CHAPTER - II

CHAPTER-II

FUZZY SOFT TOPOLOGICAL SPACES

SECTION 2.1

FUZZY SOFT SETS

Notation 2.1.1

Let U be an initial universe set and E be the set of parameters and $\widetilde{P}(U)$, the set of all fuzzy subset of U and also by (U, E), we mean the universal set U and parameter set E.

Definition 2.1.2

A pair (F, A) is called a **fuzzy soft set** over U where $F: A \rightarrow \widetilde{P}(U)$ is a mapping from A into$\widetilde{P}(U)$.

The fuzzy soft set (F, A) is also denoted as F_A.

Definition 2.1.3

For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft class (U, E) we say that (F, A) is a **fuzzy soft subset** of(G, B), if

i. $A \subseteq B$

ii. For all$\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$ and is written as $(F, A) \widetilde{\subseteq} (G, B)$.

Definition 2.1.4

Union of two fuzzy soft sets (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H, C) where $C = AUB$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), \text{ if } \varepsilon \in A \backslash B, \\ G(\varepsilon) \text{ if } \varepsilon \in B \backslash A \text{ and} \\ F(\varepsilon) \cup G(\varepsilon) \text{ if } \varepsilon \in A \cap B \end{cases}$$

and is denoted as $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

Definition 2.1.5

Let (F, A) and (G, B) be two fuzzy soft sets in a soft class (U, E) with $A \cap B \neq \phi$.

Then **intersection** of two fuzzy soft sets (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H, C) where $C=A \cap B$ and $\forall \varepsilon \in C$, $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. We write $(F, A) \tilde{\cap} (G, B) = (H, C)$.

Definition 2.1.6

The **complement** of a fuzzy soft set (F, A) is denoted by $(F, A)^C$ and is defined by $(F, A)^C = (F^C, A)$ where $F^C: A \rightarrow \tilde{P}(U)$ is a mapping given by $F^C(\alpha) = [F(\alpha)]^C$, $\forall \alpha \in A$.

The set of all fuzzy soft set over (U, E) is denoted by $FS(U, E)$.

Definition 2.1.7

The fuzzy soft set $F_\Phi \in FS(U, E)$ is called **null fuzzy soft set** and is denoted by $\tilde{\Phi}$,if $F_\Phi(e) = \bar{0}$ where $\bar{0}$ is the constant fuzzy set zero, for every $e \in E$.

Definition 2.1.8

The fuzzy soft set $F_E \in FS(U, E)$ is called **absolute fuzzy soft set** and is denoted by $\tilde{E}$, if $F_E(e) = \bar{1}$, where $\bar{1}$ is the constant fuzzy set one, for every $e \in E$.

Theorem 2.1.9

Let $F_A, G_B, H_C \in FS(U, E)$. It can be verified that the following hold according to our notion of fuzzy soft sets.

i. $\tilde{\Phi} \tilde{\subseteq} F_A \tilde{\subseteq} \tilde{E}$

ii. $F_A \tilde{\cup} G_B = G_B \tilde{\cup} F_A$, $F_A \tilde{\cap} G_B = G_B \tilde{\cap} F_A$

iii. $F_A \tilde{\cup} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cup} H_C$, $F_A \tilde{\cap} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cap} H_C$

iv. $F_A \tilde{\cup} (G_B \tilde{\cap} H_C) = (F_A \tilde{\cup} G_B) \tilde{\cap} (F_A \tilde{\cup} H_C)$,
$F_A \tilde{\cap} (G_B \tilde{\cup} H_C) = (F_A \tilde{\cap} G_B) \tilde{\cup} (F_A \tilde{\cap} H_C)$

v. $F_A, G_B \tilde{\subseteq} F_A \tilde{\cup} G_B$, $F_A \tilde{\cap} G_B \tilde{\subseteq} F_A, G_B$

