

Chapter 4

LMI Conditions for Delay-Probability-Distribution-Dependent Robust Stability Analysis of Markovian Jump Stochastic Neural Networks with Time-Varying Delays

4.1 Introduction

In recent years, dynamics of neural networks have been widely studied due to their extensive applications in aerospace, defense, robotic, telecommunications, signal processing, pattern recognition, static image processing, associative memory and combinatorial optimization, Haykin (1998). During the implementation of artificial neural networks, time delays often arise in the processing of information storage and transmission. Some of these applications require the equilibrium points of designed networks to be stable, see Arik (2005), Gao et al. (2008), Li (2002, 2008 b, 2009 a) and Zhao (2009). Furthermore, time delay is frequently a source of oscillation, divergence, or even instability and deterioration of neural networks. Generally speaking, the so-far obtained stability results for delayed neural networks can be classified into two types; that is, delay-independent stability as in Arik (2004), Chen Rong (2003) and Singh (2005) and delay-dependent stability as in Chen and Wu (2009), Cho and Park (2007) and Liao et al. (2002 a). Since delay-dependent methods make use of information on the length of delays, they are generally less conservative than delay-independent ones. However, in a real system, time delay often exists in a random form, that is, if some values of the time delay are very large but the probability of the delay taking such large values is very small, it may lead to a more conservative result if only the information of variation range of the time delay is

considered. In addition, its probabilistic characteristic, such as the Bernoulli distribution and the Poisson distribution, can also be obtained by statistical methods. Therefore, it is necessary and realizable to investigate the probability-distribution delay. Recently, the stability of discrete neural networks and discrete stochastic neural networks with probability-distribution delay are investigated in Yue et al. (2008) and Zhang (2010 a) respectively. In addition, the problem of neural networks with probability-distribution delay is investigated in Yang et al. (2009). But neither of them considers the information of the delay derivative. Recently Fu et al. (2009) studied the delay-probability-distribution-dependent robust stability analysis of stochastic neural networks with time-varying delays and the information of the delay derivative have been taken into account.

As time delays, there are two types of disturbances that is, parametric uncertainty and stochastic perturbations. First, uncertainties are frequently encountered in various engineering and communication systems. The characteristics of dynamic systems are significantly affected by the presence of the uncertainty, even to the extend of instability in extreme situation. The desired stability properties of neural networks are customarily based on imposing constraint conditions on the network parameters neural system. It is desired that the stability properties of neural networks should be affected by the small deviations in the values of the parameters. In other words, the neural networks must be globally robust stable, refer Wenwu and Cao (2007). Next, in real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes, as stated in Blythe et al.(2001), Chen et al.(2009 a) and Haykin (1998). Practically, the stochastic phenomenon usually appear in the electrical circuit design of neural networks, see Haykin (1998). Recently, some results on stability of stochastic neural networks with time varying delays have been reported in Chen and Lu (2008), Feng et al. (2009 b), Huang and Feng (2008), Huang and Cao (2007), Huang et al. (2009), Li et al (2008 a), Li and Cao (2007),Liu et al. (2008), Su and Chen (2009),Wa et al. (2009),Yu et al. (2009),Yang et al. (2009)and Zhang et al (2007 a, 2008 b,2009 a). To the best of authors knowledge, very few authors have studied the delay-probability-distribution-dependent robust stability analysis of stochastic neural networks with time-varying delays, which is very important in both theories and applications and also is a very challenging problem.

Modern industrial applications are come upon with numerous hybrid behavior of the processes. For example, any malfunction of sensors or actuators can cause a jumping behaviour in process performance. This type of jumping behavior may be

modelled as a Markov jump systems. In other words, the neural networks may have finite modes and the mode may jump from one to another at different times. The jumping between different modes can be governed by a Markov chain as in Huang and Du (2013), Yi et al.(2013), Zhu and Cao (2012) and Zhang and Yu (2012). Thus, Markovian jump systems correspond to an important class of systems that are subject to abrupt process changes. The abrupt changes in the systems are discrete events and are assumed to be modelled by a Markov chain taking values in a finite value set. Practical motivations as well as many theoretical results for Markovian jump system can be found, as in Li et al. (2011), Wu et al.(2013) and Zheng et al.(2013). More recently, Luo et al. (2013) studied the Robust fault detection of Markovian jump systems with different system modes. Wang et al. (2013) investigated the Delay-dependent H_∞ control for singular Markovian jump systems with time delay. Therefore, neural networks with Markovian jump parameters have received a great deal of attention. So studies of the stability criteria and the performance for Markovian jump systems with delays are more important to theoretical and practical applications.

Motivated by the above discussions, LMI conditions for delay probability distribution dependent robust stability analysis of stochastic neural networks with time-varying delays are considered in this chapter. The information of delay probability distribution is introduced into the stochastic neural networks model and a new method is proposed to remove the constraint on the upper bound of the delay derivative. Delay-dependent stability criteria are presented such that the stochastic neural networks with probability distribution delay is robustly globally asymptotically stable in the mean square for all admissible parameter uncertainties. By constructing a novel Lyapunov-Krasovskii functional, employing some analysis techniques and introducing some free-weighting matrices, sufficient conditions are derived for the considered stochastic neural networks in terms of LMIs, which can be easily calculated by MATLAB LMI control Toolbox. Numerical examples are given to illustrate the effectiveness and less conservativeness of the proposed method.

Notations: Throughout this chapter, \mathbb{R}^n and $\mathbb{R}^{n \times n}$ denote respectively, the n -dimensional Euclidean space and the set of all $n \times n$ real matrices. The superscript T denotes the transposition and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I_n is the $n \times n$ identity matrix. $|\cdot|$ is the Euclidean norm in \mathbb{R}^n . Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. That is the filtration contains all

P-null sets and is right continuous. The notation $*$ always denotes the symmetric block in one symmetric matrix. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

4.2 Problem Description and Preliminaries

Let $\{r(t), t \geq 0\}$ is a right-continuous Markov chain on the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ taking values in a finite state space $\mathcal{S} = \{1, 2, \dots, N\}$ with generator $\mathcal{Q} = (q_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$, $q_{ij} \geq 0$ is the transition rate from i to j , if $i \neq j$ while $q_{ii} = -\sum_{j \neq i} q_{ij}$.

In this chapter, the following uncertain Markovian jump stochastic Hopfield neural networks with time-varying delays are considered:

$$\begin{aligned} dx(t) &= \left[-A_i(t)x(t) + B_i(t)f(x(t)) + W_i(t)f(x(t - \tau(t))) \right] dt \\ &\quad + \left[H_{0i}x(t) + H_{1i}x(t - \tau(t)) \right] dw(t) \\ x(t) &= \phi(t), \quad \forall t \in [-\bar{\tau}, 0], \end{aligned} \tag{4.2.1}$$

where $x(t) \in \mathbb{R}^n$ is the neural state vector, $f(x(t)) = [f_1(x_1(t)), \dots, f_n(x_n(t))]^T \in \mathbb{R}^n$ is the neuron activation function vector with initial condition $f(0) = 0$. $w(t) = [w_1(t), \dots, w_n(t)] \in \mathbb{R}^n$ is an n -dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t \geq 0}, P)$.

The time-varying delays $\tau(t)$ satisfies

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \dot{\tau}(t) \leq \mu, \tag{4.2.2}$$

where $\bar{\tau}$ and μ are constants. In (4.2.1), $A_i(t) = A_i + \Delta A_i(t)$, $B_i(t) = B_i + \Delta B_i(t)$, $W_i(t) = W_i + \Delta W_i(t)$ and where the diagonal matrices $A_i > 0$ ($i = 1, 2, \dots, N$) and B_i, W_i, H_{0i}, H_{1i} are connection weight matrices with appropriate dimensions. Further $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta W_i(t)$, denote the time-varying and norm-bounded uncertainties.

