CHAPTER - II

2. An M/G/1 Retrial G-Queue with Server Breakdown

An M/G/1 retrial queue with positive and negative customers is considered. If the server is idle upon the arrival of a customer, then the customer receives service immediately. Otherwise he joins the orbit. The arrival of negative customer brings the server down and removes the customer in service from the system. The server is subject to random breakdown while it is working. Using supplementary variable technique various performance measures are derived. Stochastic decomposition property is established. Some special cases are discussed and numerical results are presented.

2.1 Model Description

Consider a single server retrial queueing system with two types of arrivals-positive and negative. Positive customers arrive according to Poisson process with rates λ^{+}. Negative customers arrive with Poisson arrival rate λ^{-}. There is no waiting space in front of the server and therefore if an arriving positive customer finds the server idle, then he receives the service. If the server is busy, then the arriving customer enters the orbit. The retrial time of the customers is generally distributed with distribution function A(x), density function a(x), Laplace-Stieltjes transform $A^{*}(s)$ and hazard rate function $\eta(x)=\frac{a(x)}{1-A(x)}$. The service time of positive customers is generally distributed with distribution function B(x), density function b(x), Laplace-Stieltjes transform $B^{*}(s)$ and hazard rate function $\mu(x)=\frac{b(x)}{1-B(x)}$. The arrival of a negative customer removes the positive customer in service from the system and causes the server breakdown. The server is subject to unpredictable breakdown while it is working. The life time of the server is exponentially distributed with rate α.The repair time of the failed server is generally distributed with distribution function R(x), density function r(x) and hazard rate function $\beta(x)=\frac{r(x)}{1-R(x)}$. All stochastic processes involved in the system are assumed to be independent of each other.

Throughout the rest of the paper, we also denote $\overline{F}(x) = 1 - F(x)$ the tail of distribution function F(x). We also denote $F^{*}(s) = \int_{0}^{\infty} e^{-sx} dF(x)$, $\tilde{\overline{F}}(s) = \int_{0}^{\infty} e^{-sx}\overline{F}(x)dx = \frac{1-F^{*}(s)}{s}$

2.2 Stability Condition

Let $N(t_n^{+})$ be the number of customers in the orbit just after the time t_n. Then the sequence of random variables $Y_n = N(t_n^{+})$ form a Markov chain, which is the embedded Markov chain for this queueing system.

Theorem 2.1

The embedded Markov chain $\{Y_n, n \in N\}$ is ergodic if and only if

$$(1 - B^{*}(\lambda^{-} + \alpha))(\lambda^{+}(1 + \beta_1(\lambda^{-} + \alpha)) + \alpha + (\lambda^{-} + \alpha)(1 - A^{*}(\lambda^{+})))$$
$$< (\lambda^{-} + \alpha)(1 - (1 - A^{*}(\lambda^{+}))B^{*}(\lambda^{-} + \alpha))$$

The theorem can be proved along similar lines as in Gomez-Corral (1999).

2.3 Steady State Distribution

In this section, by treating elapsed service time and elapsed repair time of the server as supplementary variables, the steady state probability generating functions of the orbit size distribution are derived.

Define the states of the server as

$$C(t) = \begin{cases} 0, & \text{if the server is idle at time t,} \\ 1, & \text{if the server is busy at time t,} \\ 2, & \text{if the server is under repair at time t.} \end{cases}$$

For $t \geq 0$, we define the random variable $\xi(t)$ as follows:

(i) If $C(t) = 0$, then $\xi(t)$ represents the elapsed retrial time;

(ii) If $C(t) = 1$, then $\xi(t)$ represents the elapsed service time;

(iii) If $C(t) = 2$, then $\xi(t)$ represents the elapsed repair time.

Then, the process $\{X(t); t \geq 0\} = \{C(t), N(t), \xi(t), t \geq 0\}$ is a Markov process.

For the process $\{X(t); t \geq 0\}$, we define the following probabilities.