vi. $F_A \tilde{\subseteq} G_B \Rightarrow G_B^C \tilde{\subseteq} F_A^C$

vii. $\tilde{\Phi}^C = \tilde{E}$, $\tilde{E}^C = \tilde{\Phi}$

viii. $\left(F_A^C\right)^C = F_A$

ix. $(\tilde{\cap}_\alpha F^{\alpha}_{A_\alpha})^C = \tilde{\cup}_\alpha \left(F^{\alpha}{}^{C}_{A_\alpha}\right)$

x. $(\tilde{\cup}_\alpha F^{\alpha}_{A_\alpha})^C = \tilde{\cap}_\alpha \left(F^{\alpha}{}^{C}_{A_\alpha}\right)$

SECTION 2.2

DEFINITIONS AND BASIC PROPERTIES OF FUZZY SOFT TOPOLOGICAL SPACES

Definition 2.2.1

A **fuzzy soft topology** τ on (U, E) is a family of fuzzy soft sets over (U, E) satisfying the following properties

i. $\tilde{\Phi}, \tilde{E} \in \tau$

ii. If $F_A, G_B \in \tau$ then $F_A \tilde{\cap} G_B \in \tau$

iii. If $F^{\alpha}_{A_\alpha} \in \tau$ for all $\alpha \in \Delta$, an index set, then $\tilde{\cup}_{\alpha\in\Delta} F^{\alpha}_{A_\alpha} \in \tau$.

Definition 2.2.2

If τ is a fuzzy soft topology on (U, E), the triple (U, E, τ) is said to be a **fuzzy soft topological space.** Also each member of τ is called a **fuzzy soft open** set in (U, E, τ).

Example 2.2.3

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_4\} \subseteq E$, $B = \{e_2, e_4\} \subseteq E$.

$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0)\}$

$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$

$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$

$F(e_4) = \{(a, 0.2), (b, 0.6), (c, 0.3)\}\}$

$G_B = \{G(e_1) = \{(a, 0), (b, 0), (c, 0)\}$

$G(e_2) = \{(a, 0.5), (b, 0.2), (c, 0.1)\}$

$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$

$G(e_4) = \{(a, 0.1), (b, 0.3), (c, 0.2)\}\}$

We consider a collection τ of fuzzy soft sets over (U, E) as $\tau = \{\tilde{\Phi}, \tilde{E}, F_A, G_B\}$. We see that

i. $\tilde{\Phi}, \tilde{E} \in \tau$

ii. $\tilde{\Phi} \tilde{\cap} \tilde{E} = \tilde{\Phi}$, $\tilde{\Phi} \tilde{\cap} F_A = \tilde{\Phi}$, $\tilde{\Phi} \tilde{\cap} G_B = \tilde{\Phi}$,

$\tilde{E} \tilde{\cap} F_A = F_A$, $\tilde{E} \tilde{\cap} G_B = G_B$, $F_A \tilde{\cap} G_B = G_B$

iii. $\tilde{\Phi} \tilde{\cup} \tilde{E} = \tilde{E}$, $\tilde{\Phi} \tilde{\cup} F_A = F_A$, $\tilde{\Phi} \tilde{\cup} G_B = G_B$,

$\widetilde{E} \widetilde{\cup} F_A = \widetilde{E}, \widetilde{E} \widetilde{\cup} G_B = \widetilde{E}, F_A \widetilde{\cup} G_B = F_A,$

$\widetilde{\Phi} \widetilde{\cup} \widetilde{E} \widetilde{\cup} F_A = \widetilde{E}, \widetilde{\Phi} \widetilde{\cup} \widetilde{E} \widetilde{\cup} G_B = \widetilde{E}, \widetilde{E} \widetilde{\cup} F_A \widetilde{\cup} G_B = \widetilde{E}$ and

$\widetilde{\Phi} \widetilde{\cup} \widetilde{E} \widetilde{\cup} F_A \widetilde{\cup} G_B = \widetilde{E}.$

Thus τ is a fuzzy soft topology on (U, E) and the triple (U, E, τ) is a fuzzy soft topological space. The open sets in (U, E, τ) are $\widetilde{\Phi}$, $\widetilde{E}$, F_A, G_B.

