CHAPTER – 1

PRELIMINARY DEFINITIONS AND RESULTS IN CI-ALGEBRAS

SECTION 1.1

PRELIMINARIES ON CI-ALGEBRAS

Definition : 1.1.1

A **BCI-algebra** $(X ; *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms : For all $x, y, z \in X$,

(BCI 1) $((x * y) * (x * z)) * (z * y) = 0$

(BCI 2) $(x * (x * y)) * y = 0$

(BCI 3) $x * x = 0$

(BCI 4) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

Definition : 1.1.2

A BCI-algebra $(X ; *, 0)$ is called a BCK-algebra if

(BCK 1) $0 * x = 0$ for all $x \in X$.

Definition : 1.1.3

A **BCH-algebra** $(X ; *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms. For all $x, y, z \in X$,

(BCH 1) $x * x = 0$

(BCH 2) $(x * y) * z = (x * z) * y$

(BCH 3) If $x * y = 0$ and $y * x = 0 \Rightarrow x = y \ \forall \ x, y, z \in X$.

Note

In BCI / BCK / BCH-algebra $(X ; *, 0)$ or simply denoted by X, define a binary relation "$\leq$" by $x \leq y$ iff $x * y = 0$.

Definition : 1.1.4

A **BE-algebra** $(X ; *, 1)$ is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms : For all $x, y, z \in X$,

(BE 1) $x * x = 1$

(BE 2) $x * 1 = 1$

(BE 3) $1 * x = x$

(BE 4) $x * (y * z) = y * (x * z)$

Definition : 1.1.5

A **dual BCK-algebra** $(X ; *, 1)$ is a non empty set X with a constant 1 and a binary operation $*$ satisfying (BE 1), (BE 2) and the following axioms : For all $x, y, z \in X$.

(dBCK 1) $x * y = y * x = 1$ implies $x = y$

(dBCK 2) $(x * y) * ((y * z) * (x * z)) = 1$

(dBCK 3) $x * ((x * y) * y) = 1$.

Definition : 1.1.6

A non-empty set X with a constant 1 and a binary operation $*$ denoted by $(X ; *, 1)$ or simply by X is called a **CI-algebra** if it satisfies the following axioms : For all $x, y, z \in X$

(CI 1) $x * x = 1$

(CI 2) $1 * x = x$

(CI 3) $x * (y * z) = y * (x * z)$

Note

In a BE / dual BCK / CI algebra X, define a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$ for all $x, y, \in X$.

Note

(i) Obviously, every dual BCK / BCI / BCH-algebra is a CI-algebra.

(ii) For any CI-algebra X, denote $B(X) = \{x \in X / x * 1 = 1\}$

Now, a CI-algebra is a BE-algebra if and only if $X = B(X)$.

Note :

Hereafter X will denote CI-algebra unless it is specified.

Definition : 1.1.7

A non-empty subset S of an CI-algebra X is said to be a **subalgebra of X** if $x * y \in S$, whenever $x, y \in S$.

Note

For any CI-algebra X, {1} and X are trivial subalgebra of X.

Definition : 1.1.8

A CI-algebra X is said to be **self-distributive** if for any $x, y, z \in X$,

$x * (y * z) = (x * y) * (x * z)$.

Definition : 1.1.9

A CI-algebra X is said to be **commutative** if it satisfies for any $x, y \in X$,

$(x * y) * y = (y * x) * x$.

Proposition : 1.1.10

In an CI-algebra X, the following properties holds good :For all $x, y \in X$.

(i) $y * ((y * x) * x) = 1$

(ii) $(x * 1) * (y * 1) = (x * y) * 1$

(iii) $1 \leq x \Rightarrow x = 1$.

Proof

(i) Let X be a CI-algebra. For any $x, y \in X$ we have,

$y * ((y * x) * x) = (y * x) * (y * x) = 1$. By (CI 3) and (CI 1).

