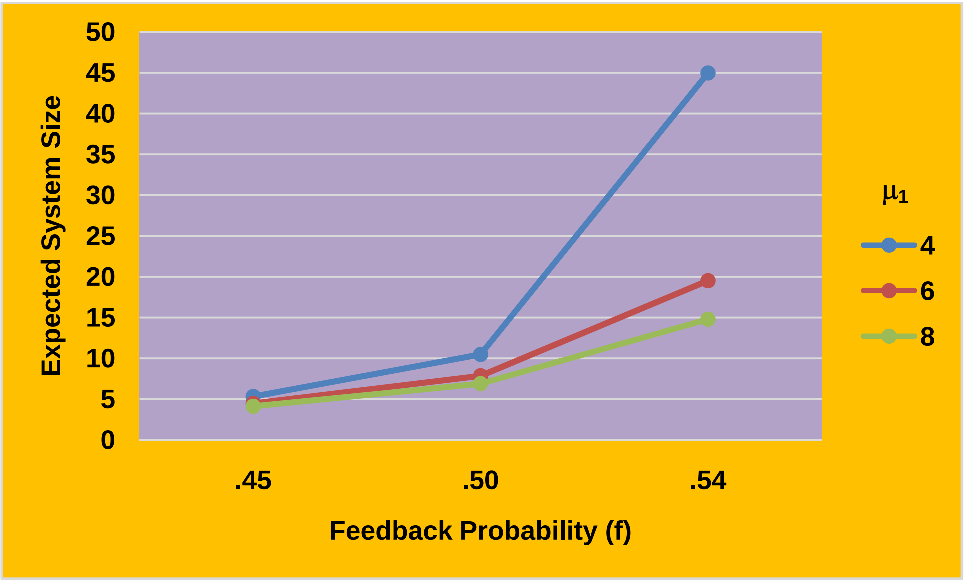


Table 6.2 Expected System Size Vs. Breakdown Rates (a_i) and Repair rate E(R_i) (i = 1, 2) and Arrival Rate λ

The parametric values are same as Table 6.1 with μ_1 = 4, f = 0.54, μ_{21} = 5

	(a_1, a_2)	**λ = 0.16**	**0.020**	**0.024**
E(R_1) = 0.80 E(R_2) = 0.33	(1, 2)	1.311	2.251	4.298
	(1.2, 5)	1.738	3.387	9.146
	(2, 1)	1.256	2.124	3.924
	(3, 1)	1.990	4.204	16.020
	E(R_1), E(R_2)	**λ**		
a_1 = 1, a_2 = 2	(1, 0.3)	1.374	2.400	4.766
	(1, 0.1)	1.290	2.200	4.146
	(0.8, 0.5)	1.372	2.397	4.754
	(0.4, 0.5)	1.263	2.140	3.975

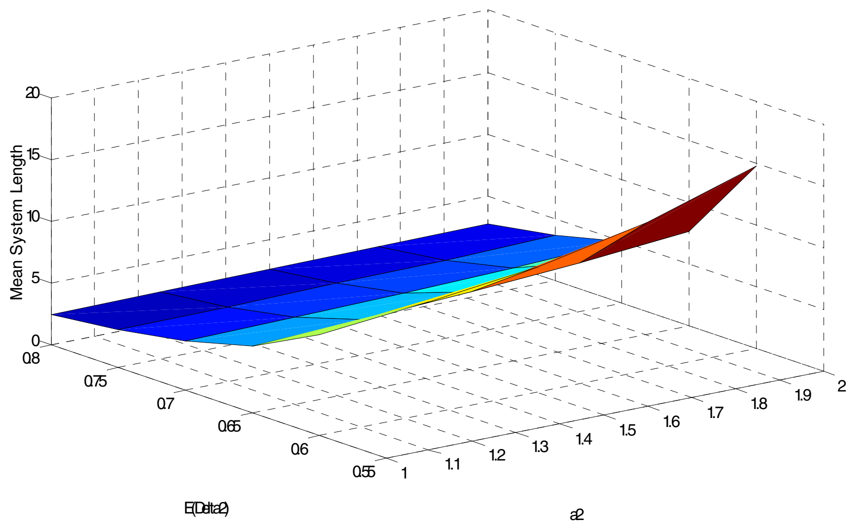
It is illustrated in Table 6.3 that L increases with the increase of mean delayed repair time $E(\Delta_i)$ corresponding to different breakdown rates (a_1, a_2) for two different distributions of the batch size (Geometric and Binomial). The increase in L w.r. to $E(\Delta_2)$ and a_2 is shown in Figure 6.3 for the geometric distribution.