Definition 2.2.4

A fuzzy soft set F_A over (U, E) is called a **fuzzy soft closed** set in (U, E, τ) if and only if its complement F_A^C is a fuzzy soft open set in (U, E, τ).

Definition 2.2.5

A fuzzy soft topological space (U, E, τ) is called **indiscrete** if it contains only $\widetilde{\Phi}$ and $\widetilde{E}$ while the **discrete** fuzzy soft topology consists of all fuzzy soft sets over (U, E).

Theorem 2.2.6

If $\{ \tau_\lambda : \lambda \in I\}$ is a family of fuzzy soft topologies on (U, E), then $\cap_\lambda \{ \tau_\lambda : \lambda \in I\}$ is also a fuzzy soft topology on (U, E).

Proposition 2.2.7

Let (U, E, τ) be a fuzzy soft topological space and Γ denote the collection of all closed fuzzy soft sets in (U, E, τ). Then

i. $\widetilde{\Phi}, \widetilde{E} \in \Gamma$
ii. If F_{A_1}, $F_{A_2} \in \Gamma$ then $F_{A_1} \widetilde{\cup} F_{A_2} \in \Gamma$
iii. If $F_{A_\alpha}^\alpha \in \Gamma$ for all $\alpha \in \Delta$, an index set, then $\widetilde{\cap}_{\alpha \in \Delta} F_{A_\alpha}^\alpha \in \Gamma$

Definition 2.2.8

Let τ_1 and τ_2 be two fuzzy soft topologies on (U, E). We say that τ_1 is **coarser** than τ_2 or that τ_2 is **finer** than τ_1 if and only if $\tau_1 \subseteq \tau_2$ i.e. every τ_1 fuzzy soft open set is τ_2 fuzzy soft open set. If either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, we say that the topologies τ_1 and τ_2 are comparable.

If $\tau_1 \not\subset \tau_2$ and $\tau_2 \not\subset \tau_1$, we say that the topologies τ_1 and τ_2 are not comparable.

Example 2.2.9

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0)\}$$
$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$
$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$

$$G_B = \{G(e_1) = \{(a, 0.6), (b, 0.1), (c, 0)\}$$
$$G(e_2) = \{(a, 0.7), (b, 0.9), (c, 0.5)\}$$
$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$G(e_4) = \{(a, 0.5), (b, 0.3), (c, 0.9)\}\}$$

We consider a fuzzy soft topologies τ_1 and τ_2 on (U, E) as $\tau_1 = \{\widetilde{\Phi}, \widetilde{E}, F_A\}$ and $\tau_1 = \{\widetilde{\Phi}, \widetilde{E}, F_A, G_B\}$. We see that $\tau_1 \subseteq \tau_2$ and hence τ_1 is coarser than τ_2.

Definition 2.2.10

The fuzzy soft set F_A over (U, E) is called a **fuzzy soft point** in (U, E) denoted by $e(F_A)$, if for the element $e \in A$, $F(e) \neq \bar{0}$ and $F(e^1)$ for all $e^1 \in A - \{e\}$.

Example 2.2.11

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$.
Consider a fuzzy soft set F_A over (U, E) as

$$F_A = \{F(e_1) = \{(a, 0), (b, 0), (c, 0)\}$$
$$F(e_2) = \{(a, 0.1), (b, 0.2), (c, 0.7)\}$$
$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$

Here $e_2 \in A$, $F(e_2) \neq \bar{0}$ and for $e^1 \in A - \{e_2\}$, $F(e^1) = \bar{0}$. Thus F_A is a fuzzy soft point in (U, E) denoted by $e_2(F_A)$.

