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An Optimum Control of a Batch Arrival Queue with Second Optional Service and Setup time under Bernoulli Vaction Schedule

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Abstract

An $M^X/G/1$ queueing system with second optional service (SOS)is studied under N policy and Bernoulli Vacation. The system remains idle until the queue size reaches or exceeds N (≥ 1). When the queue size reaches at leastN, the server may begin his setup operation with probability s or may start the service with probability (1-s). The server provides two phases of heterogeneous services, of which, first phase of service is essential and second phase of service is optional. As soon as the first essential service (FES) of a unit is completed, the customer may leave the system with probability (1-r) or may immediately opt for SOS with probability r. Wheneverthe service of each unit is completed, the server will have the option of taking vacation(Bernoulli). Thus a customer completes his service, by undergoing FES alone, the server may take a vacation with probability $p_1$orstays idle or continue the next service to the new customer if any,with probability ($1-p_1$). If the customer who finishes FES proceeds to SOS then the server may take vacation at the end of SOS with probability p_2 or stays idle or continues with the FES for the new customer with probability ($1-p_2$) according as the system is empty or having customers in the system. The queue size distribution at a random epoch is obtained for this model using SVT and various particular cases are deduced. Further various performance measures and the optimum management policy are also derived.

Key words : Second optional service, N policy, Setup time and Bernoulli vacation.

Introduction:

This paper concerns with the steady state analysis of single server batch arrival queue with SOS service channel under Bernoulli schedule vacation and N policy. The first study of classical batch arrival queue with N policy was done by Lee and Srinivasan (1989). Later Lee etal (1994 & 1995) analysed an $M^X/G/1$ queue with N policy of multiple and single vacation respectively. In many real world production systems, setup operations are recorded in several occasions.

Recent research contribution consider thequeueing system with two phases of services. Madan (2000) introduced the concept of SOS and high lightened numerous equations of the queueing situations where all arriving customers require the main service and only some may require the subsidiary service provided by the server. Later several authors including Medhi (2003) and Choudhury (2003) analysedqueuing models with SOS

At present, most of the studies are devoted to batch arrival vacation models , under different vacation policies because of the inter disciplinary concept of Bernoulli vacation and modified service time . In Bernoulli vacation models, after each service completion the server may go for a vacation of random length V with probability p ($0 \leq p \leq 1$) or may continue to serve the next unit , if any , with probability 1-p. Otherwise, the server remains idle in the system. In most of the queueing system with Bernoulli schedule vacation, it is assumed that , the server provides , two phases of heterogeneous services one after the other to the arriving customers. In this paper we consider the Bernoulli schedule vacation for the first time along with SOS facility. One can note that the two phases of heterogeneous service of queueing model is a special case of the SOS queueing model.

Thus we have analysed the most general N policy , batch arrival queueing system with SOS channel under Bernoulli schedule vacation with or without setup operations. For this model, the steady state queue size distributions at random epoch are derived using the supplementary variable technique and various important performance measures are obtained. A cost model is proposed to obtain the optimal stationary policy under a suitable linear cost structure.Further particular cases are also derived.

Model Description:

In this paper the optimal control of $M^X/G_1\ G_2/1$ queueing system where the arrivals occur according to a compound Poisson process with arrival size of random variable X is considered under Bernoulli schedule vacation process. The server provides two types of heterogeneous services of which one is essential and the other is optional.

The server is deactivated as soon as the system becomes empty. If the queue length reaches or exceeds N, the server starts the setup operation with probability "s" whose length is a generally distributed random variable D. The customers who arrive during the idle period (or) setup period will join the queue. Immediately after the setup the server is turned on and begins to serve the first phase of essential service (FES) one ata time according to the FCFS queue discipline. After the completion of FES of a customer, the customer may leave the system with probability $(1-r)$ (or) may opt for a SOS in an additional channel with probability r $(0 \le r \le 1)$.

If a customer leaves the system soon after the FES, the server may take a vacation (Bernoulli schedule) of random length v with probability p_1 $(0 \le p_1 \le 1)$ or may continue to serve the next customer if any with probability $(1-p_1)$. On the other hand, if a customer finishes FES and opts for the SOS then the server takes vacation only after finishing the SOS for the customer, with probability p_2 (i.e.) thus after completing each service and sending the customer out of the system, the sever takes a vacation with probability p_j $(j = 1, 2)$ (or continue to stay in the system with probability $(1-p_j)$. The vacation time in either case is a random variable and follows the same general distribution with finite moments. It is assumed that the server takes only a single vacation which means that whenever the vacation period of the serves ends, then he joins the system irrespective of whether there are customers waiting for the service or not .

Thus a cycle begins, when the system length reaches atleast N, and the server starts setup operation with probability s if necessary and continues with FES and SOS along with Bernoulli schedule vacation. This process continues and the cycle ends when the system becomes empty again. Thus in this model, some cycles may start directly with FES as soon as the queue length reaches or exceeds N with probability $(1-s)$. The model under consideration is a general N-policy queueing system with heterogeneous FES and SOS facilities under Bernoulli vacation with (or) without server's setup. The random variables including service times of FES, SOS and Bernoulli vacation time are assumed to follow general law of distribution with finite moments and independent of each other. The model is denoted by $M^X/G_1\ G_2/1/V(BS)$, where $V(BS)$ represents the Bernoulli schedule vacation and $(G_1\ G_2)$ denotes the service provided in two stages of which one is essential and the other is optional.