Assumption 1: Considering the information of probability distribution of the time delay $\tau(t)$, two sets and functions are defined

$$\Omega_1 = \{t : \tau(t) \in [0, \tau_0)\} \quad \text{and} \quad \Omega_2 = \{t : \tau(t) \in [\tau_0, \bar{\tau}]\}$$

$$\begin{aligned} \tau_1(t) &= \begin{cases} \tau(t), & \text{for } t \in \Omega_1 \\ \bar{\tau}_1, & \text{for } t \in \Omega_2, \end{cases} \\ \text{and } \tau_2(t) &= \begin{cases} \tau(t), & \text{for } t \in \Omega_2 \\ \bar{\tau}_2, & \text{for } t \in \Omega_1, \end{cases} \end{aligned} \quad (4.2.3)$$

$$\dot{\tau}_1(t) \leq \mu_1 < \infty, \quad \dot{\tau}_2(t) \leq \mu_2 < \infty \quad (4.2.4)$$

where $\tau_0 \in [0, \bar{\tau}]$, $\bar{\tau}_1 \in [0, \tau_0)$ and $\bar{\tau}_2 \in [\tau_0, \bar{\tau}]$. It is easy to know $t \in \Omega_1$ means the event $\tau(t) \in [0, \tau_0)$ occurs and $t \in \Omega_2$ means the event $\tau(t) \in [\tau_0, \bar{\tau}]$ occurs. Therefore, a stochastic variable $\alpha(t)$ can be defined as

$$\alpha(t) = \begin{cases} 1, & \text{for } t \in \Omega_1 \\ 0, & \text{for } t \in \Omega_2. \end{cases} \quad (4.2.5)$$

Assumption 2: $\alpha(t)$ is a Bernoulli distributed sequence with

$\text{Prob}\{\alpha(t) = 1\} = \mathbb{E}\{\alpha(t)\} = \alpha_0$, $\text{Prob}\{\alpha(t) = 0\} = 1 - \mathbb{E}\{\alpha(t)\} = 1 - \alpha_0$, where $0 \leq \alpha_0 \leq 1$ is a constant and $\mathbb{E}\{\alpha(t)\}$ is the expectation of $\alpha(t)$.

Remark 4.2.1. From Assumption 2, it is easy to know that

$$\mathbb{E}\{\alpha(t) - \alpha_0\} = 0, \quad \mathbb{E}\{(\alpha(t) - \alpha_0)^2\} = \alpha_0(1 - \alpha_0).$$

By Assumptions 1 and 2, the system (4.2.1) can be rewritten as

$$\begin{aligned} dx(t) &= \left(-A_i(t)x(t) + B_i(t)f(x(t)) + \alpha(t)W_i(t)f(x(t - \tau_1(t))) \right. \\ &\quad \left. + (1 - \alpha(t))W_i(t)f(x(t - \tau_2(t))) \right) dt \\ &\quad + \left(H_{0i}x(t) + \alpha(t)H_{1i}x(t - \tau_1(t)) + (1 - \alpha(t))H_{1i}x(t - \tau_2(t)) \right) dw(t) \end{aligned} \quad (4.2.6)$$

$$x(t) = \xi(t), \quad t \in [-\bar{\tau}, 0],$$

which is equivalent to

$$\begin{aligned}
dx(t) &= \left(-A_i(t)x(t) + B_i(t)f(x(t)) + \alpha_0 W_i(t)f(x(t - \tau_1(t))) \right. \\
&\quad \left. + (1 - \alpha_0)W_i(t)f(x(t - \tau_2(t))) \right. \\
&\quad \left. + (\alpha(t) - \alpha_0)(W_i(t)f(x(t - \tau_1(t))) - W_i(t)f(x(t - \tau_2(t)))) \right) dt \\
&\quad + \left(H_{0i}x(t) + \alpha_0 H_{1i}x(t - \tau_1(t)) \right. \\
&\quad \left. + (1 - \alpha_0)H_{1i}x(t - \tau_2(t)) + (\alpha(t) - \alpha_0)(H_{1i}x(t - \tau_1(t)) - H_{1i}x(t - \tau_2(t))) \right) dw(t) \\
x(t) &= \xi(t), \quad t \in [-\bar{\tau}, 0].
\end{aligned} \tag{4.2.7}$$

Remark 4.2.2. *The probability distribution of the delay taking values in some interval is assumed to be known in advance in this chapter, and then a new model of the markovian jump stochastic neural networks (4.2.7) has been derived, which can be seen as an extension of the common markovian jump stochastic neural networks (4.2.1). Specially, when $\alpha(t) \equiv 1$, system (4.2.7) becomes system (4.2.1). When the probability of time delay taking values is known a priori, the possible values that the delay takes may be larger than those obtained based on the traditional methods, which will be illustrated via example in section 4.*

Assumption 3: The neural activation function $f_i(x_i)$ satisfies

$$l_i^- \leq \frac{f_i(x_i) - f_i(y_i)}{x_i - y_i} \leq l_i^+ \quad \forall x_i, y_i \in \mathbb{R}, \quad x_i \neq y_i, i = 1, \dots, n \tag{4.2.8}$$

which implies that

$$(f_i(x_i) - l_i^+ x_i)(f_i(x_i) - l_i^- x_i) \leq 0, \tag{4.2.9}$$

where l_i^-, l_i^+ are some constant.

The parameter uncertainties $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta W_i(t)$ are of the forms

$$\begin{bmatrix} \Delta A_i(t) & \Delta B_i(t) & \Delta W_i(t) \end{bmatrix} = H_i F_i(t) \begin{bmatrix} E_{1i} & E_{2i} & E_{3i} \end{bmatrix}, \tag{4.2.10}$$

where H_i , E_{1i} , E_{2i} and E_{3i} are given known matrices. $F_i(t)$ is an uncertain matrix satisfying

$$F_i^T(t)F_i(t) \leq I. \tag{4.2.11}$$

Definition 4.2.1. For system (4.2.7) and any $\xi \in L^2_{\mathcal{F}_0}([-\bar{\tau}, 0]; \mathbb{R}^n)$, the trivial solution is robustly, globally, asymptotically stable in the mean-square sense for all admissible uncertainties, if

$$\lim_{t \rightarrow \infty} \mathbb{E}|x(t, \xi)|^2 = 0.$$

To obtain the main results of this chapter, the following lemmas will be essential for proofs.

Lemma 4.2.1. (Schur Complement) Boyd et al. (1994); Given constant matrices Ω_1 , Ω_2 and Ω_3 with appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$, the inequality

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0,$$

holds, if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

Lemma 4.2.2. Zhang et al.(2008) For any constant matrix $M > 0$, any scalars a and b with $a < b$ and a vector function $x(t) : [a, b] \rightarrow \mathbb{R}^n$ such that the integrals concerned are well defined, the following holds

$$\left[\int_a^b x(s) ds \right]^T M \left[\int_a^b x(s) ds \right] \leq (b - a) \left[\int_a^b x(s)^T M x(s) ds \right].$$

Lemma 4.2.3. Xie et al.(1992) Let $U, V(t), W$ and Z be real matrices of appropriate dimensions with Z satisfying $Z = Z^T$, then

$$Z + UV(t)W + W^T V^T(t)U^T < 0, \quad V^T(t)V(t) \leq I$$

if and only if there exists a scalar $\epsilon > 0$ such that

$$Z + \epsilon^{-1}UU^T + \epsilon W^T W < 0,$$

4.3 Main Results

Defining two new state variables for the Markovian jump stochastic neural networks (4.2.7),

$$y(t) = -A_i(t)x(t) + B_i(t)f(x(t)) + \alpha_0 W_i(t)f(x(t - \tau_1(t))) + (1 - \alpha_0)W_i(t)f(x(t - \tau_2(t))) \\ + (\alpha(t) - \alpha_0)[W_i(t)f(x(t - \tau_1(t))) - W_i(t)f(x(t - \tau_2(t)))], \quad (4.3.1)$$

and

$$g(t) = H_{0i}x(t) + \alpha_0 H_{1i}(x(t - \tau_1(t))) + (1 - \alpha_0)H_{1i}x(t - \tau_2(t)) \\ + (\alpha(t) - \alpha_0)[H_{1i}x(t - \tau_1(t)) - H_{1i}x(t - \tau_2(t))], \quad (4.3.2)$$

the stochastic neural networks (4.2.7) can be written as

$$dx(t) = y(t)dt + g(t)d\omega(t). \quad (4.3.3)$$

Moreover, the following equality holds,

$$x(t) - x(t - \tau(t)) = \int_{t-\tau(t)}^t y(s)ds + \int_{t-\tau(t)}^t g(s)d\omega(s). \quad (4.3.4)$$