Definition 2.2.12

The **fuzzy soft point** $e(F_A)$ is said to be in the fuzzy soft set G_B if $A \subseteq B$ and for the element $e \in A$, $F(e) \subseteq G(e)$. Symbolically we write $e(F_A) \widetilde{\in} G_B$.

Example 2.2.13

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

Clearly $A \subseteq B$. We consider the fuzzy soft point $e_2(F_A)$ given in Example 2.2.11 and a fuzzy soft set G_B over (U, E) as

$$G_B = \{G(e_1) = \{(a, 0.1), (b, 0.7), (c, 0.9)\}$$
$$G(e_2) = \{(a, 0.1), (b, 0.3), (c, 0.8)\}$$
$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$G(e_4) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}\}$$

Here $e_2 \in A$ and $F(e_2) \subseteq G(e_2)$. Thus by our definition $e_2(F_A) \tilde{\in} G_B$.

Definition 2.2.14

A fuzzy soft set H_C in a fuzzy soft topological space (U, E, τ) is called a **fuzzy soft neighborhood** of the fuzzy soft point $e(F_A) \tilde{\in} (U, E)$ if there is a fuzzy soft open set G_B such that $e(F_A) \tilde{\in} G_B \tilde{\subseteq} H_C$.

Example 2.2.15

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

Clearly $A \subseteq B$. We consider the fuzzy soft point $e_2(F_A)$ given in Example 2.2.11 and the fuzzy soft set G_B over (U, E) given in Example 2.2.13 above.

Consider the fuzzy soft topology $\tau = \{\tilde{\Phi}, \tilde{E}, G_B\}$.

Now let us consider the fuzzy soft set H_B over (U, E) as

$$H_B = \{H(e_1) = \{(a, 0.1), (b, 0.8), (c, 0.9)\}$$
$$H(e_2) = \{(a, 0.2), (b, 0.3), (c, 0.8)\}$$
$$H(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$H(e_4) = \{(a, 0.7), (b, 0.6), (c, 0.2)\}\}$$

We see that $e(F_A) \tilde{\in} G_B \tilde{\subseteq} H_C$. Thus by our definition the fuzzy soft set H_B is a fuzzy soft neighborhood of the fuzzy soft point $e_2(F_A)$.

Definition 2.2.16

The neighborhood system of a fuzzy soft point $e(F_A) \tilde{\in} (U, E)$ is the family of all neighborhoods of $e(F_A)$ and is written as $N_\tau(e(F_A))$.

Definition 2.2.17

A fuzzy soft set H_C in a fuzzy soft topological space (U, E, τ) is called a **fuzzy soft neighborhood** of the fuzzy soft open set G_B such that $F_A \tilde{\subseteq} G_B \tilde{\subseteq} H_C$.

Example 2.2.18

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

Clearly $A \subseteq B$. We consider the fuzzy soft set G_B over (U, E) given in Example 2.2.13 and the fuzzy soft set H_B over (U, E) given in Example 2.2.15 above.

Consider the fuzzy soft topology $\tau = \{\tilde{\Phi}, \tilde{E}, G_B\}$.

Let us consider the fuzzy soft set I_A over (U, E) as

$$I_A = \{I(e_1) = \{(a, 0.1), (b, 0.6), (c, 0.7)\}$$
$$I(e_2) = \{(a, 0), (b, 0.1), (c, 0.8)\}$$
$$I(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$I(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$

We see that $I_A \tilde{\subseteq} G_B \tilde{\subseteq} H_B$. It follows that the fuzzy soft set H_B is a fuzzy soft neighborhood of the fuzzy soft set I_A.