$I_0(t) = P\{C(t) = 0, N(t) = n\}$

$I_n(x, t)\,dx = P\{C(t) = 0, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 1$

$P_n(x, t)\,dx = P\{C(t) = 1, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 0$

$R_n(x, t)\,dx = P\{C(t) = 2, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 0$

The system of equation that governs the model under supplementary variable technique are given below

$$\frac{d}{dt} I_0(t) = -\lambda^+ I_0(t) + \int_0^\infty P_0(x,t)\mu(x)dx + \int_0^\infty R_0(x,t)\beta(x)dx, \tag{2.1}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) I_n(x,t) = -(\lambda^+ + \eta(x))\, I_n(x,t), \; n \geq 1 \tag{2.2}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) P_n(x,t) = -(\lambda^+ + \lambda^- + \alpha + \mu(x))\, P_n(x,t) + \lambda^+ P_{n-1}(x,t), \; n \geq 0 \tag{2.3}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) R_n(x,t) = -(\lambda^+ + \beta(x)) R_n(x,t) + \lambda^+ R_{n-1}(x,t), \; n \geq 0 \tag{2.4}$$

with boundary conditions

$$I_n(0,t) = \int_0^\infty P_n(x,t)\,\mu(x)dx + \int_0^\infty R_n(x,t)\,\beta(x)dx, \; n \geq 1 \tag{2.5}$$

$$P_n(0,t) = \lambda^+ \int_0^\infty I_n(x,t)dx + \int_0^\infty I_{n+1}(x,t)\,\eta(x)dx, \; n \geq 0 \tag{2.6}$$

$$R_0(0,t) = \lambda^- \int_0^\infty P_0(x,t)dx \tag{2.7}$$

$$R_n(0,t) = \lambda^- \int_0^\infty P_n(x,t)dx + \alpha \int_0^\infty P_{n-1}(x,t)dx, \; n \geq 1 \tag{2.8}$$

Define the steady state probabilities

$$I_0 = \lim_{t\to\infty} I_0(t);$$

$$I_n(x) = \lim_{t\to\infty} I_n(x,t),\ x \ge 0,\ n \ge 1\ ;$$

$$P_n(x) = \lim_{t\to\infty} P_n(x,t),\ x \ge 0,\ n \ge 0 \text{ and}$$

$$R_n(x) = \lim_{t\to\infty} R_n(x,t), x \ge 0, n \ge 0\ .$$

Taking limit as $t \to \infty$ on both sides of the equations, we get the following steady state equations.

$$\lambda^+ I_0 = \int_0^\infty P_0(x)\mu(x)dx + \int_0^\infty R_0(x)\beta(x)dx, \tag{2.9}$$

$$\frac{d}{dx} I_n(x) = -(\lambda^+ + \eta(x)) I_n(x),\ \ n \ge 1 \tag{2.10}$$

$$\frac{d}{dx} P_n(x) = -(\lambda^+ + \lambda^- + \alpha + \mu(x)) P_n(x) + \lambda^+ P_{n-1}(x),\ \ n \ge 0 \tag{2.11}$$

$$\frac{d}{dx} R_n(x) = -(\lambda^+ + \beta(x)) R_n(x) + \lambda^+ R_{n-1}(x),\ \ n \ge 0 \tag{2.12}$$

with boundary conditions

$$I_n(0) = \int_0^\infty P_n(x)\mu(x)dx + \int_0^\infty R_n(x)\beta(x)dx,\ \ n \ge 1 \tag{2.13}$$

$$P_n(0) = \lambda^+ \int_0^\infty I_n(x)dx + \int_0^\infty I_{n+1}(x)\,\eta(x)dx,\ \ n \ge 0 \tag{2.14}$$

$$R_0(0) = \lambda^- \int_0^\infty P_0(x)dx \tag{2.15}$$

$$R_n(0) = \lambda^- \int_0^\infty P_n(x)dx + \alpha \int_0^\infty P_{n-1}(x)dx,\ \ n \ge 1 \tag{2.16}$$

The normalising condition is

$$I_0 + \sum_{n=1}^{\infty}\int_0^{\infty} I_n(x)dx + \sum_{n=0}^{\infty}\int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty}\int_0^{\infty} R_n(x)dx = 1 \quad (2.17)$$

Define the probability generating functions for $|z| \le 1$:

$$I(x,z) = \sum_{n=1}^{\infty} I_n(x) z^n; \; P(x,z) = \sum_{n=0}^{\infty} P_n(x) z^n; \; R(x,z) = \sum_{n=0}^{\infty} R_n(x) z^n$$