Theorem 4.3.1. *For given scalars $\tau_0 \geq 0$, $\bar{\tau}_0 > 0$, μ_1 , $0 < \alpha_0 < 1$ satisfying $\alpha_0 \mu_1 < 1$, the markovian jump stochastic neural networks (4.2.7) without uncertain parameters is asymptotically stable in the mean square if there exist matrices $P_i > 0$, $Q_i > 0$, $i = 1, 2, 3$, $R_l > 0$, $Z_l > 0$, $l = 1, 2$, for any matrices $N_k, M_k, S_k, U_k, V_k, Y_k$ ($k = 1, 2$) and there exist positive diagonal matrices $K_1 > 0$,*

$K_2 > 0$ and $K_3 > 0$ such that the following LMIs are feasible

$$\Xi_1 = \begin{bmatrix} \Pi & -\alpha_0\tau_0 M \\ * & -\alpha_0\tau_0 Z_1 \end{bmatrix} < 0, \quad (4.3.5)$$

$$\Xi_2 = \begin{bmatrix} \Pi & -\tau_0(1 - \alpha_0)N \\ * & -\tau_0(1 - \alpha_0)Z_1 \end{bmatrix} < 0, \quad (4.3.6)$$

$$\Xi_3 = \begin{bmatrix} \Pi & -\tau_0 S \\ * & -\tau_0 Z_1 \end{bmatrix} < 0, \quad (4.3.7)$$

$$\Xi_4 = \begin{bmatrix} \Pi & -(\bar{\tau} - \tau_0)U \\ * & -(\bar{\tau} - \tau_0)Z_2 \end{bmatrix} < 0, \quad (4.3.8)$$

$$\Xi_5 = \begin{bmatrix} \Pi & -(\bar{\tau} - \tau_0)V \\ * & -(\bar{\tau} - \tau_0)Z_2 \end{bmatrix} < 0, \quad (4.3.9)$$

where

$$H = \begin{bmatrix} \tilde{\Omega} & M & N & S & U & V & \Gamma \\ * & -Z_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & -Z_1 & 0 & 0 & 0 \\ * & * & * & * & -Z_2 & 0 & 0 \\ * & * & * & * & * & -Z_2 & 0 \\ * & * & * & * & * & * & -\bar{P} \end{bmatrix}, \quad (4.3.10)$$

and $\tilde{\Omega} = (\tilde{\Omega}_{i,j})_{10 \times 10}$

with

$$\begin{aligned} \tilde{\Omega}_{1,1} &= Q_1 + Q_2 + Q_3 + \sum_{j=1}^N q_{ij} P_j + M_1 + M_1^T - Y_1 A_i - A_i^T Y_1^T - K_1 L_1, \tilde{\Omega}_{1,2} = -M_1 + M_2^T, \\ \tilde{\Omega}_{1,3} &= 0, \tilde{\Omega}_{1,4} = 0, \tilde{\Omega}_{1,5} = 0, \tilde{\Omega}_{1,6} = 0, \tilde{\Omega}_{1,7} = -Y_1 - A_i^T Y_2^T + P_i, \tilde{\Omega}_{1,8} = Y_1 B_i + K_1 L_2, \\ \tilde{\Omega}_{1,9} &= \alpha_0 Y_1 W_i, \tilde{\Omega}_{1,10} = (1 - \alpha_0) Y_1 W_i, \tilde{\Omega}_{2,2} = -(1 - \alpha_0 \mu_1) Q_1 - M_2 - M_2^T + N_1 + N_1^T, \\ \tilde{\Omega}_{2,3} &= -N_1 + N_2^T, \tilde{\Omega}_{2,4} = 0, \tilde{\Omega}_{2,5} = 0, \tilde{\Omega}_{2,6} = 0, \tilde{\Omega}_{2,7} = 0, \tilde{\Omega}_{2,8} = 0, \tilde{\Omega}_{2,9} = 0, \\ \tilde{\Omega}_{2,10} &= 0, \tilde{\Omega}_{3,3} = -N_2 - N_2^T + S_1 + S_1^T - K_2 L_1 + \alpha_0 (1 - \alpha_0) H_{1i}^T \bar{P} H_{1i}, \tilde{\Omega}_{3,4} = -S_1 + S_2^T, \\ \tilde{\Omega}_{3,5} &= -\alpha_0 (1 - \alpha_0) H_{1i}^T \bar{P} H_{1i}, \tilde{\Omega}_{3,6} = 0, \tilde{\Omega}_{3,7} = 0, \tilde{\Omega}_{3,8} = 0, \tilde{\Omega}_{3,9} = K_2 L_2, \tilde{\Omega}_{3,10} = 0, \\ \tilde{\Omega}_{4,4} &= -Q_2 - S_2 - S_2^T + U_1 + U_1^T, \tilde{\Omega}_{4,5} = -U_1 + U_2^T, \tilde{\Omega}_{4,6} = 0, \tilde{\Omega}_{4,7} = 0, \tilde{\Omega}_{4,8} = 0, \end{aligned}$$

$$\begin{aligned}
\tilde{\Omega}_{4,9} &= 0, \quad \tilde{\Omega}_{4,10} = 0, \quad \tilde{\Omega}_{5,5} = -U_2 - U_2^T + V_1 + V_1^T - K_3 L_1 + \alpha_0(1 - \alpha_0) H_{1i}^T \bar{P} H_{1i}, \\
\tilde{\Omega}_{5,6} &= -V_1 + V_2^T, \quad \tilde{\Omega}_{5,7} = 0, \quad \tilde{\Omega}_{5,8} = 0, \quad \tilde{\Omega}_{5,9} = 0, \quad \tilde{\Omega}_{5,10} = K_3 L_2, \\
\tilde{\Omega}_{6,6} &= -Q_3 - V_2 - V_2^T, \quad \tilde{\Omega}_{6,7} = 0, \quad \tilde{\Omega}_{6,8} = 0, \quad \tilde{\Omega}_{6,9} = 0, \\
\tilde{\Omega}_{6,10} &= 0, \quad \tilde{\Omega}_{7,7} = \tau_0 R_1 + (\bar{\tau} - \tau_0) R_2 - Y_2 - Y_2^T, \quad \tilde{\Omega}_{7,8} = Y_2 B_i, \quad \tilde{\Omega}_{7,9} = \alpha_0 Y_2 W_i, \\
\tilde{\Omega}_{7,10} &= (1 - \alpha_0) Y_2 W_i, \quad \tilde{\Omega}_{8,8} = -K_1 I, \quad \tilde{\Omega}_{8,9} = \tilde{\Omega}_{8,10} = 0, \quad \tilde{\Omega}_{9,9} = -K_2 I, \quad \tilde{\Omega}_{10,10} = -K_3 I,
\end{aligned}$$

$$\begin{aligned}
M &= \begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
N &= \begin{bmatrix} 0 & N_1^T & N_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
S &= \begin{bmatrix} 0 & 0 & S_1^T & S_2^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
U &= \begin{bmatrix} 0 & 0 & 0 & U_1^T & U_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
V &= \begin{bmatrix} 0 & 0 & 0 & 0 & V_1^T & V_2^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
Y &= \begin{bmatrix} Y_1^T & 0 & 0 & 0 & 0 & 0 & Y_2^T & 0 & 0 & 0 \end{bmatrix}^T, \\
\Gamma^T &= [\bar{P} H_{0i} 0 \quad \alpha_0 \bar{P} H_{1i} \quad 0 \quad (1 - \alpha_0) \bar{P} H_{1i} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \\
\bar{P} &= \tau_0 Z_1 + (\bar{\tau} - \tau_0) Z_2 + P_i.
\end{aligned}$$