Notations

The arrivals occur in batches of size X whose probability distribution is given by $Pr(X = k)\ g_k$, $k = 1, 2, 3, \ldots$ and the PGF of g_k is denoted by $X(z) = \sum_{k=1}^{\infty} p_k z^k$, with mean $E(X) = X'(1)$. The services provided in two different channels by the same server are heterogeneous whose cumulative distribution function, (probability density function), {Laplace Stieltijes transform} and [remaining service time] of FEs and SOS are denoted respectively by $S_i(x)$, $(s_i(x))$, $\{S_i^*(\theta)\}$ and $[S_i^0(x)]$ for $i=1, 2$. $V(x)$, $v(x)$, $\{V^*(\theta)\}$ and $[V^0(x)]$; $D(x)$, $d(x)$ $\{D^*(\theta)\}$ and $[D^0(x)]$ are the corresponding notations for vacation time and setup time random variables respectively.

Let N(t) denote the number of customers present in the system at time t, the system states are denoted by $C(t) = 0, 1, 2, 3$ and 4 according as the server is idle, doing preparatory work, doing FES, doing SOS and on vacation respectively. The time interval between the consecutive services of each customer in this model is given by

$$
\begin{aligned}
S &= S_1 + S_2 + V && \text{with probability } p_2\, r \\
&= S_1 + S_2 && \text{with probability } (1-p_2)\, r \\
&= S_1 + V && \text{with probability } p_1\,(1-r) \\
&= S_1 && \text{with probability } (1-p_1)(1-r)
\end{aligned}
$$

Thus the Laplace Stieltijes transforms LST of S is

$$S^*(\theta) = S_1^*(\theta)\ [(1-p_2)\, r\, S_2^*(\theta) + (1-p_1)(1-r) + V^*(\theta)(r\, p_2 S_2^*(\theta) + (1-r)\, p_1)].$$

And the first two moments of the random variable S are given by

$$\left.\frac{-d}{d\theta}(S^*(\theta))\right|_{\theta=0} = E(S); \quad \left.\frac{-d^2}{d\theta^2}(s^*(\theta))\right|_{\theta=0} = E(S^2).$$

It isassumed that the arrival process are independent of vacation times, setup time and service time and these random variables are also independent of each other. Further the system state probabilities are defined by using the remaining vacation time $V^o(t)$, remaining setup time $D^o(t)$, and remaining service time $S_i^o(t)$ as supplementary variables.

Thus $R_n(t) = \Pr\{N(t) = n, c(t) = 0\}$ $0 \le n \le m-1$

$D_n(x, t) = \Pr\{N(t) = n, x \le D_o(t) \le x + dt, c(t) = 1\}$ $n \ge m$

$P_{n1}(t) = \Pr\{N(t) = n, x \le S_1^o(t) \le x + dt, c(t) = 2\}$ $n \ge 1$

$P_{n2}(t) = \Pr\{N(t) = n, x \le S_2^o(t) \le x + dt, c(t) = 3\}$ $n \ge 1$

$Qn(x, t) = \Pr\{N(t) = n, x \le Vo(t) \le x + dt, c(t) = 4\}$ $n \ge 0$

where$R_n(t)$ denotes the probability that there are n customers in the system at a time t when the system is idle. $Q_n(x, t)$, $D_n(x, t)$ and $P_{ni}(x, t)$, $i = 1, 2$ respectively denote the probability that there are n customers in the system and the remaining vacation time, setup time and service time lie in the interval $[x, x + \Delta t]$ and $P_{ni}(0)$ $i = 1, 2$ $(Q_n(0), D_n(0))$ denote the probability that there are n customers in the system at the terminations of service time, setup time and vacation time.

Assuming that the steady state probabilities ,

$$\lim_{t\to\infty} R_n(t) = R_n, \quad \lim_{t\to\infty} Q_n(x, t) = Q_n, \quad \lim_{t\to\infty} D_n(x, t) = D_n(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x}(P_{ni}(x, t)) = \frac{d}{dt} P_n(x)\ i = 1, 2 \text{and} \qquad \lim_{t\to\infty} \frac{\partial}{\partial t}(P_{ni}(x, t)) = 0 \qquad i = 1, 2\text{exist}$$

Now,by following the argument of Cox [1955] and observing the changes of states during the interval $(t, t + \Delta t)$ at any time t, the steady state system equations are written.

$$\lambda R_0 = P_{12}(0)(1-p_2) + Q_0(0) + (1-r)\, p_{11}(1-p_1) \tag{1}$$

$$\lambda R_n = \lambda \sum_{k=1}^{n} R_{n-k}\, g_k \qquad 1 \le n \le N-1 \tag{2}$$

$$\frac{-d}{dx}(P_{11}(x)) = -\lambda P_{11}(x) + (1-p_2) S_1(x) P_{22}(0) + Q_1(0) S_1(x) + P_{21}(0)(1-r) S_1(x)(1-p_1) \tag{3}$$

$$\frac{-d}{dx}(P_{n1}(x)) = -\lambda P_{n1}(x) + (1-p_2) S_1(x) P_{n+12}(0) + Q_n(0) S_1(x)$$
$$+ P_{n+11}(0)(1-r) S_1(x)(1-p_1) + \lambda \sum_{k=1}^{n-1} P_{n-k1}\, g_k \qquad 2 \le n \le N-1 \tag{4}$$