Multiplying equation (2.10) by z^n and summing over n, we get

$$\left(\frac{d}{dx} + \lambda^+ + \eta(x)\right) I(x,z) = 0 \quad (2.18)$$

Solving the partial differential equation (2.18), we get

$$I(x,z) = Ce^{-\lambda^+ x} e^{-\int \eta(x)\,dx}$$

$$= Ce^{-\lambda^+ x} e^{\log(1-A(x))}$$

$$= Ce^{-\lambda^+ x}(1 - A(x))$$

Eliminating C by taking x = 0, we get

$$I(x,z) = I(0,z)e^{-\lambda^+ x}(1 - A(x)) \quad (2.19)$$

The partial differential equations in (2.11) and (2.12) yield

$$P(x,z) = P(0,z)e^{-(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)x}(1 - B(x)) \quad (2.20)$$

$$R(x,z) = R(0,z)e^{-(\lambda^+ - \lambda^+ z)x}(1 - R(x)) \quad (2.21)$$

Multiplying equations (2.13) to (2.16) by z^n and summing over n, we get

$$I(0,z) = \int_0^{\infty} P(x,z)\,\mu(x)dx + \int_0^{\infty} R(x,z)\,\beta(x)dx - \lambda^+ I_0 \quad (2.22)$$

$$P(0,z) = \lambda^+ I_0 + \frac{1}{z}\int_0^{\infty} I(x,z)\,\eta(x)dx + \lambda^+ \int_0^{\infty} I(x,z)dx \quad (2.23)$$

$$R(0,z) = (\lambda^- + \alpha z)\int_0^\infty P(x,z)\,dx \tag{2.24}$$

Using equations (2.20) and (2.21) in equation (2.22), we get

$$I(0,z) = P(0,z)[B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) + (\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ z)] - \lambda^+ I_0 \tag{2.25}$$

where $K(z) = \dfrac{1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)}{(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)}$

Substituting the expression in equation (2.19) in equation (2.23) and simplifying we obtain

$$P(0,z) = \lambda^+ I_0 + \frac{I(0,z)}{z}(z + (1-z)A^*(\lambda^+)) \tag{2.26}$$

Solving equations (2.25) and (2.26), we get

$$I(0,z) = \lambda^+ I_0[zB^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) - z + z(\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ z)]/T(z) \tag{2.27}$$

and

$$P(0,z) = \lambda^+ I_0 A^*(\lambda^+)(z-1)/T(z) \tag{2.28}$$

where

$$T(z) = z - (z + (1-z)A^*(\lambda^+))[B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) + (\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ z)]$$

Substituting the result in equation (2.20) in equation (2.24), we get

$$R(0,z) = (\lambda^- + \alpha z)P(0,z)K(z) \tag{2.29}$$

Using equation (2.28), the equation (2.29) yields

$$R(0,z) = \lambda^+(\lambda^- + \alpha z)I_0(z-1)A^*(\lambda^+)K(z)/T(z) \tag{2.30}$$

Substituting the expressions of I(0,z), P(0,z) and R(0,z), in equations (2.19), (2.20), and (2.21) respectively, we get

$$I(x,z) = \lambda^+ I_0 z[B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) - 1 + (\lambda^- + \alpha z)K(z)R^*(\lambda^+ - \lambda^+ z)]e^{-\lambda^+ x}(1 - A(x))/T(z) \tag{2.31}$$

$$P(x,z) = \lambda^+ I_0 A^*(\lambda^+)(z-1)e^{-(\lambda^+(1-z)+\lambda^- +\alpha)x}(1-B(x))/T(z) \tag{2.32}$$

$$R(x,z) = \lambda^+(\lambda^- + \alpha z) I_0 A^*(\lambda^+)(z-1)e^{-(\lambda^+(1-z))x} K(z)(1-R(x))/T(z) \tag{2.33}$$

The partial probability generating function of the orbit size when the server is idle is

$$I(z) = \int_0^\infty I(x,z)dx$$

$$= I_0(1-A^*(\lambda^+))[(zB^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) - z)(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) + z(\lambda^- + \alpha z)$$

$$(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))\, R^*(\lambda^+ - \lambda^+ z)]/T_1(z) \tag{2.34}$$

The partial probability generating function of the orbit size when the server is busy is

$$P(z) = \int_0^\infty P(x,z)dx$$

$$= \lambda^+ I_0 A^*(\lambda^+)(z-1)(1-B^*(\lambda^+ + \lambda^- + \alpha -- \lambda^+ z))/T_1(z) \tag{2.35}$$

The partial probability generating function of the orbit size when the server is under repair is

$$R(z) = \int_0^\infty R(x,z)dx$$

$$= (\lambda^- + \alpha z) I_0 A^*(\lambda^+)(1-B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))(R^*(\lambda^+ - \lambda^+ z) - 1)/T_1(z) \tag{2.36}$$

The partial probability generating function of the orbit size is

$$P_q(z) = I_0 + I(z) + P(z) + R(z)$$

$$= I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-z) + (z-1)(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))]/T_1(z) \tag{2.37}$$