Proof. Consider the Lyapunov-Krasovskii functional

$$V(x_t, t) = V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t),$$

where

$$\begin{aligned}
V_1(x_t, t) &= x^T(t) P_i x(t), \\
V_2(x_t, t) &= \int_{t-\alpha_0 \tau_1(t)}^t x^T(s) Q_1 x(s) ds + \int_{t-\tau_0}^t x^T(s) Q_2 x(s) ds + \int_{t-\bar{\tau}}^t x^T(s) Q_3 x(s) ds, \\
V_3(x_t, t) &= \int_{-\tau_0}^0 \int_{t+\theta}^t y^T(s) R_1 y(s) ds d\theta + \int_{-\bar{\tau}}^{-\tau_0} \int_{t+\theta}^t y^T(s) R_2 y(s) ds d\theta, \\
&\quad + \int_{-\tau_0}^0 \int_{t+\theta}^t g^T(s) Z_1 g(s) ds d\theta + \int_{-\bar{\tau}}^{-\tau_0} \int_{t+\theta}^t g^T(s) Z_2 g(s) ds d\theta,
\end{aligned}$$

where $x_t = \{x(t + \theta) : -\bar{\tau} \leq \theta \leq 0\}$. Then, it can be obtained by Ito's formula that

$$dV(x_t, t) = LV(x_t, t)dt + 2x^T(t) P_i g(t)d\omega(t), \quad (4.3.11)$$

where

$$\begin{aligned} LV_1(x_t, t) &= 2x^T(t) P_i y(t) + g^T(t) P_i g(t) + \sum_{j=1}^N q_{ij} x^T(t) P_j x(t), \\ LV_2(x_t, t) &\leq x^T(t) Q_1 x(t) - (1 - \alpha_0 \mu_1) x^T(t - \alpha_0 \tau_1(t)) Q_1 x(t - \alpha_0 \tau_1(t)) + x^T(t) Q_2 x(t) \\ &\quad - x^T(t - \tau_0) Q_2 x(t - \tau_0) + x^T(t) Q_3 x(t) - x^T(t - \bar{\tau}) Q_3 x(t - \bar{\tau}) \\ LV_3(x_t, t) &= \tau_0 y^T(t) R_1 y(t) - \int_{t-\tau_0}^t y^T(s) R_1 y(s) ds + (\bar{\tau} - \tau_0) y^T(t) R_2 y(t) - \int_{t-\bar{\tau}}^{t-\tau_0} y^T(s) R_2 y(s) ds \\ &\quad + \tau_0 g^T(t) Z_1 g(t) - \int_{t-\tau_0}^t g^T(s) Z_1 g(s) ds + (\bar{\tau} - \tau_0) g^T(t) Z_2 g(t) - \int_{t-\bar{\tau}}^{t-\tau_0} g^T(s) Z_2 g(s) ds. \end{aligned}$$

$$\begin{aligned} &LV_3(x_t, t) \\ \leq &y^T(t) (\tau_0 R_1 + (\bar{\tau} - \tau_0) R_2) y(t) - \int_{t-\alpha_0 \tau_1(t)}^t y^T(s) R_1 y(s) ds \\ &- \int_{t-\tau_1(t)}^{t-\alpha_0 \tau_1(t)} y^T(s) R_1 y(s) ds - \int_{t-\tau_2(t)}^{t-\tau_1(t)} y^T(s) R_2 y(s) ds \\ &- \int_{t-\bar{\tau}}^{t-\tau_2(t)} y^T(s) R_2 y(s) ds + g^T(t) (\tau_0 Z_1 + (\bar{\tau} - \tau_0) Z_2) g(t) - \int_{t-\alpha_0 \tau_1(t)}^t g^T(s) Z_1 g(s) ds \\ &- \int_{t-\tau_1(t)}^{t-\alpha_0 \tau_1(t)} g^T(s) Z_1 g(s) ds - \int_{t-\tau_0}^{t-\tau_1(t)} g^T(s) Z_1 g(s) ds - \int_{t-\tau_2(t)}^{t-\tau_0} g^T(s) Z_2 g(s) ds \\ &- \int_{t-\bar{\tau}}^{t-\tau_2(t)} g^T(s) Z_2 g(s) ds. \end{aligned}$$

From (4.2.9), for any matrices $K_i = \text{diag}(k_{i1}, k_{i2}, \dots, k_{in}) \geq 0, i = 1, 2, 3$, it is

easy to obtain

$$\begin{aligned}
& - \sum_{j=1}^n k_{1j} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- e_j e_j^T & -\frac{l_j^+ + l_j^-}{2} e_j e_j^T \\ -\frac{l_j^+ + l_j^-}{2} e_j e_j^T & e_j e_j^T \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \\
& - \sum_{i=1}^2 \sum_{j=1}^n k_{(i+1)j} \begin{bmatrix} x(t - \tau_i(t)) \\ f(x(t - \tau_i(t))) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- e_j e_j^T & -\frac{l_j^+ + l_j^-}{2} e_j e_j^T \\ -\frac{l_j^+ + l_j^-}{2} e_j e_j^T & e_j e_j^T \end{bmatrix} \begin{bmatrix} x(t - \tau_i(t)) \\ f(x(t - \tau_i(t))) \end{bmatrix} \\
& = \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} -K_1 L_1 & K_1 L_2 \\ K_1 L_2 & -K_1 \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} + \sum_{i=1}^2 \begin{bmatrix} x(t - \tau_i(t)) \\ f(x(t - \tau_i(t))) \end{bmatrix}^T \\
& \begin{bmatrix} -K_{i+1} L_1 & K_{i+1} L_2 \\ K_{i+1} L_2 & -K_{i+1} \end{bmatrix} \begin{bmatrix} x(t - \tau_i(t)) \\ f(x(t - \tau_i(t))) \end{bmatrix} \geq 0, \tag{4.3.12}
\end{aligned}$$

where $L_1 = \text{diag}(l_1^+ l_1^-, \dots, l_n^+ l_n^-)$ and $L_2 = \text{diag}(\frac{l_1^+ + l_1^-}{2}, \dots, \frac{l_n^+ + l_n^-}{2})$ are matrices of appropriate dimensions. Now, we define the new vector

$$\begin{aligned}
\xi^T(t) = & \begin{bmatrix} x^T(t) & x^T(t - \alpha_0 \tau_1(t)) & x^T(t - \tau_1(t)) & x^T(t - \tau_0) & x^T(t - \tau_2(t)) & x^T(t - \bar{\tau}) & y^T(t) \\ f^T(x(t)) & f^T(x(t - \tau_1(t))) & f^T(x(t - \tau_2(t))) \end{bmatrix}.
\end{aligned}$$

From (4.3.1), (4.3.2) and (4.3.4), one can see that the following equations hold for any matrices M, N, S, U, V and Y with appropriate dimensions,

$$2\xi^T(t)M \left[x(t) - x(t - \alpha_0 \tau_1(t)) - \int_{t - \alpha_0 \tau_1(t)}^t y(s) ds - \int_{t - \alpha_0 \tau_1(t)}^t g(s) d\omega(s) \right] = 0, \tag{4.3.13}$$

$$2\xi^T(t)N \left[x(t - \alpha_0 \tau_1(t)) - x(t - \tau_1(t)) - \int_{t - \tau_1(t)}^{t - \alpha_0 \tau_1(t)} y(s) ds - \int_{t - \tau_1(t)}^{t - \alpha_0 \tau_1(t)} g(s) d\omega(s) \right] = 0, \tag{4.3.14}$$

$$2\xi^T(t)S \left[x(t - \tau_1(t)) - x(t - \tau_0) - \int_{t-\tau_0}^{t-\tau_1(t)} y(s)ds - \int_{t-\tau_0}^{t-\tau_1(t)} g(s)d\omega(s) \right] = 0, \quad (4.3.15)$$

$$2\xi^T(t)U \left[x(t - \tau_0) - x(t - \tau_2(t)) - \int_{t-\tau_2(t)}^{t-\tau_0} y(s)ds - \int_{t-\tau_2(t)}^{t-\tau_0} g(s)d\omega(s) \right] = 0, \quad (4.3.16)$$

$$2\xi^T(t)V \left[x(t - \tau_2(t)) - x(t - \bar{\tau}) - \int_{t-\bar{\tau}}^{t-\tau_2(t)} y(s)ds - \int_{t-\bar{\tau}}^{t-\tau_2(t)} g(s)d\omega(s) \right] = 0, \quad (4.3.17)$$

$$2\xi^T(t)Y \left[-A_i(t)x(t) + B_i(t)f(x(t)) + \alpha_0 W_i(t)f(x(t - \tau_1(t))) + (1 - \alpha_0)W_i(t)f(x(t - \tau_2(t))) \right. \\ \left. + (\alpha(t) - \alpha_0)[W_i(t)f(x(t - \tau_1(t))) - W_i(t)f(x(t - \tau_2(t)))] - y(t) \right]. \quad (4.3.18)$$