$$\frac{-d}{dx}(P_{n1}(x)) = -\lambda P_{n1}(x) + (1-p_2) S_1(x) P_{n+12}(0) + Q_n(0) S_1(x) + \lambda \sum_{k=1}^{n-1} P_{n-k1}(x)\, g_k$$
$$+ (1-r) P_{n+11}(0)(1-p_1) S_1(x) + \lambda(1-S) \sum_{k=n-N+1}^{n} R_{n-k} \text{ of } \mu\ S_1(x) + D_n(0) S_1(x) \quad n \ge N \tag{5}$$

$$\frac{-d}{dx}(P_{12}(x)) = -\lambda P_{12}(x) + P_{11}(0) S_2(x)\, r \tag{6}$$

$$\frac{-d}{dx}(P_{n2}(x)) = -\lambda P_{n2}(x) + P_{n1}(0)\, r\, S_2(x) + \lambda \sum_{k=1}^{n-1} P_{n-k2}(x)\, g_k \qquad n \ge 2 \tag{7}$$

$$\frac{-d}{dx}(Q_0(x)) = -\lambda Q_0(x) + p_2 P_{12}(0) V(x) + (1-r) P_{11}(0)\, p_1 V(x) \tag{8}$$

$$\frac{-d}{dx}(Q_n(x)) = -\lambda Q_n(x) + p_2 P_{n+12}(0) V(x) + \lambda \sum_{k=1}^{n} Q_{n-k}(x)\, g_k + (1-r) P_{n+11}(0)\, p_1 V(x) \quad n \ge 1 \tag{9}$$

$$\frac{-d}{dx}(D_N(x)) = -\lambda D_N(x) + \lambda \sum_{k=1}^{N} R_{N-k}\, g_k\, dx(S) \qquad (10)$$

$$\frac{-d}{dx}(D_n(x)) = -\lambda D_n(x) + \lambda s \sum_{k=n-N+1}^{n} R_{N-k}(x)\, g_k\, dx + \lambda \sum_{k=1}^{n-N} D_{N-k}(x)\, g_k \qquad n \geq N+1 \qquad (11)$$

The LST of the equations are obtained by following the definition of Laplace Stieltijes transformation and their properties. Further,the following partial probability generating functions are also defined to obtain the analytical solution. (i.e.,)

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{ni}^*(\theta) z^n;\quad P_i(z,0) = \sum_{n=1}^{\infty} P_{ni}(0) z^n (i=1,2);\quad D^*(z,\theta) = \sum_{n=m}^{\infty} D_n^*(\theta) z^n;$$

$$D(z,0) = \sum_{n=m}^{\infty} D_n(0) z^n;\quad Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta) z^n;\quad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n;\ R(z) = \sum_{n=0}^{m-1} R_n z^n$$

By consideringthe suitablepartial PGFs and algebraic techniques, we obtain,

$$R(z) = \lambda R_0 \sum_{n=0}^{N-1} \frac{\pi_n z_n}{\lambda};\ P_2^*(z,0) = (1 - S_2^*(w_x(z)))\frac{P_1(z,0)}{(w_x(z))};\ P_1^*(z,0) = \frac{(1-S_1^*(w_x(z)))}{S_1^*(w_x(z)))}\cdot\frac{P_1(z,0)}{(w_x(z))}\quad Q^*(z,0) =$$

$$\frac{(1-V^*(w_x(z)))}{z}(p_1(1-r)+rS_2^*(w_x(z))p_2)\frac{P_1(z,0)}{(w_x(z))}\ \text{and}\ D^*(z,0) = \frac{s(1-D^*(w_x(z)))}{(w_x(z))}(\lambda R_0 - R(z)\, w_x(z)).$$

Then total PGF is given by,

$$P(z) = P_1^*(z,0) + P_2^*(z,0) + D^*(z,0) + R(z) + Q^*(z,0)$$

$$(i.e)P(z) = \left(\frac{z-1}{z - S^*(w_x(z))}\right) S_1^*(w_x(z))((1-r)+r\, S_2^*(w_x(z))(\lambda R_0 s\left(\frac{1-D^*(w_x(z))}{w_x(z)}\right) + R(z)((1-s)+sD^*(w_x(z))) \quad (12)$$

Performance Measures

Let P_v, P_{build}, P_D and P_{busy} denotethe probability that the system is in vacation, buildup, setup and busy period respectively. Then, their corresponding system size probabilities are given by

$$P_v = \lim_{z\to1} Q^*(z,0) = \lambda R_0\ p.\lambda\ EV\ EX\ \frac{\left(sED + \Sigma \frac{\pi_n}{\lambda}\right)}{(1-\rho)};\quad P_{build} = \lim_{z\to1} R(z) = \sum_{n=0}^{N-1} \frac{\pi_n}{\lambda}.\lambda R_0$$

$$P_D = \lim_{z\to1} D^*(z,0) = s(ED)\lambda R_0 \text{ and } P_{busy} = \lim_{z\to1} P_1^*(z,0) + P_2^*(z,0) = \rho_1$$

Further the value of λR_0 can be evaluated by equating the total probability to 1 and it is found that,

$$(i.e.)\lambda R_0 = \frac{\lambda(1-\rho)}{\left(\lambda sED + \sum_{n=0}^{N-1} \pi_n\right)}$$

Decomposition Property

Now, we describe the decomposition structure. Equation (12) implies that the total PGF of the system size probabilities of the model is decomposed into the product of two random variables one of which is P_{M^X}/G_1

$$G_2/I/BV = \frac{(1-\rho)(z-1)S_1^*(w_x(z))((1-r)+rS_2^*w_x(z))}{(z - S^*(w_x(z)))}$$

This gives the PGF of the SOS $M^X/G_1\ G_2/I/BV$ without N policy by Choudhury [2002]

and the other as

$$\psi(z) = \frac{1}{\left(\lambda sED + \sum_{n=0}^{N-1} \frac{\pi_n}{\lambda}\right)}\left(\frac{s(1-D^*(w_x(z))}{w_x(z)} + (sD^*(w_x(z)) + (1-s))\sum_{n=0}^{N-1} \frac{\pi_n z^n}{\lambda}\right)$$

gives the conditional system size distribution during the servers idle period (vacation + build up + setup).