(ii) Let X be a CI-algebra. For any $x, y \in X$ we have,

$$\begin{aligned} (x * 1) * (y * 1) &= (x * 1) * \{y * [(x * y) * (x * y)]\} \\ &= (x * 1) * \{(x * y) * [x * (y * y)]\} \text{ (By CI 3)} \\ &= (x * 1) * [(x * y) * (x * 1)] \text{ (By CI 1)} \\ &= (x * y) * [(x * 1) * (x * 1)] \text{ (By CI 3)} \\ &= (x * y) * 1 \text{ (By CI 1)} \end{aligned}$$

(iii) Obvious.

Example : 1.1.11

Let X = {1, a, b} be a set with the following cayley table :

∗	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then (X ; ∗, 1) is a **CI-algebra**.

Example : 1.1.12

Let X = {1, 2, 3, 4, 5, 6} be a set with the following cayley table :

∗	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	1	4	4	4
3	1	1	1	4	4	4
4	4	5	1	1	2	3
5	4	4	4	1	1	1
6	4	4	4	1	1	1

Then $(X \,;\, *, 1)$ is a CI-algebra. But it is not a BE-algebra because $4 * 1 = 5 * 1 = 6 * 1 = 4 \neq 1$.

Therefore, the class of BE-algebras is a proper subclass of the class of CI-algebras.

Example : 1.1.13

Let $X = \{1, a, b, c, d\}$ be a set with the following cayley table :

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then $(X \,;\, *, 1)$ is a **CI-algebra**.

Definition : 1.1.14

Let $(X \,;\, *, 1)$ be an CI-algebra and F a non empty subset of X. Then F is said to be a **filter of X** if

(F 1) $1 \in F$,

(F 2) If $x \in F$ and $x * y \in F$, then $y \in F$.

The set of all filters in X is denoted by Fil (x). Obviously $\{1\}, X \in$ Fil (X).

Example : 1.1.15

In Example (1.1.13) of CI-algebra X, consider $F_1 = \{1, a\}$ and $F_2 = \{1, b\}$.

Then F_1 and F_2 are filters of CI-algebra X.

Proposition : 1.1.16

If $F_i (i \in \Lambda)$ are filters of a CI-algebra X. Then $\bigcap_{i \in \Lambda} F_i$ is a filter of X.

Proof : Obvious.

Definition : 1.1.17

Let X be a CI-algebra. A non empty subset I of X is called an **ideal of X** if

(I 1) If $x \in X$ and $a \in \mathrm{I}$, then $x * a \in \mathrm{I}$

i.e, $X * \mathrm{I} \subseteq \mathrm{I}$,

(I 2) If $x \in X$ and $a, b \in \mathrm{I}$, then $(a * (b * x)) * x \in \mathrm{I}$.

Note

For any CI-algebra X, {1} and X are trivial ideals of X. Obviously every ideal in a CI-algebra is a subalgebra.

Lemma : 1.1.18

Let X be an CI-algebra. Then

(i) Every ideal of X contains 1,

(ii) If I is an ideal of X, then $(a * x) * x \in \mathrm{I}$ for all $a \in \mathrm{I}$ and $x \in X$.

Proof

Let X be an CI-algebra.

To Prove (i), Let I be an ideal of X.

For $x \in \mathrm{I}$, $1 = x * x \in \mathrm{I} * \mathrm{I} \subseteq X * \mathrm{I}$. Thus $1 \in \mathrm{I}$.

To Prove (ii), Let I be an ideal of X.

Then $(a * (b * x)) * x \in \mathrm{I}$ for $x \in X$ and $a, b \in \mathrm{I}$ (1)

Put b = 1 in equation (1). Then $(a * (1 * x)) * x \in \mathrm{I}$.

Hence $(a * x) * x \in \mathrm{I}$.

Definition : 1.1.19

Let I be an **ideal** of a CI-algebra X. Define I_w by $I_w = \{x \in X / w * x \in I\}$ for any $w \in X$.

Proposition : 1.1.20

Let X be a self-distributive CI-algebra and I an ideal of X. Then I_w is a subalgebra of an CI-algebra X.

Proof

Let X be a self-distributive CI-algebra. And let I be an ideal of X.

Let $a, b \in I_w$.

Then $w * a \in I$ and $w * b \in I$.

And, $w * (a * b) = (w * a) * (w * b)$

$\subseteq I * I$

$\subseteq X * I \subseteq I$

This implies $a * b \in I_w$.