From the above equations (4.3.13)-(4.3.17), we have

$$-2\xi^T(t)M \int_{t-\alpha_0\tau_1(t)}^t g(s)d\omega(s) \leq \xi^T(t)MZ_1^{-1}M^T\xi(t) \quad (4.3.19)$$

$$+ \int_{t-\alpha_0\tau(t)}^t g^T(s)d\omega(s)Z_1 \int_{t-\alpha_0\tau(t)}^t g(s)d\omega(s),$$

$$-2\xi^T(t)N \int_{t-\tau_1}^{t-\alpha_0\tau_1(t)} g(s)d\omega(s) \leq \xi^T(t)NZ_1^{-1}N^T\xi(t) \\ + \int_{t-\tau_1}^{t-\alpha_0\tau_1(t)} g^T(s)d\omega(s), Z_1 \int_{t-\tau_1}^{t-\alpha_0\tau_1(t)} g(s)d\omega(s) \quad (4.3.20)$$

$$-2\xi^T(t)S \int_{t-\tau_0}^{t-\tau_1(t)} g(s)d\omega(s) \leq \xi^T(t)SZ_1^{-1}S^T\xi(t) \quad (4.3.21)$$

$$+ \int_{t-\tau_0}^{t-\tau_1(t)} g^T(s)d\omega(s)Z_1 \int_{t-\tau_0}^{t-\tau_1(t)} g(s)d\omega(s),$$

$$-2\xi^T(t)U \int_{t-\tau_2(t)}^{t-\tau_0} g(s)d\omega(s) \leq \xi^T(t)UZ_2^{-1}U^T\xi(t) \\ + \int_{t-\tau_2(t)}^{t-\tau_0} g^T(s)d\omega(s)Z_2 \int_{t-\tau_2(t)}^{t-\tau_0} g(s)d\omega(s) \quad (4.3.22)$$

$$-2\xi^T(t)V \int_{t-\bar{\tau}}^{t-\tau_2(t)} g(s)d\omega(s) \leq \xi^T(t)VZ_2^{-1}V^T\xi(t) \quad (4.3.23)$$

$$+ \int_{t-\bar{\tau}}^{t-\tau_2(t)} g^T(s)d\omega(s)Z_2 \int_{t-\bar{\tau}}^{t-\tau_2(t)} g(s)d\omega(s).$$

By Remark 4.2.1 , it is easy to derive the following equality

$$\begin{aligned}
& \mathbb{E}\{g^T(t) \left(P_i + \tau_0 Z_1 + (\bar{\tau} - \tau_0) Z_2 \right) g(t)\} \\
= & \mathbb{E}\left\{ [H_{0i}x(t) + \alpha_0 H_{1i}x(t - \tau_1(t)) + (1 - \alpha_0) H_{1i}x(t - \tau_2(t))]^T \right. \\
& \times \bar{P} [H_{0i}x(t) + \alpha_0 H_{1i}x(t - \tau_1(t)) + (1 - \alpha_0) H_{1i}x(t - \tau_2(t))] \\
& + 2(\alpha(t) - \alpha_0) [H_{0i}x(t) + \alpha_0 H_{1i}x(t - \tau_1(t)) + (1 - \alpha_0) H_{1i}x(t - \tau_2(t))]^T \\
& \times \bar{P} [H_{1i}x(t - \tau_1(t)) - Dx(t - \tau_2(t))] + (\alpha(t) - \alpha_0)^2 \\
& \times [H_{1i}x(t - \tau_1(t)) - Dx(t - \tau_2(t))]^T \bar{P} [H_{1i}x(t - \tau_1(t)) \\
& \left. - H_{1i}x(t - \tau_2(t))] \right\} \\
= & [H_{0i}x(t) + \alpha_0 H_{1i}x(t - \tau_1(t)) \\
& + (1 - \alpha_0) H_{1i}x(t - \tau_2(t))]^T \bar{P} [H_{0i}x(t) + \alpha_0 H_{1i}x(t - \tau_1(t)) \\
& + (1 - \alpha_0) H_{1i}x(t - \tau_2(t))] + \alpha_0(1 - \alpha_0) [H_{1i}x(t - \tau_1(t)) \\
& - H_{1i}x(t - \tau_2(t))]^T \bar{P} [H_{1i}x(t - \tau_1(t)) - H_{1i}x(t - \tau_2(t))]. \tag{4.3.24}
\end{aligned}$$

Since

$$\mathbb{E}\left\{ \int_{t-\alpha_0\tau_1(t)}^t g^T(s) d\omega(s) Z_1 \int_{t-\alpha_0\tau_1(t)}^t g(s) d\omega(s) \right\} = \mathbb{E}\left\{ \int_{t-\alpha_0\tau_1(t)}^t g^T(s) Z_1 g(s) ds \right\}, \tag{4.3.25}$$

$$\mathbb{E}\left\{ \int_{t-\tau_1(t)}^{t-\alpha_0\tau_1(t)} g^T(s) d\omega(s) Z_1 \int_{t-\tau_1(t)}^{t-\alpha_0\tau_1(t)} g^T(s) d\omega(s) \right\} = \mathbb{E}\left\{ \int_{t-\alpha_0\tau_1(t)}^t g^T(s) Z_1 g(s) ds \right\}, \tag{4.3.26}$$

$$\mathbb{E}\left\{ \int_{t-\tau_0}^{t-\tau_1(t)} g^T(s) d\omega(s) Z_2 \int_{t-\tau_0}^{t-\tau_1(t)} g^T(s) d\omega(s) \right\} = \mathbb{E}\left\{ \int_{t-\tau_0}^{t-\tau_1(t)} g^T(s) Z_2 g(s) ds \right\}, \tag{4.3.27}$$

$$\mathbb{E}\left\{ \int_{t-\tau_2(t)}^{t-\tau_0} g^T(s) d\omega(s) Z_1 \int_{t-\tau_2(t)}^{t-\tau_0} g^T(s) d\omega(s) \right\} = \mathbb{E}\left\{ \int_{t-\tau_2(t)}^{t-\tau_0} g^T(s) Z_2 g(s) ds \right\}, \tag{4.3.28}$$

$$\mathbb{E}\left\{ \int_{t-\bar{\tau}}^{t-\tau_2(t)} g^T(s) d\omega(s) Z_2 \int_{t-\bar{\tau}}^{t-\tau_2(t)} g^T(s) d\omega(s) \right\} = \mathbb{E}\left\{ \int_{t-\bar{\tau}}^{t-\tau_2(t)} g^T(s) Z_2 g(s) ds \right\}. \tag{4.3.29}$$