Theorem 2.2.19

In a fuzzy soft topological space (U, E, τ) the following hold.

i. $\tilde{E} \in N_\tau(e(F_A))\ \forall e(F_A)$ and
$G_B \in N_\tau(e(F_A)) \Rightarrow e(F_A) \tilde{\in} G_B$

ii. If $G_B \in N_\tau(e(F_A))$ and $G_B \tilde{\subseteq} H_C$ then $H_C \in N_\tau(e(F_A))$

iii. If $G_B, H_C \in N_\tau(e(F_A))$ then $G_B \tilde{\cap} H_C \in N_\tau(e(F_A))$

iv. If $G_B \in N_\tau(e(F_A))$ then there is a $H_C \in N_\tau(e(F_A))$ such that $G_B \in N_\tau(e^1(M_D))$ for each $e^1(M_D) \tilde{\in} H_C$

Proof

i. Let $\tilde{E} = H_E$

Then $\forall \varepsilon \in E$, $H(\varepsilon) = \bar{1}$

Now $A \subseteq E$ and $F(e) = H(e)$

It follows that $e(F_A) \tilde{\in} H_E \tilde{\subseteq} H_E$

i.e. $e(F_A) \tilde{\in} \tilde{E} \tilde{\subseteq} \tilde{E}$

Thus $\tilde{E} \in N_\tau(e(F_A))\ \forall e(F_A)$

For the second part, we have $G_B \in N_\tau(e(F_A))$

So there is a fuzzy soft set H_C such that $e(F_A) \tilde{\in} H_C \tilde{\subseteq} G_B$. It follows that $e(F_A) \tilde{\in} G_B$

ii. We have $G_B \in N_\tau(e(F_A))$

So there is a fuzzy soft open sets I_D such that $e(F_A) \tilde{\in} I_D \tilde{\subseteq} G_B$

Also $G_B \tilde{\subseteq} H_C$

Thus $e(F_A) \tilde{\in} I_D \tilde{\subseteq} G_B \tilde{\subseteq} H_C$ and hence

$H_C \in N_\tau(e(F_A))$

iii. We have $G_B, H_C \in N_\tau(e(F_A))$

So there is a fuzzy soft open sets I_D, J_E such that

$e(F_A) \tilde{\in} I_D \tilde{\subseteq} G_B$ and $e(F_A) \tilde{\in} J_E \tilde{\subseteq} H_C$

Thus $e(F_A) \tilde{\in} I_D \tilde{\cap} J_E \tilde{\subseteq} G_B \tilde{\cap} H_C$ and

Since $I_D \tilde{\cap} J_E \in \tau$, it follows that

$G_B \tilde{\cap} H_C \in N_\tau(e(F_A))$

iv. We have $G_B \in N_\tau(e(F_A))$

So there is an $I_P \in \tau$ such that $e(F_A) \tilde{\in} I_P \tilde{\subseteq} G_B$.

We take $H_C = I_P$. Then for each $e^1(M_D) \tilde{\in} H_C$, $e^1(M_D) \tilde{\in} H_C \tilde{\subseteq} G_B$.

It follows that $G_B \in N_\tau(e^1(M_D))$

Definition 2.2.20

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E). The **fuzzy soft closure** of F_A is defined as the intersection of all fuzzy soft closed sets which contain F_A and is denoted by $\overline{F_A}$. We write

$\overline{F_A} = \tilde{\cap} \{G_B : G_B$ is a fuzzy soft closed and $F_A \tilde{\subseteq} G_B\}$

It is obvious that

i. $\overline{F_A}$ is fuzzy soft closed and

ii. $F_A \tilde{\subseteq} \overline{F_A}$.

Example 2.2.21

. Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0)\}$$
$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$
$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$

$$G_B = \{G(e_1) = \{(a, 0.6), (b, 0.1), (c, 0)\}$$
$$G(e_2) = \{(a, 0.7), (b, 0.9), (c, 0.5)\}$$
$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$G(e_4) = \{(a, 0.5), (b, 0.3), (c, 0.9)\}\}$$

We consider a fuzzy soft topology τ on (U, E) as $\tau = \{\widetilde{\Phi}, \widetilde{E}, F_A, G_B\}$.