Proposition : 1.1.21

Let X be a self-distributive CI-algebra and I an ideal of X. Then I_w is an ideal of an CI-algebra X.

Proof

Let X be a self-distributive CI-algebra and I be an ideal of X.

Let $x \in X$ and $a \in I_w$.

Then, we have $w * a \in I$ and $w * (x * a) = (w * x) * (w * a)$

$\in X * I$

$\subseteq I$ (by I 1)

This implies $x * a \in I_w$ (1)

Now let $a, b \in I_w$ and $x \in X$.

Then, $w * a \in I$ and $w * b \in I$.

Thus, $w * ((a * (b * x)) * x)$

$= w * (((a * (b * x)))) * (w * x)$

$= ((w * a) * (w * (b * x))) * (w * x)$

$= ((w * a) * ((w * b) * (w * x))) * (w * x)$

$\in I$ [by I 2]

This implies $(a * (b * x)) * x \in I_w$ (2)

By (1) and (2), I_w is an ideal of X.

Proposition : 1.1.22

Let X be a self-distributive CI-algebra and I an ideal of X. If $a \in I_w$ and $a * b = 1$ then $b \in I_w$.

Proof

Let $a \in I_w$ and $a \subseteq b$.

Then $w * a \in I$ and $a * b = 1$.

Hence, $w * b = w * (1 * b)$

$= w * ((a * b) * b)$

$= (w * (a * b)) * (w * b)$

$= ((w * a) * (w * b)) * (w * b)$

$\in I$ by lemma (1.1.18 (ii))

This implies $b \in I_w$.

SECTION 1.2

FILTERS AND UPPER SETS IN CI-ALGEBRAS

Definition : 1.2.1

A CI-algebra X is said to be **transitive** if for all x, y, z $\in$ X,
$(y * z) * [(x * y) * (x * z)] = 1$

Example : 1.2.2

Let X = {1, a, b, c} in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a transitive CI-algebra.

Lemma : 1.2.3

If a CI-algebra X is transitive, then for all x, y, z $\in$ X, $x \leq y$ implies $z * x \leq z * y$.

Proposition : 1.2.4

If X is a self-distributive CI-algebra, then it is transitive.

Proof

Let X be a self-distributive CI-algebra. For any x, y $\in$ X we have

$$\begin{aligned}(y * z) * ((x * y) * (x * z)) &= (y * z) * (x * (y * z)) \\ &= x * ((y * z) * (y * z)) = x * 1 = 1.\end{aligned}$$

Proposition : 1.2.5

Let X be a transitive CI-algebra and A a non empty subset of X. Then A is an ideal of X if and only if A is a filter of X.

Proof

Let X be a transitive CI-algebra. Suppose A is an ideal of X.

Take any $a \in I$, then $1 = a * a \in A$ (1)

Let $x, y \in X$ be such that $x * y \in A$ and $x \in A$.

Since $(x * y) * y = [1 * (x * y)] * y$, it follows from (I2) that $(x * y) * y \in A$.

Denote $\alpha = (x * y) * y$ and $\beta = x * y$

By (I2) we have $y = 1 * y$

$= \{[(x * y) * y] * [(x * y) * y]\} * y$

$= [\alpha * (\beta * y)] * y$

$\in A$ (2)

By (1) and (2), A is a filter of X.

Conversely, let A be a filter of X. Assume that $x \in X$ and $a \in A$.

Since $a * (x * a) = x * (a * a) = x * 1$

$= 1 \in A$ [by F1]

It follows from (F2) that $x * a \in A$.

Hence (I1) is true.

Let $a, b \in A$ and $x \in X$. Because $a * [(a * x) * x]$

$= (a * x) * (a * x)$

$= 1 \in A$ [by F1]

And so $(a * x) * x \in A$ [by F2]

using transitivity of X we have

$[(a * x) * x] * \{[b * (a * x)] * (b * x)\} = 1 \in A$,

and so $[b * (a * x)] * (b * x) \in A$ [by F2]

Hence $b * \{[b * (a * x)] * x\} \in A$

By $b \in A$ and (F2) we obtain $[b * (a * x)] * x \in A$

i.e, (I2) holds

Therefore, A is an ideal of X.