Further it is noted that

$$\psi(z) = \frac{Q^*(z,0) + D^*(z,0) + R(z)}{Q^*(1,0) + D^*(1,0) + R(1)}$$

Mean System Size

Let L_S denote the expected system size of $M^X/G_1\ G_2/1$ BV queueing system under second optional service with (or) without setup, then L_S is given by $L_S = \frac{d}{dz}(P(z))_{z=1}$

$$\text{(i.e) } L_S = L_{M^X/G_1 G_2/1/BV} + \frac{1}{\left(\lambda s\ ED + \sum_{n=0}^{N-1} \pi_n\right)}\left(\lambda\ s\ EX\left(\frac{\lambda ED^2}{z} + ED\sum_{n=0}^{N-1}\pi_n\right) + \sum_{n=0}^{N-1} n\,\pi_n\right) \quad (13) \text{ where}$$

$L_{M^X/G_1 G_2/1/BV}$ gives the expected system size of $M^X/G_1G_2/1/BV$ queueingsystem without N-policy.

Optional Management Policy

In this section the main objective is to determine the optimal management policy to minimize the linear cost function while maintainingthe minimal service quantity to customers. Let C_y, C_h, C_D, C_v and C_{build}denote the cycle cost, holdingcost, setup cost, vacation cost and build up cost per unit time and $T_C(N)$ denote the average cost per unit time.

$$\text{Then} T_C(N) = \frac{C_y}{E_{Cy}} + C_D P_D + C_{build}P_{build} + C_{busy}P_{busy} + C_h L_S - C_V P_V$$

By substituting the values of P_D, P_{build}, P_{busy}, P_V and E_{Cy} it is found that

$$T_C(N) = \frac{1}{D(N)}\left(\pi + B\sum_{n=0}^{N-1}\pi_n + C_h\sum_{n=0}^{N-1} n\,\pi_n\right) + A' \text{ where}$$

$B = C_{build}(1-\rho) + C_h\lambda\ s\ EX\ ED$; $A' = C_h L_1 + C_{busy}\rho. - C_V P_V$ and $D(N) = s\ \lambda\ ED + \sum_{n=0}^{N-1}\pi_n$

Thus to calculate the optimal value of $T_C(N^*)$. consider, $T_C(k+1) - T_C(k) = \frac{\pi_n}{D(k+1)\,D(k)} h_N(k)$

where $h_N(k) = (B + k\,C_h)\,\lambda\ s\ ED - C_h\Sigma\, n\,\pi_n - \overline{A}$ and $\overline{A} = \lambda\,(1-\rho)\,(C_v + C_D\ s\ ED)$

Thus the sign of $h_N(k)$ determines whether $T_C(N)$ increases (or) decreases, since $\frac{\pi_n}{D(k+1)\,D(k)} > 0$

It is observed that, $T_C(N) > T_C(N+1)$**(i.e.)** $\mathbf{N^* = \min\{k / h_N(k) > 0\}}$

Particular Cases

Case 1 : By taking $s = 1$, the equations (12) and (13) corresponding to the total PGF and the mean system size L_S coincides with $M^X/G_1\ G_2/1/BV$ with N-policy and setup time.

(i.e.) The equation (12) becomes

$$P(z) = \left(\frac{z-1}{z - S^*w_x(z)}\right) S_1^*(w_x(z))((1-r) + r\ S_2^*(w_x(z))\ \frac{(1-\rho)}{\left(ED + \Sigma\frac{\pi_n}{\lambda}\right)}\left(\frac{(1-D^*(w_x(z))}{w_x(z)}\right)\sum_{n=0}^{N-1}\frac{\pi_n z^n}{\lambda} D^* w_x(z)\right)$$

$$\text{And(13) yields.} L_S = L_{M^X}/G_1\ G_2/1 + \frac{\lambda^2 EX\left(\frac{ED^2}{2} + ED\sum_{n=0}^{N-1}\frac{\pi_n}{\lambda}\right) + \sum_{n=0}^{N-1} n\,\pi_n}{\lambda ED + \sum_{n=0}^{N-1}\frac{\pi_n}{\lambda}}$$

Case 2 : By letting $s = 0$,equation (12) becomes, $P(z) = \left(\frac{z-1}{z - S^*w_x(z)}\right)(S_1^*\ w_x(z))((1-r) + r\ S_2^*(w_x(z))\,R(z)$

and the equation (13) results in $L_S = L_1 + \frac{\sum_{n=0}^{N-1} n\,\pi_n}{\sum_{n=0}^{N-1}\frac{\pi_n}{\lambda}}$.(i.e.) The PGF and L_S agrees with that of M^X/G_1 $G_2/1/BV$ with N policy and without setup time.Thus the model described in this paper is the generalization of the BV SOS queueing system with or without setup facility.