Then, substituting inequalities (4.3.12)-(4.3.29) into (4.3.11), it is obtained that

$$\begin{aligned}
LV(x_t, t) \leq & -\frac{1}{\alpha_0\tau_0} \int_{t-\alpha_0\tau_1(t)}^t \eta^T(t, s) \begin{bmatrix} \Pi & -\alpha_0\tau_0 M \\ * & -\alpha_0\tau_0 Z_1 \end{bmatrix} \eta(t, s) ds \\
& -\frac{1}{\tau_0(1-\alpha_0)} \int_{t-\tau_1(t)}^{t-\alpha_0\tau_1(t)} \eta^T(t, s) \begin{bmatrix} \Pi & -\tau_0(1-\alpha_0)N \\ * & -\tau_0(1-\alpha_0)Z_1 \end{bmatrix} \eta(t, s) ds \\
& -\frac{1}{\tau_0} \int_{t-\tau_0}^{t-\tau_1(t)} \eta^T(t, s) \begin{bmatrix} \Pi & -\tau_0 S \\ * & -\tau_0 Z_1 \end{bmatrix} \eta(t, s) ds \\
& -\frac{1}{\bar{\tau}-\tau_0} \int_{t-\tau_2(t)}^{t-\tau_0} \eta^T(t, s) \begin{bmatrix} \Pi & -(\bar{\tau}-\tau_0)U \\ * & -(\bar{\tau}-\tau_0)Z_2 \end{bmatrix} \eta(t, s) ds \\
& -\frac{1}{\bar{\tau}-\tau_0} \int_{t-\bar{\tau}}^{t-\tau_2(t)} \eta^T(t, s) \begin{bmatrix} \Pi & -(\bar{\tau}-\tau_0)V \\ * & -(\bar{\tau}-\tau_0)Z_2 \end{bmatrix} \eta(t, s) ds, \quad (4.3.30)
\end{aligned}$$

where $\eta(t, s) = \begin{bmatrix} \xi^T(s) & y^T(s) \end{bmatrix}$ and

$$\Pi = \bar{\Omega} + MZ_1^{-1}M^T + NZ_1^{-1}N^T + SZ_1^{-1}S^T + UZ_2^{-1}U^T + VZ_2^{-1}V^T + \Gamma\bar{P}^{-1}\Gamma^T.$$

Therefore, if (4.3.5)-(4.3.9) are satisfied, (4.3.30) implies that

$$\begin{aligned}
LV(x_t, t) &\leq -\frac{1}{\alpha_0 \tau_0} \int_{t-\alpha_0 \tau_1(t)}^t \lambda \|x(t)\|^2 ds - \frac{1}{\tau_0(1-\alpha_0)} \int_{t-\tau_1(t)}^{t-\alpha_0 \tau_1(t)} \lambda \|x(t)\|^2 ds \\
&\quad - \frac{1}{\tau_0} \int_{t-\tau_0}^{t-\tau_1(t)} \lambda \|x(t)\|^2 ds \\
&\quad - \frac{1}{\bar{\tau} - \tau_0} \int_{t-\tau_2(t)}^{t-\tau_0} \lambda \|x(t)\|^2 ds - \frac{1}{\bar{\tau} - \tau_0} \int_{t-\bar{\tau}}^{t-\tau_2(t)} \lambda \|x(t)\|^2 ds \\
&= -\lambda \|x(t)\|^2
\end{aligned} \tag{4.3.31}$$

where $\lambda = \min\{\lambda_{\min}(\Xi_i)\}, i = 1, \dots, 5$. Taking the expectation of both sides of (4.3.31) yields

$$\mathbb{E}\{LV(x_t, t)\} \leq -\lambda \mathbb{E}\|x(t)\|^2 \tag{4.3.32}$$

which indicates from the Lyapunov stability theory that the stochastic neural networks (4.2.7) is asymptotically stable in the mean square. \square

Remark 4.3.1. *When it is not considered Markov jump parameters, i.e, the Markov chain $\{r(t), t \geq 0\}$ only takes a unique value 1(i.e, $S = \{1\}$), the system (4.2.7) will be reduced to the following time-varying delayed neural networks:*

$$\begin{aligned}
dx(t) &= [-A(t)x(t) + B(t)f(x(t)) + \alpha_0 W(t)f(x(t - \tau_1(t))) + (1 - \alpha_0)W(t)f(x(t - \tau_2(t)))] \\
&\quad + (\alpha(t) - \alpha_0)(W(t)f(x(t - \tau_1(t))) - W(t)f(x(t - \tau_2(t))))]dt \\
&\quad + [H_0 x(t) + \alpha_0 H_1 x(t - \tau_1(t)) + (1 - \alpha_0)H_1 x(t - \tau_2(t))] \\
&\quad + (\alpha(t) - \alpha_0)(H_1 x(t - \tau_1(t)) - H_1 x(t - \tau_2(t)))]dw(t)
\end{aligned} \tag{4.3.33}$$

$$x(t) = \xi(t), \quad t \in [-\bar{\tau}, 0].$$

For system (4.3.33), we have the following result by Theorem 4.3.1.

Theorem 4.3.2. *For given scalars $\tau_0 \geq 0, \bar{\tau}_0 > 0, \mu_1, 0 < \alpha_0 < 1$ satisfying $\alpha_0 \mu_1 < 1$, the stochastic neural networks (4.3.33) without uncertain parameters is*

asymptotically stable in the mean square if there exist matrices $P > 0$, $Q_i > 0, i = 1, 2, 3$, $R_l > 0$, $Z_l > 0, l = 1, 2$, for any matrices $N_k, M_k, S_k, U_k, V_k, Y_k (k = 1, 2)$ and there exist positive diagonal matrices $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ such that the following LMIs are feasible

$$\Xi_1 = \begin{bmatrix} \Pi & -\alpha_0 \tau_0 M \\ * & -\alpha_0 \tau_0 Z_1 \end{bmatrix} < 0, \quad (4.3.34)$$

$$\Xi_2 = \begin{bmatrix} \Pi & -\tau_0(1 - \alpha_0)N \\ * & -\tau_0(1 - \alpha_0)Z_1 \end{bmatrix} < 0, \quad (4.3.35)$$

$$\Xi_3 = \begin{bmatrix} \Pi & -\tau_0 S \\ * & -\tau_0 Z_1 \end{bmatrix} < 0, \quad (4.3.36)$$

$$\Xi_4 = \begin{bmatrix} \Pi & -(\bar{\tau} - \tau_0)U \\ * & -(\bar{\tau} - \tau_0)Z_2 \end{bmatrix} < 0, \quad (4.3.37)$$

$$\Xi_5 = \begin{bmatrix} \Pi & -(\bar{\tau} - \tau_0)V \\ * & -(\bar{\tau} - \tau_0)Z_2 \end{bmatrix} < 0, \quad (4.3.38)$$

where

$$\Pi = \begin{bmatrix} \bar{\Omega} & M & N & S & U & V & \Gamma \\ * & -Z_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & -Z_1 & 0 & 0 & 0 \\ * & * & * & * & -Z_2 & 0 & 0 \\ * & * & * & * & * & -Z_2 & 0 \\ * & * & * & * & * & * & -\bar{P} \end{bmatrix}, \quad (4.3.39)$$

and $\bar{\Omega} = (\Omega_{i,j})_{10 \times 10}$

with

$$\begin{aligned}
\Omega_{1,1} &= Q_1 + Q_2 + Q_3 + M_1 + M_1^T - Y_1 A - A^T Y_1^T - K_1 L_1, & \Omega_{1,2} &= -M_1 + M_2^T & \Omega_{1,3} &= 0, \\
\Omega_{1,4} &= 0, & \Omega_{1,5} &= 0, & \Omega_{1,6} &= 0, & \Omega_{1,7} &= -Y_1 - A^T Y_2^T + P, & \Omega_{1,8} &= Y_1 B + K_1 L_2, \\
\Omega_{1,9} &= \alpha_0 Y_1 W, & \Omega_{1,10} &= (1 - \alpha_0) Y_1 W, & \Omega_{2,2} &= -(1 - \alpha_0 \mu_1) Q_1 - M_2 - M_2^T + N_1 + N_1^T, \\
\Omega_{2,3} &= -N_1 + N_2^T, & \Omega_{2,4} &= 0, & \Omega_{2,5} &= 0, & \Omega_{2,6} &= 0, & \Omega_{2,7} &= 0, & \Omega_{2,8} &= 0, & \Omega_{2,9} &= 0, \\
\Omega_{2,10} &= 0, & \Omega_{3,3} &= -N_2 - N_2^T + S_1 + S_1^T - K_2 L_1 + \alpha_0 (1 - \alpha_0) H_1^T \bar{P} H_1, & \Omega_{3,4} &= -S_1 + S_2^T, \\
\Omega_{3,5} &= -\alpha_0 (1 - \alpha_0) H_1^T \bar{P} H_1, & \Omega_{3,6} &= 0, & \Omega_{3,7} &= 0, & \Omega_{3,8} &= 0, & \Omega_{3,9} &= K_2 L_2, & \Omega_{3,10} &= 0, \\
\Omega_{4,4} &= -Q_2 - S_2 - S_2^T + U_1 + U_1^T, & \Omega_{4,5} &= -U_1 + U_2^T, & \Omega_{4,6} &= 0, & \Omega_{4,7} &= 0, & \Omega_{4,8} &= 0,
\end{aligned}$$