Then fuzzy soft closed sets are

$$F_A^C = \{F^C(e_1) = \{(a, 0.6), (b, 0.9), (c, 1)\}$$
$$F^C(e_2) = \{(a, 0.4), (b, 0.5), (c, 0.8)\}$$
$$F^C(e_3) = \{(a, 1), (b, 1), (c, 1)\}$$
$$F^C(e_4) = \{(a, 1), (b, 1), (c, 1)\}\}$$

$$G_B^C = \{G^C(e_1) = \{(a, 0.4), (b, 0.9), (c, 1)\}$$
$$G^C(e_2) = \{(a, 0.3), (b, 0.1), (c, 0.5)\}$$
$$G^C(e_3) = \{(a, 1), (b, 1), (c, 1)\}$$
$$G^C(e_4) = \{(a, 0.5), (b, 0.7), (c, 0.1)\}\}$$

We consider a fuzzy soft set H_C over (U, E) as

$$H_C = \{H(e_1) = \{(a, 0.5), (b, 0.7), (c, 0.8)\}$$
$$H(e_2) = \{(a, 0.3), (b, 0.5), (c, 0.6)\}$$
$$H(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$H(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$

Then

$\overline{H_C}$ = Intersection of all fuzzy soft closed sets containing H_C

$= F_A^C \,\widetilde{\cap}\, \widetilde{E} = F_A^C = \{\, F^C(e_1) = \{(a, 0.6), (b, 0.9), (c, 1)\}$

$$F^C(e_2) = \{(a, 0.4), (b, 0.5), (c, 0.8)\}$$
$$F^C(e_3) = \{(a, 1), (b, 1), (c, 1)\}$$
$$F^C(e_4) = \{(a, 1), (b, 1), (c, 1)\}\}$$

Definition 2.2.22

Let (U, E, τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U, E). The **fuzzy soft interior** of F_A is defined as the union of all fuzzy soft open sets contained in F_A and is denoted by F_A^o. We write

$F_A^o = \tilde{\cup} \{G_B : G_B$ is a fuzzy soft open and $G_B \tilde{\subseteq} F_A\}$.

It is obvious that

i. F_A^o is fuzzy soft open
ii. $F_A^o \tilde{\subseteq} F_A$
iii. F_A^o is the largest fuzzy soft open set contained in F_A.

Example 2.2.23

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$F_A = \{F(e_1) = \{(a, 0.4), (b, 0.1), (c, 0.1)\}$$
$$F(e_2) = \{(a, 0.6), (b, 0.5), (c, 0.2)\}$$
$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$F(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$
$$G_B = \{G(e_1) = \{(a, 0.6), (b, 0.1), (c, 0.1)\}$$
$$G(e_2) = \{(a, 0.7), (b, 0.9), (c, 0.5)\}$$
$$G(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$
$$G(e_4) = \{(a, 0.5), (b, 0.3), (c, 0.9)\}\}$$

We consider a fuzzy soft topology τ on (U, E) as $\tau = \{\tilde{\Phi}, \tilde{E}, F_A, G_B\}$.

We consider a fuzzy soft set H_C over (U, E) as

$$H_C = \{H(e_1) = \{(a, 0.6), (b, 0.3), (c, 0.2)\}$$
$$H(e_2) = \{(a, 0.7), (b, 0.5), (c, 0.4)\}$$
$$H(e_3) = \{(a, 0.2), (b, 0.8), (c, 0.6)\}$$
$$H(e_4) = \{(a, 0), (b, 0), (c, 0)\}\}$$

Then

H_C^o = Union of all fuzzy soft open sets contained in H_C

$$= F_A \,\tilde{\cup}\, \tilde{\Phi} = F_A = \{\, F(e_1) = \{(a,0.4),(b,0.1),(c,0)\}$$
$$F(e_2) = \{(a,0.6),(b,0.5),(c,0.2)\}$$
$$F(e_3) = \{(a,0),(b,0),(c,0)\}$$
$$F(e_4) = \{(a,0),(b,0),(c,0)\}\}$$

Theorem 2.2.24

Let (U,E,τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U,E). Then F_A is a fuzzy soft closed set iff $\overline{F_A} = F_A$.