Proposition : 1.2.6

Let X be a transitive CI-algebra and A a non empty subset of X, then A is a filter of X if and only if A satisfies : for any a, b $\in$ A and x $\in$ X, $a * (b * x) = 1$ implies x $\in$ A.

Proof

Let A be a non empty subset of an CI-algebra X. Assume that for any a, b $\in$ A and x $\in$ X, $a * (b * x) = 1$ implies x $\in$ A.

To Prove : A is a filter of X.

Let a $\in$ A. Since $a * (a * 1) = 1$, it follows that 1 $\in$ A.

Hence (F1) holds for A.

Suppose $a * x \in A$ and a $\in$ A.

Because $a * [(a * x) * x] = 1$, and so x $\in$ A, (F2) is true.

Therefore A is a filter of X.

Conversely, let A be a filter of X.

Assume a, b $\in$ A and x $\in$ X such that $a * (b * x) = 1$.

By (F1) we have $a * (b * x) \in A$.

Then applying (F2) twice we obtain x $\in$ A. This completes the proof.

Definition : 1.2.7

Let X be an CI-algebra. For any x_1, … x_n, a $\in$ X define

$$\prod_{i=1}^{n} (x_i * a) = x_n * (\ldots * (x_1 * a) \ldots).$$

Corollary : 1.2.8

Let X be a transitive CI-algebra and A a non empty subset of X. Then A is a filter of X if and only if A satisfies : For any $a_i \in A$ ($i \in N$) and $x \in X$, $a_n * (\ldots * (a_1 * x) \ldots) = 1$ implies $x \in A$.

Lemma : 1.2.9

Let X be a transitive CI-algebra and let $x, y \in X$ such that $x * y = 1$. Then for all $a_1, \ldots, a_n \in X$, $\prod_{i=1}^{n} a_i * x = 1$ implies $\prod_{i=1}^{n} a_i * y = 1$.

Proof

Let X be a transitive CI-algebra.

We have $x \leq y$ implies $a_i * x \leq a_i * y$.

$\therefore\ 1 = \prod_{i=1}^{n} a_i * x \leq \prod_{i=1}^{n} a_i * y$

By proposition 1.1.10 (iii), $\prod_{i=1}^{n} a_i * y = 1$.

Definition : 1.2.10

For every subset $A \subseteq X$, the smallest filter of X which contains A, that is, the intersection of all filters $F \supseteq A$, is said to be **the filter generated by A**, and will be denoted [A) obviously, $[\phi) = \{1\}$.

Theorem : 1.2.11

Let A be a nonvoid subset of a transitive CI-algebra X.

Thus $[A) = \{x \in X \,/\, x = 1 \text{ or } \prod_{i=1}^{n} a_i * x = 1 \text{ for some } a_1, \ldots a_n \in A\}$.

Proof

Let A be a nonvoid subset of a transitive CI-algebra X.

Let $F = \{x \in X \,/\, x = 1 \text{ or } \prod_{i=1}^{n} a_i * x = 1 \text{ for some } a_1, \ldots, a_n \in A\}$.

Since $a * a = 1$ for all $a \in A$, we obtain $A \subseteq F$.

Obviously, $1 \in F$.

Let $x * y \in F$ and $x \in F$. Then to prove that $y \in F$.

We consider three cases :

Case 1 : x = 1

Then $1 * y = y \in F$

Case 2 : $x * y = 1$ and $x \neq 1$

Since $x \in F$ and $x \neq 1$, we have that $\prod_{i=1}^{n} a_i * x = 1$ for some $a_1, \ldots a_n \in A$. From lemma (1.2.9) it follows that $\prod_{i=1}^{n} a_i * y = 1$. Therefore $y \in F$.

Case 3 : $x * y \neq 1$ and $x \neq 1$

Then there are $a_1, \ldots, a_n, b_1, \ldots b_m \in A$. Such that $\prod_{i=1}^{n} a_i * (x * y) = 1$ and $\prod_{j=1}^{m} b_j * x = 1$.