The partial probability generating function of the system size is

$$P_s(z) = I_0 + I(z) + zP(z) + R(z)$$

$$= I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-z)B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) + (z-1)(\lambda^- + \alpha B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))]/T_1(z)$$

$$\tag{2.38}$$

where

$$T_1(z) = (z - (z + (1 - z)A^*(\lambda^+))B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) - (\lambda^- + \alpha z)$$
$$(z + (1 - z)A^*(\lambda^+))(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))R^*(\lambda^+ - \lambda^+ z)$$

2.4 Performance Measures

- The probability that the server is idle during retrial time is given by

$$I = \lim_{z \to 1} I(z)$$

$$= I_0(1 - A^*(\lambda^+))[(1 - B^*(\lambda^- + \alpha))(\lambda^+(1 + \beta_1(\lambda^- + \alpha)) + \alpha + (\lambda^- + \alpha)) + (\lambda^- + \alpha)$$

$$(B^*(\lambda^- + \alpha) - 1)] / T_1'(1) \quad (2.39)$$

where

$$T_1'(1) = (B^*(\lambda^- + \alpha) - 1)[\lambda^+(1 + \beta_1(\lambda^- + \alpha)) + \alpha + (\lambda^- + \alpha)(1 - A^*(\lambda^+))] + (\lambda^- + \alpha)$$
$$(1 - (1 - A^*(\lambda^+))B^*(\lambda^- + \alpha))$$

- The probability that the server is busy is given by

$$P = \lim_{z \to 1} P(z)$$

$$= \lambda^+ I_0 A^*(\lambda^+)(1 - B^*(\lambda^- + \alpha)) / T_1'(1) \quad (2.40)$$

- The probability that the server is down is given by

$$R = \lim_{z \to 1} R(z)$$

$$= \lambda^+(\lambda^- + \alpha)\beta_1 I_0 A^*(\lambda^+)(1 - B^*(\lambda^- + \alpha)) / T_1'(1) \quad (2.41)$$

The normalising equation (2.17) is equivalent to

$$I_0 + I + P + R = 1 \quad (2.42)$$

Using equations (2.39) to (2.41), equation (2.42) yields

$$I_0 = (B^*(\lambda^- + \alpha) - 1)(\lambda^+(1 + \beta_1(\lambda^- + \alpha)) + \alpha + (\lambda^- + \alpha)(1 - A^*(\lambda^+))) + (\lambda^- + \alpha)$$

$$(1 - (1 - A^*(\lambda^+))B^*(\lambda^- + \alpha)) / A^*(\lambda^+)((\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)) \quad (2.43)$$

- The mean number of customers in the orbit is given by

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z) \quad (2.44)$$

$$= \frac{Dr'(1)Nr''(1) - Nr'(1)Dr''(1)}{2Dr'(1)^2}$$

where Nr(z) and Dr(z) are the Numerator and Denominator of $P_q(z)$

$$Nr'(1) = I_0 A^*(\lambda^+)((\lambda^- + \alpha) - \alpha(1 - B^*(\lambda^- + \alpha)))$$

$$Nr''(1) = I_0 A^*(\lambda^+)[2\lambda^+(\alpha \int_0^\infty e^{-(\lambda^- + \alpha)x} x b_1(x)dx - 1)]$$

$$Dr'(1) = T_1'(1)$$

$$Dr''(1) = (B^*(\lambda^- + \alpha) - 1)[(\lambda^- + \alpha)\lambda^+\beta_2 + 2\lambda^+\beta_1\alpha + 2(1 - A^*(\lambda^+))$$

$$((\lambda^- + \alpha)\lambda^+\beta_1 + \alpha)] + 2h_1[\lambda^+\beta_1(\lambda^- + \alpha) + \alpha + \lambda^+] - 2\lambda^+$$

$$(1 - (1 - A^*(\lambda^+))B^*(\lambda^- + \alpha))$$

where $h_1 = \lambda^+ \int_0^\infty e^{-(\lambda^- + \alpha)x} x\, b_1(x)dx$

- The mean number of customers in the system is given by

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

$$= L_q + P \quad (2.45)$$

2.5 Stochastic Decomposition

The stochastic decomposition property of the system size distribution is verified. The classical stochastic decomposition property shows that the steady state system size at an arbitrary point can be represented as the sum of two independent random variables, one of which is the system size of the corresponding queueing system without server vacations and the other is the orbit size given that the server is on vacations. Stochastic decomposition has also been held for retrial queues.

Theorem 2.2

The number of customers in the system (L_s) can be expressed as the sum of two independent random variables, one of which is the mean number of customers G-queue with server breakdown (L) and the other is the mean number of customers in the orbit given that the server is idle (L_I).