Proof

Let (U,E,τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U,E) such that $\overline{F_A} = F_A$. To prove that F_A is fuzzy soft closed. We have

$$\overline{F_A} = \tilde{\cap}\,\{G_B : G_B \text{ is a fuzzy soft closed and } F_A \,\tilde{\subseteq}\, G_B\}$$

$\overline{F_A}$ is fuzzy soft closed, being an arbitrary intersection of fuzzy soft closed sets. Also $\overline{F_A}$ is fuzzy soft closed and $\overline{F_A} = F_A \Longrightarrow F_A$ is fuzzy soft closed.

Conversely, suppose that F_A is fuzzy soft closed in (U,E,τ). To prove that $\overline{F_A} = F_A$. It is clear from definition that any fuzzy soft closed set G_B, $F_A \,\tilde{\subseteq}\, G_B \Longrightarrow \overline{F_A} \,\tilde{\subseteq}\, G_B$.

Since $F_A \,\tilde{\subseteq}\, F_A \Longrightarrow \overline{F_A} \,\tilde{\subseteq}\, F_A$ and $F_A \,\tilde{\subseteq}\, \overline{F_A}$, it follows that $\overline{F_A} = F_A$.

Theorem 2.2.25

Let (U,E,τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U,E). Then F_A is a fuzzy soft open set if and only if $F_A^o = F_A$.

Proof

Let (U,E,τ) be a fuzzy soft topological space. Let F_A be a fuzzy soft set over (U,E) such that $F_A^o = F_A$. To prove that F_A is fuzzy soft open. We have

$$F_A^o = \tilde{\cup}\,\{G_B : G_B \text{ is a fuzzy soft open and } G_B \,\tilde{\subseteq}\, F_A\}$$

F_A^o is fuzzy soft open, being an arbitrary union of fuzzy soft open sets. Also F_A^o is fuzzy soft open and $F_A^o = F_A \Longrightarrow F_A$ is fuzzy soft open.

Conversely, suppose that F_A is fuzzy soft open in (U, E, τ). To prove that $F_A^o = F_A$. It is clear from definition that any fuzzy soft open set $G_B \tilde{\subseteq} F_A \Longrightarrow G_B \tilde{\subseteq} F_A^o$.

Since $F_A \tilde{\subseteq} F_A \Longrightarrow F_A \tilde{\subseteq} F_A^o$ and $F_A^o \tilde{\subseteq} F_A$, it follows that $F_A^o = F_A$.

Theorem 2.2.26

Let (U, E, τ) be a fuzzy soft topological space. Let F_A, G_B be two fuzzy soft sets over (U, E). Then

i. $\overline{\tilde{\Phi}} = \tilde{\Phi}$

ii. $F_A \tilde{\subseteq} \overline{F_A}$

iii. $F_A \tilde{\subseteq} G_B \Longrightarrow \overline{F_A} \tilde{\subseteq} \overline{G_B}$

iv. $\overline{F_A \tilde{\cup} G_B} = \overline{F_A} \tilde{\cup} \overline{G_B}$

v. $\overline{F_A \tilde{\cap} G_B} = \overline{F_A} \tilde{\cap} \overline{G_B}$

vi. $\overline{\overline{F_A}} = F_A$

Theorem 2.2.27

Let (U, E, τ) be a fuzzy soft topological space. Let F_A, G_B be two fuzzy soft sets over (U, E). Then

i. $\tilde{\Phi}^o = \tilde{\Phi}$

ii. $\tilde{E}^o = \tilde{E}$

iii. $F_A \tilde{\subseteq} G_B \Longrightarrow F_A^o \tilde{\subseteq} G_B^o$

iv. $(F_A^o)^o = F_A^o$

Definition 2.2.28

Let (U, E, τ) be a fuzzy soft topological space. Let Y be an ordinary subset of U and H_E be a fuzzy soft set over (U, E) such that

$$\forall \varepsilon \in E,\ \mu^{\varepsilon} H_E(x) = \begin{cases} 1 \text{ if } x \in Y \\ 0 \text{ if } x \notin Y \end{cases}$$