Applying (CI 3) we deduce that $x \leq \prod_{i=1}^{n} a_i * y$. From lemma (1.2.3) we get that $1 = \prod_{j=1}^{m} b_j * x \leq \prod_{j=1}^{m} b_j * \left(\prod_{i=1}^{n} a_i * y\right)$.

Then by proposition 1.1.10 (iii), $\prod_{j=1}^{m} b_j * \left(\prod_{i=1}^{n} a_i * y\right) = 1$.

Hence $y \in F$, and so F is a filter of X.

Suppose that, U is any filter of X containing A.

Let $x \in F$. If $x = 1$, then obviously $x \in U$.

Assume that $x \neq 1$. Then there are $a_1, \ldots, a_n \in A$ such that $\prod_{i=1}^{n} a_i * x = 1$.

Since $A \subseteq U$, it follows that $a_1, \ldots, a_n \in U$.

Therefore $x \in U$ by corollary (1.2.8). Thus $F \subseteq U$ and hence $F = [A)$.

Theorem : 1.2.12

Let $F_1, F_2 \in \text{Fil}(X)$. Define the meet of F_1 and F_2 by $F_1 \wedge F_2 = F_1 \cap F_2$ and the join of F_1 and F_2 by $F_1 \vee F_2 = [F_1 \cup F_2)$. Then (Fil (X) ; $\wedge$ v) is a complete lattice.

Proof : Obvious.

Definition : 1.2.13

Let X be an CI-algebra and $x, y \in X$. Define A(x, y) by $A(x, y) = \{z \in X / z = 1 \text{ (or) } x * (y * z) = 1\}$ which is called an **upper set of x and y** in X. A subset A of X is called an upper set of X if $A = A(x, y)$ for some $x, y \in X$.

Note

(i) $1, x, y \in A(x, y)$ for all $x, y \in X$.

(ii) The set of all upper sets A(x, y) of x and y in X is denoted by US(X).

Example : 1.2.14

Let X = {1, a, b} and $*$ be defined by the following table :

$*$	1	a	b
1	1	a	b
a	a	1	1
b	a	1	1

Then $(X ; *, 1)$ is a CI-algebra. For $x, y \in X$, we have

$$A(x, y) = \begin{cases} X \text{ if } x \neq y \text{ and } (x = 1 \text{ or } y = 1) \\ \{1\} \text{ otherwise} \end{cases}$$

Also $Fil(X) = \{\{1\}, X\}$. Hence $Fil(X) = US(X)$.

Note

Not every filter is an upper set and not every upper set is a filter.

Example : 1.2.15

Let $X = \{1, a, b\}$ and $*$ be defined by the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	c	c	c	1

Then $(X ; *, 1)$ is a CI-algebra.

Then $Fil(X) = \{\{1\}, \{1, a, b\}, X\}$ and $US(X) = \{\{1\}, \{1, a, b\}, \{1, c\}\}$.

Therefore X is not an upper set of X and $\{1, c\}$ is not a filter in X.

Proposition : 1.2.16

Let X be an CI-algebra. If $(X ; *, 1)$ is a self distributive CI-algebra, then $A(x, y)$ is a subalgebra of X.

Proof

Let $A(x, y)$ be an upper set of x and y in a CI-algebra X.

Let $m, n \in A(x, y)$.

Then $x * (y * m) = 1$ and $x * (y * n) = 1$

and $x * (y * (m * n)) = x * ((y * m) * (y * n))$

$$= (x * (y * m)) * (x * (y * n)) = 1 * 1 = 1$$

This implies that $m * n \in A(x * y)$.

Hence $A(x, y)$ is a subalgebra of X.

Definition : 1.2.17

Let X be an CI-algebra and $a \in X$. Define $A(a)$ by $A(a) = \{x \in X / x = 1 \text{ or } a * x = 1\}$ which is called the **initial section of the element a**.

Example : 1.2.18

Let $X = \{1, a, b, c\}$ be an CI-algebra as in example (1.2.2).

Then $A(1) = \{1\}$, $A(a) = \{1, a\}$, $A(b) = \{1, a, b\}$ and $A(c) = \{1, a, c\}$.

Lemma : 1.2.19

Let X be a self-distributive CI-algebra and $x * y = 1$. If $x \in A(a)$, then $y \in A(a)$.