$$\begin{aligned}
\Omega_{4,9} &= 0, \quad \Omega_{4,10} = 0, \quad \Omega_{5,5} = -U_2 - U_2^T + V_1 + V_1^T - K_3L_1 + \alpha_0(1 - \alpha_0)H_1^T\bar{P}H_1, \\
\Omega_{5,6} &= -V_1 + V_2^T, \quad \Omega_{5,7} = 0, \quad \Omega_{5,8} = 0, \quad \Omega_{5,9} = 0, \quad \Omega_{5,10} = K_3L_2, \\
\Omega_{6,6} &= -Q_3 - V_2 - V_2^T, \quad \Omega_{6,7} = 0, \quad \Omega_{6,8} = 0, \quad \Omega_{6,9} = 0, \\
\Omega_{6,10} &= 0, \quad \Omega_{7,7} = \tau_0R_1 + (\bar{\tau} - \tau_0)R_2 - Y_2 - Y_2^T, \quad \Omega_{7,8} = Y_2B, \quad \Omega_{7,9} = \alpha_0Y_2W, \\
\Omega_{7,10} &= (1 - \alpha_0)Y_2W, \quad \Omega_{8,8} = -K_1I, \quad \Omega_{8,9} = \Omega_{8,10} = 0, \quad \Omega_{9,9} = -K_2I, \quad \Omega_{10,10} = -K_3I,
\end{aligned}$$

$$\begin{aligned}
M &= \begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
N &= \begin{bmatrix} 0 & N_1^T & N_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
S &= \begin{bmatrix} 0 & 0 & S_1^T & S_2^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
U &= \begin{bmatrix} 0 & 0 & 0 & U_1^T & U_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
V &= \begin{bmatrix} 0 & 0 & 0 & 0 & V_1^T & V_2^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
Y &= \begin{bmatrix} Y_1^T & 0 & 0 & 0 & 0 & 0 & Y_2^T & 0 & 0 & 0 \end{bmatrix}^T, \\
\Gamma^T &= [\bar{P}H_0 \quad 0 \quad \alpha_0\bar{P}H_1 \quad 0 \quad (1 - \alpha_0)\bar{P}H_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \\
\bar{P} &= \tau_0Z_1 + (\bar{\tau} - \tau_0)Z_2 + P.
\end{aligned}$$

Proof. The proof is similar as to Theorem 4.3.1. □

Remark 4.3.2. In Chen and Lu (2008) and Li et al. (2008 a), when $\mu \geq 1$, Q will no longer be helpful to improve the stability condition since $-(1 - \mu)Q$ is nonnegative definite. When $\mu_1 \geq 1$, if $\alpha_0\mu_1 < 1$ is satisfied, then $-(1 - \alpha_0\mu_1)Q_1$ is still negative definite. Therefore, the constraint on $\mu_1 < 1$ is eliminated.

Theorem 4.3.3. For given scalars $\tau_0 \geq 0$, $\bar{\tau}_0 > 0$, μ_1 , $0 < \alpha_0 < 1$ satisfying $\alpha_0\mu_1 < 1$, the stochastic neural networks (4.3.33) is asymptotically stable in the mean square if there exist matrices $P > 0$, $Q_i > 0, i = 1, 2, 3$, $R_l > 0$, $Z_l > 0$, $l = 1, 2$, for any matrices $M_k, N_k, S_k, U_k, V_k, Y_k (k = 1, 2)$ and there exist positive diagonal matrices $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ and scalar $\epsilon > 0$ such that the

following LMIs are feasible

$$\bar{\bar{\Xi}}_1 = \begin{bmatrix} \hat{\Pi} & -\alpha_0\tau_0M \\ * & -\alpha_0\tau_0Z_1 \end{bmatrix} < 0, \quad (4.3.40)$$

$$\bar{\bar{\Xi}}_2 = \begin{bmatrix} \hat{\Pi} & -\tau_0(1-\alpha_0)N \\ * & -\tau_0(1-\alpha_0)Z_1 \end{bmatrix} < 0, \quad (4.3.41)$$

$$\bar{\bar{\Xi}}_3 = \begin{bmatrix} \hat{\Pi} & -\tau_0S \\ * & -\tau_0Z_1 \end{bmatrix} < 0, \quad (4.3.42)$$

$$\bar{\bar{\Xi}}_4 = \begin{bmatrix} \hat{\Pi} & -(\bar{\tau} - \tau_0)U \\ * & -(\bar{\tau} - \tau_0)Z_2 \end{bmatrix} < 0, \quad (4.3.43)$$

$$\bar{\bar{\Xi}}_5 = \begin{bmatrix} \hat{\Pi} & -(\bar{\tau} - \tau_0)V \\ * & -(\bar{\tau} - \tau_0)Z_2 \end{bmatrix} < 0, \quad (4.3.44)$$

where

$$\hat{\Pi} = \begin{bmatrix} \hat{\Omega} & M & N & S & U & V & \Gamma & \Sigma \\ * & -Z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Z_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & -Z_2 & 0 & 0 \\ * & * & * & * & * & * & -\bar{P} & 0 \\ * & * & * & * & * & * & * & -\epsilon I \end{bmatrix},$$

with

$$\begin{aligned} \hat{\Omega} &= \bar{\Omega} + \text{diag}\left(\epsilon E_1^T E_1, 0, 0, 0, 0, 0, 0, \epsilon E_2^T E_2, \epsilon E_3^T E_3, \epsilon E_3^T E_3\right), \\ \sigma &= \sqrt{2 + \alpha_0^2 + (1 - \alpha_0)^2}, \quad \Sigma^T = [\sigma H^T Y_1^T, 0, 0, 0, 0, 0, \sigma H^T Y_2^T, 0, 0, 0], \end{aligned}$$

M, N, S, U, V and Y are defined as in Theorem 4.3.2.

Proof. Replace A, B, W in the LMI (4.3.39) with $A + \Delta A(t), B + \Delta B(t), W + \Delta W(t)$, respectively, we have

$$\Xi_i + \Theta \Psi \Upsilon + \Upsilon^T \Psi^T \Theta^T < 0 \quad i = 1, \dots, 5. \quad (4.3.45)$$

where

$$\begin{aligned}\Theta &= \left[\Theta_1, \underbrace{0, \dots, 0}_5, \Theta_2, \underbrace{0, \dots, 0}_9 \right], \\ \Theta_1 &= \left[Y_1 H, \underbrace{0, \dots, 0}_6, Y_1 H, \alpha_0 Y_1 H, (1 - \alpha_0) Y_1 H, \underbrace{0, \dots, 0}_6 \right], \\ \Theta_2 &= \left[Y_2 H, \underbrace{0, \dots, 0}_6, Y_2 H, \alpha_0 Y_1 H, (1 - \alpha_0) Y_2 H, \underbrace{0, \dots, 0}_6 \right], \\ \Psi &= \text{diag}(F(t), \dots, F(t)), \quad \Upsilon = \text{diag}\left(-E_1 \underbrace{0, \dots, 0}_6, E_2, E_3, E_3, \underbrace{0, \dots, 0}_6 \right).\end{aligned}$$

Using Lemma 4.2.3 that the matrix inequality (4.3.45) is equivalent to the following inequality.

$$\Xi_i + \epsilon^{-1} \Theta \Theta^T + \epsilon \Upsilon^T \Upsilon < 0. \quad (4.3.46)$$

Using Schur complement, (4.3.46) is equivalent to (4.3.40)-(4.3.44) for a scalar $\epsilon > 0$. Then, similar to the proof of the Theorem 4.3.2, the results of Theorem 4.3.3 is obtained. \square

4.4 Numerical Examples

In this section, examples are provided to show the effectiveness of established results.