Conclusion:

The model discussed in this paper is among the most general queueing system with threshold policies and includes many previous works as special cases.

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ANALYSIS OF A BATCH ARRIVAL $M^X/G/1$ QUEUE WITH SINGLE WORKING VACATION

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ABSTRACT

In this paper, a batch arrival $M^X/G/1$ queue with exponentially distributed working vacations is analyzed. Using supplementary variable technique, the probability generating function of the steady state system size probabilities is derived and the expected system size probabilities are presented in closed form. Further, the results obtained are illustrated numerically and the effect of system parameters on system performance measures is discussed. Some particular cases are also discussed.

2000 Mathematics subject classification: 60K25

Keywords: $M^X/G/1$ queue, single working vacation, supplementary variable technique.

INTRODUCTION:

Queueing system with server vacations is useful to model a system in which the server has additional task during vacation is eminent. Thus, it has wide applicability in analyzing the performance of computer systems, data communication networks and production systems. During the last two decades, the queueing systems with vacations have been studied extensively. The details can be seen in the monographs of Takagi [15], the surveys of Doshi[4,5] and Tegham[16]. In these studies, it is assumed that the server stops primary service completely during the vacations. In 2002, Servi and Finn [14] according to which a customer is served at a lower rate during vacations. They have analyzed an M/M/1 working vacations queue in which the server works at a different rate rather than completely stopping the main service during vacation. They tried to approximate a multi-queue system whose service rate is one of two service speeds such that the fast speed mode cyclically moves from queue to queue with exhaustive service. They tried to apply the M/M/1 working vacation queue, to model a wave length division multiplexing (WDM) optical access network using multiple wave lengths which can be reconfigured. Subsequently, Kim et al [8] have analyzed the M/G/1 queue with exponentially distributed working vacations and obtained the steady-state queue length distribution through the decomposition approach. Later Wu and Takagi[17] extended Servi and Finn's model to an M/G/1 working vacation in which, both service times – regular service and the service in working vacation are assumed to be generally distributed. An imbedded Markov chain that describes the queue size process in an M/G/1 working vacation queue is introduced and the probability generating function for the steady state queue size is derived. Later, based on Servi and Finn's model, Liu et al [11] gave explicit expressions of distributions for the stationary queue length and waiting time which have intuitionistic probability sense for M/M/1 multiple working vacation. Tian et al [13] in their paper studied an M/M/1 queue with single working vacation. Using quasi birth and death process and matrix geometric method, they have given the distributions for the number of customers and the virtual time in system in steady state. GI/M/1 queue with working vacations was studied by Baba [1] using Matrix geometric method. Baink et al [2] analyzed the finite GI/M/1 N queue with working vacations. Later Li and Tian[9] considered two types of discrete time GI/Geo/1 queues with working vacations and vacation interruption. Li et al[10] in their paper considered the M/G/1 queue with exponentially distributed working vacations, which is a special case of that in Wu and Takagi[17]. Later Liu et al[12] extended the M/M/1 working vacation model to bulk input model $M^X/M/1$ working vacations.

Recently, Jemila parveen et al[6] analyzed M/M/1 queue with working vacations and derived the steady state solutions in a closed form by directly solving the difference differential equations. Later they have discussed the waiting time distribution of an arbitrary customer for the model and verified the classical relation between PGF of queueing system and L.S.T of the waiting time distribution. The steady state results of M/M/1 working vacation are also extended to $M^X/M/1$ working vacations queueing model for both multiple and single vacations by Julia Rose Mary and Afthab begum [7].

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$$-\frac{d}{dt}Q_1(x) = -(\lambda+\eta)Q_1(x) + Q_2(0)s_v(x) + \lambda g_1 Q_0 s_v(x)$$

$$-\frac{d}{dt}Q_n(x) = -(\lambda+\eta)Q_n(x) + Q_{n+1}(0)s_v(x) + \lambda g_n Q_0 s_v(x) + \lambda\sum_{k=1}^{n-1} Q_{n-k}(x)g_k, n \geq 2$$

$$-\frac{d}{dt}P_1(x) = -\lambda P_1(x) + P_2(0)s_b(x) + \lambda g_1 P_0 s_b(x) + \int_0^\infty Q_1(y)dy\,\eta s_b(x)$$

$$-\frac{d}{dt}P_n(x) = -\lambda P_n(x) + P_{n+1}(0)s_b(x) + \lambda g_n P_0 s_b(x) + \int_0^\infty Q_n(y)dy\,\eta s_b(x) + \lambda\sum_{k=1}^{n-1} P_{n-k}(x)g_k, n \geq 2$$

For further simplification, we define the following L.S.T

$$Q_n^* = \int_0^\infty e^{-\theta x}Q_n(x)dx \text{ and } P_n^* = \int_0^\infty e^{-\theta x}P_n(x)dx.$$

Taking the L.S.T on both sides of the above equations, we have

$$\theta Q_1^*(\theta) - Q_1(0) = (\lambda+\eta)Q_1^*(\theta) - Q_2(0)S_v^*(\theta) - \lambda Q_0 g_1 S_v^*(\theta) \quad (3)$$

$$\theta Q_n^*(\theta) - Q_n(0) = (\lambda+\eta)Q_n^*(\theta) - Q_{n+1}(0)S_v^*(\theta) - \lambda Q_0 g_n S_v^*(\theta) - \lambda\sum_{k=1}^{n-1} Q_{n-k}^*(\theta)g_k, n \geq 2 \quad (4)$$

$$\theta P_1^*(\theta) - P_1(0) = \lambda P_1^*(\theta) - P_2(0)S_b^*(\theta) - \lambda P_0 g_1 S_b^*(\theta) - \int_0^\infty Q_1(y)dy\,\eta S_b^*(\theta) \quad (5)$$

$$\theta P_n^*(\theta) - P_n(0) = \lambda P_n^*(\theta) - P_{n+1}(0)S_b^*(\theta) - \lambda P_0 g_n S_b^*(\theta) - \int_0^\infty Q_n(y)dy\,\eta S_b^*(\theta) - \lambda\sum_{k=1}^{n-1} P_{n-k}^*(\theta)g_k, n \geq 2 \quad (6)$$