Let $T_Y = \{H_E \tilde{\cap} G_B : G_B \in \tau\}$.

It can be verified that T_Y is a fuzzy soft topology on (Y, E). We would call T_Y the **fuzzy soft subspace** topology for (Y, E).

Example 2.2.29

. Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2\} \subseteq E$, $B = \{e_1, e_2, e_4\} \subseteq E$.

$$
\begin{aligned}
F_A = \{F(e_1) &= \{(a, 0.4), (b, 0.1), (c, 0)\} \\
F(e_2) &= \{(a, 0.6), (b, 0.5), (c, 0.2)\} \\
F(e_3) &= \{(a, 0), (b, 0), (c, 0)\} \\
F(e_4) &= \{(a, 0), (b, 0), (c, 0)\}\}
\end{aligned}
$$

$$
\begin{aligned}
G_B = \{G(e_1) &= \{(a, 0.6), (b, 0.1), (c, 0)\} \\
G(e_2) &= \{(a, 0.7), (b, 0.9), (c, 0.5)\} \\
G(e_3) &= \{(a, 0), (b, 0), (c, 0)\} \\
G(e_4) &= \{(a, 0.5), (b, 0.3), (c, 0.9)\}\}
\end{aligned}
$$

We consider a fuzzy soft topology τ on (U, E) as $\tau = \{\widetilde{\Phi}, \widetilde{E}, F_A, G_B\}$.

Let $Y = \{a, b\} \subseteq U$.

We consider a fuzzy soft set H_E over (Y, E) as

$$
\begin{aligned}
H_E = \{H(e_1) &= \{(a, 1), (b, 1), (c, 0)\} \\
H(e_2) &= \{(a, 1), (b, 1), (c, 0)\} \\
H(e_3) &= \{(a, 1), (b, 1), (c, 0)\} \\
H(e_4) &= \{(a, 1), (b, 1), (c, 0)\}\}
\end{aligned}
$$

Then

i. $\widetilde{\Phi} \, \widetilde{\cap} \, H_E = \widetilde{\Phi}$

ii. $\widetilde{E} \, \widetilde{\cap} \, H_E = \widetilde{E}$

iii. $F_A \, \widetilde{\cap} \, H_E = F_A = \{$
$$
\begin{aligned}
F(e_1) &= \{(a, 0.4), (b, 0.1), (c, 0)\} \\
F(e_2) &= \{(a, 0.6), (b, 0.5), (c, 0.2)\} \\
F(e_3) &= \{(a, 0), (b, 0), (c, 0)\} \\
F(e_4) &= \{(a, 0), (b, 0), (c, 0)\}\}
\end{aligned}
$$

iv. $G_B \, \widetilde{\cap} \, H_E = G_B = \{$
$$
\begin{aligned}
G(e_1) &= \{(a, 0.6), (b, 0.1), (c, 0)\} \\
G(e_2) &= \{(a, 0.7), (b, 0.9), (c, 0.5)\} \\
G(e_3) &= \{(a, 0), (b, 0), (c, 0)\} \\
G(e_4) &= \{(a, 0.5), (b, 0.3), (c, 0.9)\}\}
\end{aligned}
$$

Thus the collection $T_Y = \{H_E \, \widetilde{\cap} \, G_B : G_B \in \tau\}$ is a fuzzy soft topology on (Y, E).