Proof

The probability generating function $\Phi(z)$ of the number of customers in the G-queue with server breakdown is given by

$$\Phi(z) = I_0[\lambda^+(z-1)(1-z)B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) + (\lambda^- + \alpha)(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))$$

$$-(\lambda^- + \alpha z)(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))]/T_2(z) \qquad (2.46)$$

where

$$T_2(z) = (\lambda^+ + \lambda^- + \alpha - \lambda^+ z)(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)) - (\lambda^- + \alpha z)(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)) \\ R^*(\lambda^+ - \lambda^+ z)$$

The probability generating function $\psi(z)$ of the number of customers in the orbit given that the server is idle, is given by

$$\Psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= \frac{[I_0 A^*(\lambda^+)((\lambda^+ + \lambda^- + \alpha - \lambda^+ z)(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)) - (1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)) (\lambda^- + \alpha z)R^*(\lambda^+ - \lambda^+ z))][(\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)]}{T_1(z)[(\lambda^- + \alpha) + (B^*(\lambda^- + \alpha) - 1)(\lambda^+ + \alpha + \lambda^+ \beta_1(\lambda^- + \alpha))]}$$

(2.47)

From equations (2.38), (2.46) and (2.47), we get

$$P_s(z) = \Phi(z)\Psi(z)$$

Differentiating $P_s(z)$ with respect to z and taking limit as $z \to 1$, we get $L_s = L + L_I$

2.6 Reliability Analysis

Let A(t) be the system availability at time t, that is, the probability that the server is idle or busy. Then under steady state condition, the availability of the server is given by

$$A = I_0 + \lim_{z\to 1}\left[\int_0^\infty I(x,z)dx + \int_0^\infty P(x,z)dx\right]$$

$$= I_0 + I + P$$

$$= \frac{(\lambda^- + \alpha) + (B^*(\lambda^- + \alpha) - 1)(\alpha + \lambda^+ \beta_1(\lambda^- + \alpha))}{(\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)} \qquad (2.48)$$

The steady state failure frequency of the server is given by

$$W_f = \lambda^- P$$

$$= \frac{\lambda^- \lambda^+ (1 - B^*(\lambda^- + \alpha))}{(\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1)} \qquad (2.49)$$

Theorem 2.3

Let τ be the time to the first failure of the server. Then the Laplace transform of reliability function $\zeta(t) = P(\tau > t)$ of the server is given by

$$\tilde{\zeta}(s) = (1 - A^*(s+\lambda^+)) + (\lambda^+ + sA^*(s+\lambda^+))\tilde{\bar{B}}(s+\lambda^- + \alpha) + \tilde{I}_0(s)[(\lambda^+ B^*(s+\lambda^- + \alpha)$$
$$- (s+\lambda^+))(1 - A^*(s+\lambda^+)) - s(s+\lambda^+)A^*(s+\lambda^+)\tilde{\bar{B}}(s+\lambda^- + \alpha)]/F(1,s) \qquad (2.50)$$

where

$$F(1,s) = (s+\lambda^+) - B^*(s+\lambda^- + \alpha)(\lambda^+ + sA^*(s+\lambda^+))$$

Proof

Considering failure states of the server as absorbing states, we obtain a new system with the following governing equations.

$$\frac{d}{dt} I_0(t) = -\lambda^+ I_0(t) + \int_0^\infty P_0(x,t)\mu(x)dx, \tag{2.51}$$

$$\left(\frac{d}{dx}+\frac{d}{dt}\right) I_n(x,t) = -(\lambda^+ + \eta(x))\, I_n(x,t), \quad n \geq 1 \tag{2.52}$$

$$\left(\frac{d}{dx}+\frac{d}{dt}\right) P_n(x,t) = -(\lambda^+ + \lambda^- + \alpha + \mu(x))\, P_n(x,t) + \lambda^+ P_{n-1}(x,t), \quad n \geq 0 \tag{2.53}$$

$$I_n(0,t) = \int_0^\infty P_n(x,t)\mu(x)dx, \quad n \geq 1 \tag{2.54}$$

$$P_n(0,t) = \lambda^+ \int_0^\infty I_n(x,t)dx + \int_0^\infty I_{n+1}(x,t)\,\eta(x)dx, \quad n \geq 0 \tag{2.55}$$