STEADY STATE SOLUTIONS:

In order to derive the distribution of the system size probabilities, we define the following pgfs.

$$Q_1^*(z,\theta) = \sum_{n=1}^\infty Q_n^*(\theta)z^n,\ Q_1(z,0) = \sum_{n=1}^\infty Q_n(0)z^n,\ P_V^*(z,\theta) = Q_1^*(z,\theta) + Q_0,\ P_B^*(z,\theta) = \sum_{n=1}^\infty P_n^*(\theta)z^n \text{ and}$$

$$P_B(z,0) = \sum_{n=1}^\infty P_n(0)z^n$$

Multiplying equations (3) and (4) by the proper powers of z and summing up over n= 1 to ∞, we get

$$\theta Q_1^*(z,\theta) - Q_1(z,0) = (\lambda+\eta)Q_1^*(z,\theta) - \frac{S_v^*(\theta)}{z}(Q_1(z,0) - Q_1(0)z) - \lambda X(z)Q_0 S_v^*(\theta) - \lambda\sum_{n=2}^\infty z^n\left(\sum_{k=1}^{n-1} Q_{n-k}^*(\theta)g_k\right)$$

Using the identity $\sum_{n=2}^\infty z^n\left(\sum_{k=1}^{n-1} Q_{n-k}^*(\theta)g_k\right) = X(z)Q_1^*(z,\theta)$, the above equation becomes

$$(\theta - h_X(z))Q_1^*(z,\theta) = Q_1(z,0)(1 - \frac{S_v^*(\theta)}{z}) - S_v^*(\theta)(\lambda X(z)Q_0 - Q_1(0))$$

At $\theta = h_X(z) = \eta + \lambda(1 - X(z))$, $Q_1(z,0) = \dfrac{zS_v^*(h_X(z))}{z - S_v^*(h_X(z))}(\lambda X(z)Q_0 - Q_1(0))$.

By similar argument of Li et al[10], the unique root z_1 of $z - S_v^*(h_X(z))$ lies inside (0,1).

Therefore, $Q_1(0) = \lambda X(z_1)Q_0$ (7)

Substituting for $Q_1(0)$ in $Q_1(z,0)$, $Q_1(z,0) = \dfrac{\lambda z Q_0 S_v^*(h_X(z))(X(z) - X(z_1))}{z - S_v^*(h_X(z))}$ (8)

And $Q_1^*(z,\theta) = \dfrac{\lambda z Q_0(X(z) - X(z_1))(S_v^*(h_X(z)) - S_v^*(\theta))}{(\theta - h_X(z))(z - S_v^*(h_X(z)))}$

At θ =0, $Q_1^*(z,0) = \dfrac{\lambda z Q_0(X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))}$ (9)

Similarly multiplying equations (5) and (6) by appropriate powers of z and then adding, we have

$$\theta P_B^*(z,\theta) - P_B(z,0) = \lambda P_B^*(z,\theta) - \frac{S_b(\theta)}{z}(P_B(z,0) - P_1(0)z) - \lambda X(z)P_0 S_b^*(\theta)\sum_{n=1}^{\infty}\int_0^{\infty} Q_n(y)dyz^n = \sum_{n=1}^{\infty} Q_n^*(0)z^n = Q_1^*(z,0),$$

we have

$$(\theta - w_X(z))P_B^*(z,\theta) = P_B(z,0)(1 - \frac{S_b^*(\theta)}{z}) - S_b^*(\theta)(\eta Q_1^*(z,0) + \lambda X(z)P_0 - P_1(0))$$

At $\theta = w_X(z) = \lambda(1 - X(z))$,

$$P_B(z,0) = \frac{z S_b(w_X(z))}{z - S_b^*(w_X(z))}(\eta Q_1(z,0) + \lambda X(z)P_0 - P_1(0)) \tag{10}$$

$$\text{And } P_B^*(z,\theta) = \frac{z(S_b^*(w_X(z)) - S_b^*(\theta))}{(\theta - w_X(z))(z - S_b(w_X(z)))}(\eta Q_1^*(z,0) + \lambda X(z)P_0 - P_1(0)) \tag{11}$$

Equation (2) implies $(\lambda + \eta)Q_0 = P_1(0) + Q_1(0) = P_1(0) + \lambda X(z)Q_0$ (from (7))

Therefore $P_1(0) = Q_0(\eta + \lambda(1 - X(z_1)))$ (12)

Substituting $P_1(0)$ and $Q_1^*(z,0)$ in equation (11) and on further simplification

$$P_B^*(z,\theta) = \frac{Q_0 z(S_b^*(w_X(z)) - S_b(\theta))}{(\theta - w_X(z))(z - S_b(w_X(z)))}\left[\frac{z\lambda\eta(X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - \eta(1 - X(z)) - \lambda(1 - X(z_1))\right]$$