Proof

Let X be a self-distributive CI-algebra and $x * y = 1$.

Since $x \in A(a)$, $a * x = 1$

$$\begin{aligned}
\text{Now } a * y &= a * (1 * y) = a * ((x * y) * y) \\
&= (a * (x * y)) * (a * y) \\
&= ((a * x) * (a * y)) * (a * y) \\
&= (1 * (a * y)) * (a * y) \\
&= (a * y) * (a * y) \\
&= 1
\end{aligned}$$

This implies $y \in A(a)$.

Proposition : 1.2.20

Let X be a self-distributive CI-algebra and $a \in X$. Then $A(a)$ is a filter of X.

Proof

Let X be a self-distributive CI-algebra and $a \in X$ where A(a) is the initial section of a. Clearly, $1 \in A(a)$ because $a * a = 1$ (1)

Let $x \in A(a)$ and $x * y \in A(a)$.

Then $a * x = 1$ and $a * (x * y) = 1$.

And so $a * (x * y) = (a * x) * (a * y) = 1 * (a * y) = a * y = 1$

This implies $y \in A(a)$ (2)

By (1) and (2), A(a) is a filter of X.

Proposition : 1.2.21

Let X be a self-distributive CI-algebra and $x, y, z \in X$. If $z * (x * y) = 1$ and $z * x = 1$ then $z * y = 1$.

Proof

Let X be a self-distributive CI-algebra and $x, y, z \in X$.

Suppose that $z * (x * y) = 1$ and $z * x = 1$ for all $x, y, z \in X$.

Then $x * y \in A(z)$ and $x \in A(z)$.

Since A(z) is a filter, it follows that $y \in A(z)$.

This implies $z * y = 1$.

Proposition : 1.2.22

Let X be an CI-algebra, F a filter and $x \in F$. Then $A(x) \subset F$.

Proof

Let F be a filter of an CI-algebra X. Let $x \in F$.

If $y \in A(x)$, then $x * y = 1$.

Since F is a filter of X and $x \in X$, $y \in F$.

Therefore $A(x) \subset F$.

Lemma : 1.2.23

Let X be an CI-algebra. For every $x, y \in X$

(i) $x \in A(x)$

(ii) $1 \in A(x, y)$ and $1 \in A(x)$,

(iii) if $y * 1 = 1$, then $A(x) \subseteq A(x, y)$

(iv) if $y * 1 \neq 1$, then $A(x) - \{1\} \subseteq X - A(x, y)$,

(v) if A(x) is a filter of X and $y \in A(x)$, then $A(x, y) \subseteq A(x)$.

Proof

Let X be an CI-algebra.

$A(x) = \{x \in X / x = 1 \text{ or } x * x = 1\}$

$A(x, y) = \{z \in X / z = 1 \text{ or } x * (y * z) = 1\}$

To Prove (i) : Let $x \in X$

Since $x * x = 1$, we have $x \in A(x)$.

(ii) is obvious, by the definition of upper sets.

To Prove (iii) : Let $y * 1 = 1$ and let $z \in A(x)$

If $z = 1$, then obviously $z \in A(x, y)$. Suppose that $x * z = 1$.

Hence $y * (x * z) = y * 1 = 1$.

And therefore $z \in A(y, x) = A(x, y)$.

Consequently, $A(x) \subseteq A(x, y)$.

To Prove (iv) : Let $y * 1 \neq 1$ and $z \in A(x) - \{1\}$.

Then $x * z = 1$ and by (CI 3) we get

$$\begin{aligned} x * (y * z) &= y * (x * z) \\ &= y * 1 \neq 1 \end{aligned}$$

Thus $z \notin A(x, y)$ and we conclude that $A(x) - \{1\} \subseteq X - A(x, y)$.

To Prove (v) : Let $A(x)$ be a filter of X and $y \in A(x)$

If $z \in A(x, y)$, then $z = 1$ (or) $x * (y * z) = 1$. In the first case $z = 1 \in A(x)$ and in the second one $x * (y * z) \in A(x)$.

Since $A(x)$ is a filter and $x, y \in A(x)$. We obtain $z \in A(x)$.