Let the initial condition be

$$I_n(0) = \delta_{n,0}, \; I_n(x,0) = 0, \; P_n(x,0) = 0.$$

Taking Laplace transforms of equations (2.51) to (2.55), we obtain

$$(s+\lambda^+)\,\tilde{I}_0(s) - 1 = \int_0^\infty \tilde{P}_0(x,s)\mu(x)dx \tag{2.56}$$

$$\frac{d}{dx}\tilde{I}_n(x,s) = -(s+\lambda^+ + \eta(x))\tilde{I}_n(x,s), \quad n \geq 1 \tag{2.57}$$

$$\frac{d}{dx}\tilde{P}_n(x,s) = -(s+\lambda^+ + \lambda^- + \alpha + \mu(x))\tilde{P}_n(x,s) + \lambda^+\tilde{P}_{n-1}(x,s), \quad n \geq 0 \tag{2.58}$$

$$\tilde{I}_n(0,s) = \int_0^\infty \tilde{P}_n(x,s)\mu(x)dx, \quad n \geq 1 \tag{2.59}$$

$$\tilde{P}_n(0,s) = \int_0^\infty \tilde{I}_{n+1}(x,s)\,\eta(x)dx + \lambda^+ \int_0^\infty \tilde{I}_n(x,s)dx, \quad n \geq 0 \tag{2.60}$$

Define the following probability generating functions for $|z| \leq 1$:

$$\tilde{I}(z,x,s) = \sum_{n=1}^{\infty} \tilde{I}_n(x,z)\,z^n \;;\; \tilde{P}(z,x,s) = \sum_{n=0}^{\infty} \tilde{P}_n(x,z)\,z^n .$$

Then equations (2.57) to (2.60) yield

$$\frac{d}{dx}\tilde{I}(z,x,s) = -(s+\lambda^{+}+\eta(x))\tilde{I}(z,x,s) \tag{2.61}$$

$$\frac{d}{dx}\tilde{P}(z,x,s) = -(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z+\mu(x))\tilde{P}(z,x,s) \tag{2.62}$$

$$\tilde{I}(z,0,s) = \int_0^\infty \tilde{P}(z,x,s)\,\mu(x)dx - (s+\lambda^{+})\tilde{I}_0(s)+1 \tag{2.63}$$

$$\tilde{P}(z,0,s) = \lambda^{+}\tilde{I}_0(s) + \frac{1}{z}\int_0^\infty \tilde{I}(z,x,s)\eta(x)dx + \lambda^{+}\int_0^\infty \tilde{I}(z,x,s)dx \tag{2.64}$$

The solutions of the partial differential equations (2.61) and (2.62) are given by

$$\tilde{I}(z,x,s) = \tilde{I}(z,0,s)\, e^{-(s+\lambda^{+})x}(1-A(x)) \tag{2.65}$$

$$\tilde{P}(z,x,s) = \tilde{P}(z,0,s)\, e^{-(s+\lambda^{+}(1-z)+\lambda^{-}+\alpha)x}(1-B(x)) \tag{2.66}$$

Using equation (2.66) in equation (2.63), we get

$$\tilde{I}(z,0,s) = \tilde{P}(z,0,s)B^{*}(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)-(s+\lambda^{+})\tilde{I}_0(s)+1 \tag{2.67}$$

Using equation (2.65) in equation (2.64) and simplifying we obtain

$$\tilde{P}(z,0,s) = \lambda^{+}\tilde{I}_0(s) + \frac{\tilde{I}(z,0,s)}{z(s+\lambda^{+})}(z\lambda^{+}+(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})) \tag{2.68}$$

Solving equation (2.67) and (2.68), we get

$$\tilde{I}(z,0,s) = (s+\lambda^{+})(z+\tilde{I}_0(s)(\lambda^{+}zB^{*}(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)-z(s+\lambda^{+})))/F(z,s) \tag{2.69}$$

$$\tilde{P}(z,0,s) = z\lambda^{+}+(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})-\tilde{I}_0(s)(s+\lambda^{+})(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})/F(z,s) \tag{2.70}$$

where

$$F(z,s) = z(s+\lambda^{+})-B^{*}(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)(\lambda^{+}z+(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+}))$$

Substituting the expressions of $\tilde{I}(z,0,s)$ and $\tilde{P}(z,0,s)$ in equations (2.65) and (2.66) we get

$$\tilde{I}(z,x,s) = (s+\lambda^{+})[z+\tilde{I}_0(s)(\lambda^{+}zB^{*}(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)-z(s+\lambda^{+}))]$$

$$e^{-(s+\lambda^{-})x}(1-A(x))/F(z,s) \tag{2.71}$$

$$\tilde{P}(z,x,s) = [z\lambda^{+}+(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})-\tilde{I}_0(s)(s+\lambda^{+})(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})]$$

$$e^{-(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)x}(1-B(x))/F(z,s) \tag{2.72}$$