And at $\theta = 0$,

$$P_B^*(z,\theta) = \frac{Q_0 z(1 - S_b^*(w_X(z)))}{w_X(z)(z - S_b^*(w_X(z)))}\left[\frac{z\lambda\eta(X(z) - X(z_1))(1 - S_v^*(h_X(z)))}{h_X(z)(z - S_v^*(h_X(z)))} - \eta(1 - X(z)) - \lambda(1 - X(z_1))\right] \tag{13}$$

Thus the total PGF P(z) of the system size probabilities is given by

$$P(z) = P_B^*(z,0) + P_1^*(z,0) + P_0$$

Using the normalizing condition P(1)=1, $Q_0 = \dfrac{1 - \rho_b}{\dfrac{h_X(z_1)}{\eta} + \dfrac{\eta}{\lambda} - \dfrac{\rho_b(1 - X(z_1))S_v^*(\eta)}{E(X)(1 - S_v^*(\eta))}}$

MEAN SYSTEM LENGTH:

Let L_v and L_b denotes the mean system size during the working vacation and regular busy respectively. Then

$$L_v = \frac{d}{dz} P_v(z,0)_{z=1}$$

$$= \lambda Q_0 \left[\frac{E(X) h_X(z_1)}{\eta^2} - \frac{(1 - X(z_1)) S_v^*(\eta)}{\eta(1 - S_v^*(\eta))}\right]$$

$$L_b = \frac{d}{dz}(P_b^*(z,0) + P_0)_{z=1}$$

$$= \left[\frac{\rho_b}{1-\rho_b} + \frac{\lambda E(X(X-1)) E(S_b) + (\lambda E(X))^2 E(S_b^2)}{2(1-\rho_b)^2}\right]\left[\frac{h_X(z_1)}{\eta} + \frac{(X(z_1)-1) S_v(\eta)}{1-S_v^*(\eta)} + \frac{\eta}{\lambda}\right]$$

$$+ \frac{\rho_b}{1-\rho_b}\left[\frac{\lambda E(X(X-1))}{2(\lambda E(X))^2}\left(\frac{\lambda(1 - X(z_1)) S_v^{*\prime}(\eta)}{1 - S_v^*(\eta)}\right) - \frac{1}{E(X)}\left(\frac{(1 - X(z_1))(1 + \lambda E(X) S_v^{*\prime}(\eta))}{(1 - S_v(\eta))^2}\right.\right.$$

$$\left.\left. - \frac{E(X) h_X(z_1)}{\eta(1 - S_v^*(\eta))} - \frac{1 - X(z_1)}{1 - S_v^*(\eta)} + \frac{h_X(z_1)(\lambda E(X))^2}{\eta^2}\right)\right]$$

Hence the mean system size of the model L is given by $L=L_v+L_b$.

Other Performance measures:

- Probability that the server is on vacation (P_v) is given by $P_v = \lim_{z\to 1} Q_1^*(z,0) = \dfrac{Q_0 h_X(z_1)}{\eta}$
- Probability that the server is busy (P_b) is given by

$$P_b = \lim_{z\to 1} P_b^*(z,0) = \frac{Q_0}{\mu_b(1-\rho_b)}\left[\lambda E(X)\left(\frac{\eta}{\lambda} + \frac{h_X(z_1)}{\eta}\right) + \frac{\lambda(X(z_1)-1) S_v^*(\eta)}{1 - S_v^*(\eta)}\right]$$

- Probability that the server is idle (P_I) is given by $P_I = \lim_{z\to 1} P_0 = \dfrac{\eta Q_0}{\lambda}$

PARTICULAR CASES:

In this section, the steady state results of $M^X/M/1$ [7] and M/G/1 are deduced as particular cases of the model.

1. **$M^X/M/1$ single working vacation** [7]:

If both the services – regular service and service during working vacation follow exponential distribution, then

$$S_b(w_X(z)) = \frac{\mu_b}{\mu_b + w_X(z)}, \; S_v(h_X(z)) = \frac{\mu_v}{\mu_v + h_X(z)} \quad \text{and} \quad P_{00}^* = Q_0$$

$$Q_1^*(z,0) = \frac{\mu_v(z - z_1) Q_0}{z_1[\mu_v(z-1) + z h_X(z)]} = P_0(z) \text{ of } M^X/M/1 \text{ single working vacation}$$

$$P_b^*(z,0) = \frac{Q_0}{[\mu_b(z-1) + z w_X(z)]}\left[\frac{\mu_v \lambda z((z-1) z_1 (X(z_1)-1) - (z_1 - 1) z (X(z) - 1))}{z_1[\mu_v(z-1) + z h_X(z)]} - \eta z(1 - X(z))\right]$$

$= P_1(z)$ of $M^X/M/1$ single working vacation

2. **M/G/1 Single working vacation:**

When both regular service time and service time during working vacation follow other than exponential distribution and by taking X(z)=z i.e, single arrival, the probability generating functions of the M/G/1 SWV model is deduced as follows.

$$Q_1(z,0) = \frac{\lambda z Q_0 (z - z_1)(1 - S_v(h(z)))}{h(z)(z - S_v(h(z)))}$$

$$P_b^*(z,0) = \frac{zQ_0(1-S_b^*(w(z)))}{w(z)(z-S_b^*(w(z)))}\left[\frac{\eta\lambda(z-z_1)(1-S_v^*(h(z)))}{h(z)(z-S_v^*(h(z)))} - \eta(1-z) - \lambda(1-z_1)\right]$$

$$L_v = \lim_{z\to 1} P_v(z,0) = \frac{h(z_1)}{\eta} \text{ and } L_b = \lim_{z\to 1} P_b^*(z,0) = \frac{\rho_b Q_0}{1-\rho_b}\frac{\eta}{\lambda} + \frac{h(z_1)}{\eta} + \frac{(z_1-1)S_v^*(\eta)}{1-S_v^*(\eta)}$$