Theorem : 1.2.24

Let F be a nonvoid subset of a CI-algebra X. Then F is a filter of X if and only if $A(x, y) \subseteq F$ for all $x, y \in F$.

Proof

Let F be a nonvoid subset of a CI-algebra X. Suppose that F is a filter of a CI-algebra X.

Let $x, y \in F$ and $z \in A(x, y)$. Then $z = 1$ or $x * (y * z) = 1$.

Obviously $z = 1 \in F$.

If $x * (y * z) = 1$, then applying twice (F2) we obtain $z \in F$.

Hence $A(x, y) \subseteq F$.

Now let $A(x, y) \subseteq F$ for all $x, y \in F$.

Since $F \neq \phi$, there exists $z \in F$.

By definition, $1 \in A(z, z) \subseteq F$ and therefore (F1) holds.

Let $x * y \in F$ and $x \in F$.

By (CI 1), $(x * y) * (x * y) = 1$ and hence $y \in A(x * y, x) \subseteq F$.

Thus (F2) also holds and consequently, F is a filter of X.

This completes the proof.

Proposition : 1.2.25

If F is a filter of a CI-algebra X, then $F = \bigcup_{x, y \in F} A(x, y)$

Proof

Let F be a filter. Then from the theorem (1.2.24) it follows that

$A(x, y) \subseteq F$ for all $x, y \in F$.

Hence $\bigcup_{x, y \in F} A(x, y) \subseteq F$.

Now let $z \in F$. Then by lemma (1.2.23 (i)).

$z \in A(z) = A(1, z) \subseteq \bigcup_{x, y \in F} A(x, y)$

Then $F \subseteq \bigcup_{x, y \in F} A(x, y)$

This completes the proof.

Proposition : 1.2.26

If F is a filter of a CI-algebra X, then $F = \bigcup_{x \in F} A(x)$

Proof

Let F be a filter of a CI-algebra X and let $z \in F$.

By lemma (1.2.23 (i)), $z \in A(z) \subseteq \bigcup_{x \in F} A(x)$.

Therefore $F \subseteq \bigcup_{x \in F} A(x)$.

From theorem (1.2.24) we conclude that $A(x) = A(1, x) \subseteq F$ for all $x \in F$.

Hence $\bigcup_{x \in F} A(x) \subseteq F$ and consequently, $F = \bigcup_{x \in F} A(x)$.

This completes the proof.

Definition : 1.2.27

A filter F in a CI-algebra X is said to be **closed** if $x \in F$ implies $x * 1 \in F$.

Example : 1.2.28

Let X be the set of all positive real numbers, $\div$ the usual division.

Define $x * y = y \div x$.

Then $(X ; *, 1)$ is a CI-algebra, $F = \{2^n / n \in z\}$ is a closed filter of X.

$F = \{2^n / n \in N\}$ is a filter of X but F is not closed.

Proposition : 1.2.29

Every filter of a finite CI-algebra is closed.

Proof

Suppose $(X ; *, 1)$ is a finite CI-algebra, 1 x 1 = n. Let F be any filter of X.

Take any $a \in F$, in the following $n + 1$ elements :

$$1, a * 1, \ldots, \underbrace{a * (\ldots * (a}_{n} * 1) \ldots),$$

There are atleast two elements to be equal, for instance,

$$\underbrace{a * (\ldots * (a}_{\ell} * 1) \ldots) = \underbrace{a * (\ldots * (a}_{k} * 1) \ldots)$$

where $0 \leq \ell < k \leq n$, when $\ell = 0$, $a * (\ldots * (a * 1) \ldots) = 1$.

Hence $\underbrace{a * (\ldots * (a}_{k-1} * 1) \ldots) = 1 \in F$, and so $a * 1 \in F$.

Proposition : 1.2.30

A filter of a CI-algebra X is closed if and only if it is subalgebra of X.

Proof

Suppose a filter F of X is closed and $x, y \in F$. Because $x * (y * x) = y * (x * x) = y * 1 \in F$. And $y * x \in F$ (by F_2). This shows that F is a subalgebra of X. Conversely, suppose a filter F of X is a subalgebra of X. For all $x \in F$, it follows from $1 \in F$ that $x * 1 \in F$, so F is closed.