From equations (2.71) and (2.72) we can obtain

$$\tilde{I}(z,s) = \int_0^{\infty}\tilde{I}(z,x,s)dx$$

$$=[z+\tilde{I}_0(s)(\lambda^{+}zB^{*}(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)-z(s+\lambda^{+}))](1-A^{*}(s+\lambda^{+}))/F(z,s) \tag{2.73}$$

$$\tilde{P}(z,s) = \int_0^{\infty}\tilde{P}(z,x,s)dx$$

$$=[z\lambda^{+}+(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})-\tilde{I}_0(s)(s+\lambda^{+})(s+\lambda^{+}(1-z))A^{*}(s+\lambda^{+})]$$

$$\tilde{\bar{B}}(s+\lambda^{+}+\lambda^{-}+\alpha-\lambda^{+}z)/F(z,s) \tag{2.74}$$

Now $\tilde{\zeta}(s)$ is given by

$$\tilde{\zeta}(s) = \tilde{I}(1,s)+\tilde{P}(1,s)$$

$$\tilde{I}(1,s) = [1+\tilde{I}_0(s)(\lambda^{+}B^{*}(s+\lambda^{-}+\alpha)-(s+\lambda^{+}))](1-A^{*}(s+\lambda^{+}))/F(1,s) \tag{2.75}$$

$$\tilde{P}(1,s) = [\lambda^{+}+sA^{*}(s+\lambda^{+})-\tilde{I}_0(s)s(s+\lambda^{+})A^{*}(s+\lambda^{+})]\tilde{\bar{B}}(s+\lambda^{-}+\alpha)/F(1,s) \tag{2.76}$$

Substituting the expressions of $\tilde{I}(1,s)$ and $\tilde{P}(1,s)$ we get the result in equation (2.50).

Corollary 2.1

The mean time to the first failure (MTTFF) of the server is given by

$$\text{MTTFF} = (1 - A^*(\lambda^+)) + \lambda^+\tilde{\tilde{B}}(\lambda^- + \alpha) + \tilde{I}_0(0)\lambda^+(B^*(\lambda^- + \alpha) - 1)(1 - A^*(\lambda^+)) / F(1,0) \tag{2.77}$$

where

$F(1,0) = \lambda^+(1 - B^*(\lambda^- + \alpha))$

Proof

$$\text{MTTFF} = \int_0^\infty \zeta(t)dt = \lim_{s\to 0}\tilde{\zeta}(s).$$

Taking limit as $s\to 0$ on both sides of equation (2.50) we get the result in equation (2.77)

2.7 Special Cases

Case (i): No negative customers ($\lambda^-\to 0$)

In this case our model reduces to M/G/1 retrial G-queue with server breakdown having,

$$P_s(z) = I_0 A^*(\lambda^+)[\lambda^+(z-1)(1-z)B^*(\lambda^+ + \alpha - \lambda^+ z) + \alpha(z-1)B^*(\lambda^+ + \alpha - \lambda^+ z)] / T_3(z)$$

where

$$T_3(z) = (\lambda^+ + \alpha - \lambda^+ z)(z - (z + (1-z)A^*(\lambda^+))B^*(\lambda^+ + \alpha - \lambda^+ z)) - \alpha z(z + (1-z)A^*(\lambda^+))$$

$$(1 - B^*(\lambda^+ + \alpha - \lambda^+ z))R^*(\lambda^+ - \lambda^+ z)$$

$$I_0 = (B^*(\alpha) - 1)(\lambda^+(1 + \beta_1\alpha) + \alpha(1 + (1 - A^*(\lambda^+))) + \alpha(1 - (1 - A^*(\lambda^+))B^*(\alpha)) / T_4(z)$$

$$T_4(z) = A^*(\lambda^+)(\alpha + \alpha(B^*(\alpha) - 1))$$

Case (ii): No retrial ($A^*(\lambda^+) \to 1$)

In this case our model reduces to M/G/1 G-queue with server breakdown and two types of customers with

$$P_s(z) = I_0[\lambda^+(z-1)(1-z)B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z) + (\lambda^- + \alpha)(z - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z)) - (\lambda^- + \alpha z)$$
$$(1 - B^*(\lambda^+ + \lambda^- + \alpha - \lambda^+ z))]/T_2(z)$$

and

$$I_0 = (\lambda^- + \alpha) + (B^*(\lambda^- + \alpha) - 1)(\lambda^+(1 + \beta_1(\lambda^- + \alpha)) + \alpha)/((\lambda^- + \alpha) + \alpha(B^*(\lambda^- + \alpha) - 1))$$

2.8 Numerical Results

Numerical results are calculated by assuming the distributions of retrial time, service time and repair time as exponential with rates η, μ and β.