NUMERICAL ANALYSIS:

In this section, we present some numerical examples to explain the influence of various parameters such as mean vacation time ($1/\eta$), mean regular service ($1/\mu_b$) and mean service during working vacation ($1/\mu_v$) on mean system size (L) and on various probabilities. For the computation, the batch arrival is assumed to follow geometric distribution and service times follow Erlang k distributions.

In the classical vacation models, since the service is stopped completely during vacation, the system size increase notably as the mean vacation time increases. But in working vacation, since the service is done with a smaller rate $\mu_v (< \mu_b)$ during vacation, the vacation parameter η have less effect on the system size. The effects of η and μ_v on the expected mean system size under two situations ($\rho_b = 0.3$ and $\rho_b = 0.6$) are presented in table 1. The table 1 values also show that as μ_v or η increases, the mean system size decrease. Also we infer that as μ_v approaches to 0, the system size of single working vacation model (SWL) approaches the system size of the corresponding classical single vacation model (CSL). The data's in table 2 shows the effect of traffic intensity (ρ_b) on the probabilities including probability that the server is idle (P_I), on vacation (P_v) and on regular busy (P_b). In all the above discussion we fix $\mu_b = 1$. Figure 1 is the graphical representation of table 2.

Table: 1 L Vs μ_v Vs η

μ_v	η	$\rho_b = 0.3$		$\rho_b = 0.6$	
		L_{SWV}	L_{SV}	L_{SWV}	L_{SV}
0.25	0.5	2.4592	4.0917	14.7156	7.1871
	1	4.3912	1.5378	18.3425	3.5917
	1.5	6.3348	1.1312	23.3371	3.1052
	2	11.4063	0.9949	29.2266	2.9532
0.5	0.5	1.9542	4.0917	11.0570	7.1871
	1	4.1285	1.5378	16.6471	3.5917
	1.5	6.1732	1.1312	22.3264	3.1052
	2	8.1750	0.9949	28.0163	2.9532
0.75	0.5	1.6429	4.0917	8.6696	7.1871
	1	3.9115	1.5378	15.2405	3.5917
	1.5	6.0174	1.1312	21.3713	3.1052
	2	8.0587	0.9949	27.3205	2.9532

Table: 2 Probabilities Vs Traffic Intensity

ρ	P_b	P_v	P_I
0.1	0.0864	0.5353	0.3963
0.2	0.1283	0.6863	0.1854
0.3	0.2043	0.7021	0.0936
0.4	0.2970	0.6530	0.0500
0.5	0.4019	0.5703	0.0278
0.6	0.5146	0.4698	0.0157
0.7	0.6322	0.3592	0.0087
0.8	0.7529	0.2427	0.0044
0.9	0.8757	0.1225	0.0017

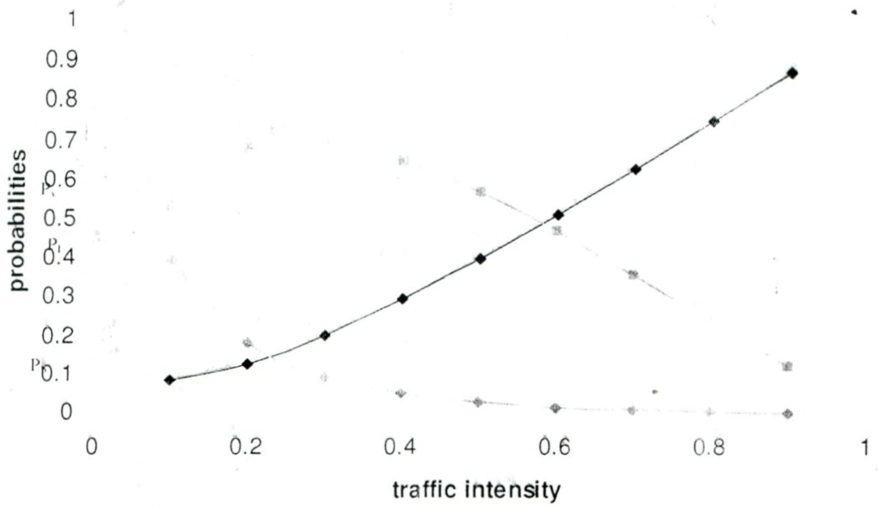

Figure: 1 Various probabilities with traffic intensity

CONCLUSION:

Over the past two decades, queueing systems with vacations have been studied by many researchers and have been applied to many situations. Working vacation is a new concept introduced by Servi and Finn [14] and the model discussed till now are M/M/1, M/G/1 multiple working vacations, GI/M/1 using matrix geometric method. In this paper we have made an attempt to discuss a batch arrival $M^X/G/1$ queue under single working vacation using supplementary variable technique and derived the steady state results in closed form. Various system performance measures are deduced from it. Further few results existing in literature are obtained as particular cases. Finally numerical examples are presented to justify the measure and to understand the model in a better way. The model analyzed in this paper may be extended to the model including the system with second optional service, with/without breakdowns, bulk service models, etc.

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