Table 6.3. The Effect of Mean Delayed Repair Time $E(\Delta_i)$ and Breakdown rates a_i (i = 1, 2) on Mean System Length with (μ_1 = 4, f = 0.54, μ_{21} = 5)

(a_1, a_2) \ $(E(\Delta_1, \Delta_2))$	(1.54, 1.43)	(1.43, 1.43)	(1.25, 1.67)	(1.25, 1.43)	(1.25, 1.25)
(2.5, 1)	37.77	22.057	15.476	12.813	11.312
	53.24	31.115	21.848	18.098	15.984
(2, 1)	9.073	7.768	6.925	6.228	5.779
	12.798	10.966	9.781	8.012	8.001
(1, 2)	14.89	12.718	14.726	10.205	8.195
	21.014	17.953	20.776	14.415	11.586
(1, 1.5)	5.21	4.874	5.038	4.400	4.00
	7.351	6.886	7.116	6.222	5.662

■ Geo (p_1) ■ B (4, p_1)

Figure 6.3 The Effect of Mean Delayed Repair Time $E(\Delta_2)$ and Breakdown Rate (a_2) on Mean System Length



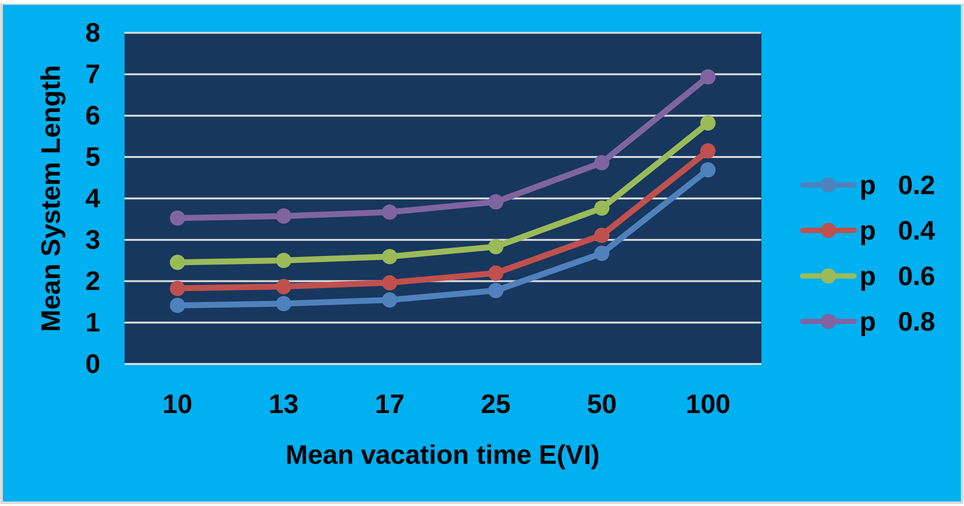
In any queueing situations the mean system size (L) is expected to increase either if the server goes for vacation for a longer duration during idle period (EVI) or taking vacation often between services (p). The Table 6.4 and Figure 6.4 justify this.

Table 6.4. The Influence of Mean Vacation Time During Idle Period and Probability of Vacation Between Services (p) on Mean System Length

The parametric values are same as in Table 6.2.

E(VI)	p = 0.2	0.4	0.6	0.8
100	4.69	5.151	5.821	6.936
50	2.674	3.112	3.767	4.865
25	1.775	2.197	2.836	3.918
17	1.547	1.964	2.596	3.672
13	1.459	1.873	2.503	3.576
10	1.416	1.829	2.457	3.529

Figure 6.4 Mean System Length Vs Mean Vacation Time E(VI) for Different Values of p



It is seen from the values of Table 6.5, that the probability that the server is idle (P_{idle}) steadily decreases as the value of λ increases and the probability that the server is busy (P_{busy}) exhibits the opposite trend that of P_{idle}.

Table 6.5. The Effect of Arrival Rate on System Size Probabilities and Mean System Length

The values of the parameters are same as in Table 6.1 with $\mu_1 = 4$, $\mu_{21} = 5$, f = 0.1, p = 0.04 and $p_1 = 0.05$.

λ	P_{busy}	P_{idle}	L_{busy}	L_{idle}	L
0.10	0.151	0.107	0.487	0.120	2.194
0.11	0.166	0.100	0.658	0.141	2.928
0.12	0.182	0.090	0.923	0.175	4.063
0.13	0.197	0.079	1.391	0.238	6.051
0.14	0.212	0.065	2.428	0.387	10.445
0.15	0.227	0.049	6.662	1.028	28.339