For the parameters $\lambda^+ = 2$, $\lambda^- = 0.5$, $\alpha = 0.7$, $\mu = 6$, $\beta = 4$, $\eta = 40$, $c_1 = 0.5$, $c_2 = 0.5$, the performance measures I_0-the probability that the system is empty, I-the probability that the server is idle in non-empty system, P-the probability that the server is busy, R-the probability that the server is under repair, A-the availability of the server, F-the failure frequency of the server and Ls-the mean number of customers in the system are calculated and presented in tables 2.1 to 2.6 respectively by varying the rates of λ^+, λ^-, α, η, μ and β.

The increasing and decreasing trends of the performance measures by varying the parameters are as expected.

Table 2.1 Performance measures by varying λ^+

λ^+	I_0	I	P	R	A	F	Ls
1.0000	0.7923	0.0077	0.1538	0.0462	0.9538	0.0769	0.2536
1.5000	0.6847	0.0153	0.2308	0.0692	0.9308	0.1154	0.5029
2.0000	0.5746	0.0254	0.3077	0.0923	0.9077	0.1538	0.8698
2.5000	0.462	0.038	0.3846	0.1154	0.8846	0.1923	1.4162
3.0000	0.3469	0.0531	0.4615	0.1385	0.8615	0.2308	2.2896

Table 2.2 Performance measures by varying λ^-

λ^-	I_0	I	P	R	A	F	Ls
1.0000	0.5675	0.0254	0.2857	0.1214	0.8786	0.2857	0.7885
2.0000	0.5559	0.0253	0.2500	0.1688	0.8313	0.5000	0.7022
3.0000	0.5469	0.0253	0.2222	0.2056	0.7944	0.6667	0.6518
4.0000	0.5398	0.0253	0.2000	0.2350	0.7650	0.8000	0.6164
5.0000	0.5339	0.0252	0.1818	0.2591	0.7409	0.9091	0.5892

Table 2.3 Performance measures by varying α

α	I_0	I	P	R	A	F	Ls
1.0000	0.5481	0.0288	0.3077	0.1154	0.8846	0.1538	0.9808
2.0000	0.4596	0.0404	0.3077	0.1923	0.8077	0.1538	1.3924
3.0000	0.3712	0.0519	0.3077	0.2692	0.7308	0.1538	1.9787
4.0000	0.2827	0.0635	0.3077	0.3462	0.6538	0.1538	2.9335
5.0000	0.1942	0.075	0.3077	0.4231	0.5769	0.1538	4.7672

Table 2.4 Performance measures by varying η

η	I_0	I	P	R	A	F	Ls
2.0000	0.0923	0.5077	0.3077	0.0923	0.9077	0.1538	10.646
4.0000	0.3462	0.2538	0.3077	0.0923	0.9077	0.1538	2.1051
6.0000	0.4308	0.1692	0.3077	0.0923	0.9077	0.1538	1.4948
8.0000	0.4731	0.1269	0.3077	0.0923	0.9077	0.1538	1.2715
10.0000	0.4985	0.1015	0.3077	0.0923	0.9077	0.1538	1.1557

Table 2.5 Performance measures by varying μ

μ	I_0	I	P	R	A	F	Ls
5.0000	0.4973	0.0300	0.3636	0.1091	0.8909	0.1818	1.1546
10.0000	0.7367	0.0157	0.1905	0.0571	0.9429	0.0952	0.4470
15.0000	0.8216	0.0106	0.1290	0.0387	0.9613	0.0645	0.2809
20.0000	0.8651	0.0080	0.0976	0.0293	0.9707	0.0488	0.2054
25.0000	0.8916	0.0065	0.0784	0.0235	0.9765	0.0392	0.1620

Table 2.6 Performance measures by varying β

β	I_0	I	P	R	A	F	Ls
1.0000	0.2838	0.0392	0.3077	0.3692	0.6308	0.1538	3.5344
2.0000	0.4777	0.0300	0.3077	0.1846	0.8154	0.1538	1.2961
3.0000	0.5423	0.0269	0.3077	0.1231	0.8769	0.1538	0.9850
4.0000	0.5746	0.0254	0.3077	0.0923	0.9077	0.1538	0.8698
5.0000	0.5940	0.0245	0.3077	0.0738	0.9262	0.1538	0.811