Example 4.4.1. Consider the Markovian jump stochastic neural networks (4.2.7) with the following matrices

$$\begin{aligned}A_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 2.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 & -0.5 \\ 0.1 & 0 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0.2 & -0.4 \\ 0.3 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.2 & -0.4 \\ 0.3 & -0.1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.1 & -0.5 \\ 0.4 & -0.2 \end{bmatrix},\end{aligned}$$

$$\begin{aligned}
H_{01} &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, & H_{02} &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, & H_{11} &= \begin{bmatrix} 0.3 & -0.3 \\ -0.3 & 0 \end{bmatrix}, \\
H_{12} &= \begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0 \end{bmatrix}, & Q &= \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}.
\end{aligned}$$

Solving the LMI in (4.3.5)-(4.3.10) by MATLAB LMI toolbox, then the feasible solution is obtained for the corresponding values $L_1 = 0$, $L_2 = 0.5I$. Meanwhile, in order to confirm the obtained results with Markovain jump time-varying delay given in (4.2.7), we gives the values for $\alpha = 0.99$, $\tau_0 = 0.4$, $\bar{\tau} = 1.2$, $\mu_1 = 0.2$, to get the feasible solution. Therefore, it follows from Theorem 4.3.1 , that the system (4.2.7) is mean square asymptotically stable.

Example 4.4.2. Consider the stochastic neural networks (4.3.33) with the following matrices

$$\begin{aligned}
A &= \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, & B &= \begin{bmatrix} 0.4 & -0.7 \\ 0.1 & 0 \end{bmatrix}, & W &= \begin{bmatrix} -0.2 & 0.6 \\ 0.5 & -0.1 \end{bmatrix}, \\
H_0 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, & H_1 &= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0 \end{bmatrix},
\end{aligned}$$

$$H = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \quad E_1 = [0.2 \quad 0.3], \quad E_2 = [0.2 \quad -0.3],$$

$$E_3 = [-0.2 \quad -0.3], \quad L_1 = 0.25I, \quad L_1 = 0$$

by Assumption 3, $L_1 = 0, L_2 = 0.25I$ equivalent to $L = 0.5I$ in Huang and Feng (2007). For various μ_1 , the computed upper bound $\bar{\tau}$, which guarantee the robust stability of system (4.3.33), are listed in Table 4.1. From Table 4.1, when the information of the delay-probability distribution is considered, for various α_0 and μ_1 the allowable upper bound $\bar{\tau}$ is larger comparing those in Chen and Lu (2008), Fu et al.(2009), Huang and Feng (2007) and Li et al.(2008 a).

Example 4.4.3. Consider the stochastic neural networks (4.3.33) with the following matrices

$$A = \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & -4 \\ 0.1 & 0.3 \end{bmatrix}, \quad W = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix},$$

$$H_0 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.5 & -0.1 \\ -0.5 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad E_1 = E_2 = E_3 = [1 \quad 1], \quad L_1 = 0.$$

For various μ_1 , the computed upper bound $\bar{\tau}$, which guarantee the robust stability of system (4.3.33), are listed in Table 4.2. From Table 4.2, when the information of

the delay-probability distribution is considered, for various α_0 and μ_1 the allowable upper bound $\bar{\tau}$ is larger than those result discussed in Fu et al.(2009).

Example 4.4.4. Consider the stochastic neural networks (4.3.33) with the following matrices

$$A = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 & -0.19 & 0.3 \\ -0.15 & 0.2 & 0.36 \\ -0.17 & 0.29 & -0.3 \end{bmatrix}, \quad W = \begin{bmatrix} 0.19 & -0.13 & 0.2 \\ 0.16 & 0.09 & 0.1 \\ 0.02 & -0.15 & 0.07 \end{bmatrix},$$

$$H_0 = H_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad E_1 = E_2 = E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

In order to compare results in this chapter with those in Chen and Lu (2008), Huang and Cao (2007) and Zhang et al. (2009 a), we assume the activation functions satisfy Assumption 3 with $l_1^- = l_2^- = l_3^- = 0$, $l_1^+ = 1.2, l_2^+ = 0.5, l_3^+ = 1.3$. In this case, the LMI-based conditions obtained in Huang and Cao (2007) are not feasible when $\mu \geq 0.7$. When the time-varying delay is differentiable and $\mu = 0.85$, by using Theorem 1 in Zhang et al.(2009 a) and Theorem 1 in Chen and Lu (2008), it is found that the maximum allowable upper bound of $\tau(t)$ as $\bar{\tau} = 9.6876$, and $\bar{\tau} = 7.7377$, respectively. However, using Theorem 4.3.3 in this chapter, the author obtains maximum allowable upper bound $\bar{\tau} = 9.7325$. When the time delay may not be differentiable; that is, μ is unknown, by using Theorem 2 in Zhang et al.(2009 a) and Theorem 2 in Chen and Lu (2008), it is found that the maximum allowable

upper bound of $\tau(t)$ as $\bar{\tau} = 2.3879$, and $\bar{\tau} = 2.314$, respectively. However, using Theorem 4.3.3 in this chapter, the author obtains the maximum allowable upper bound $\bar{\tau} = 9.7325$ ($\alpha_0 = 0.7$).

According to Theorem 4.3.3, the upper bounds are derived on the time-varying delay to guarantee the system is robustly stochastically stable in the mean square. From Table 4.3, when the information of the delay-probability distribution is considered, for various α_0 and μ_1 the allowable upper bound $\bar{\tau}$ is larger than those results discussed in the literature Chen and Lu (2008), Huang and Cao (2007) and Zhang et al. (2009 a). Hence the proposed method gives the conservative results.

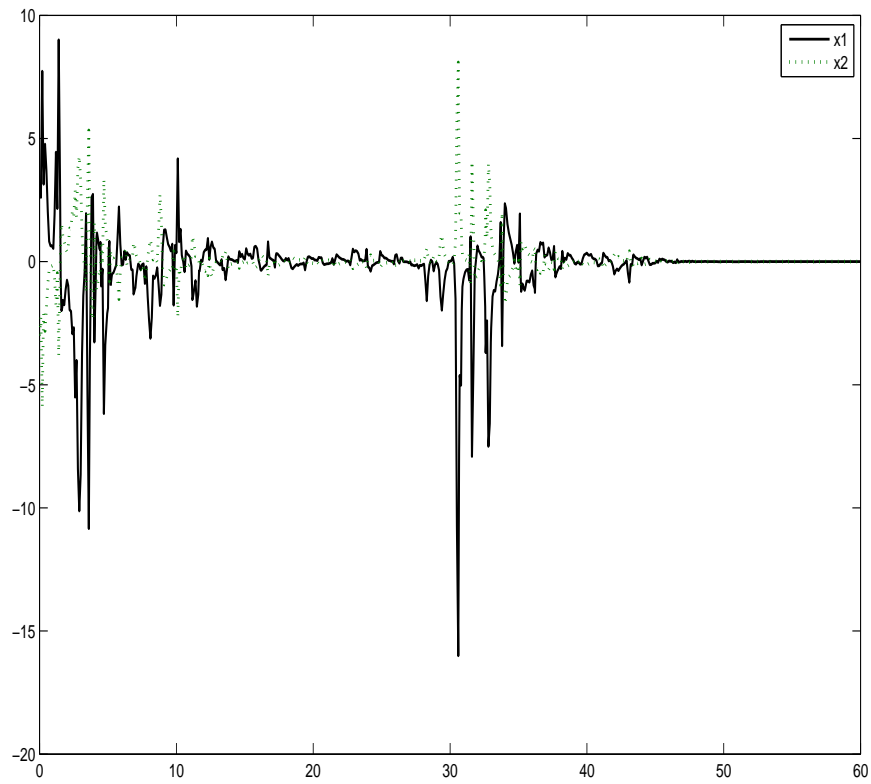


Figure 4.1: The state trajectories of Example 4.2 for $\bar{\tau} = 1$ with initial condition $(3, -3)$.

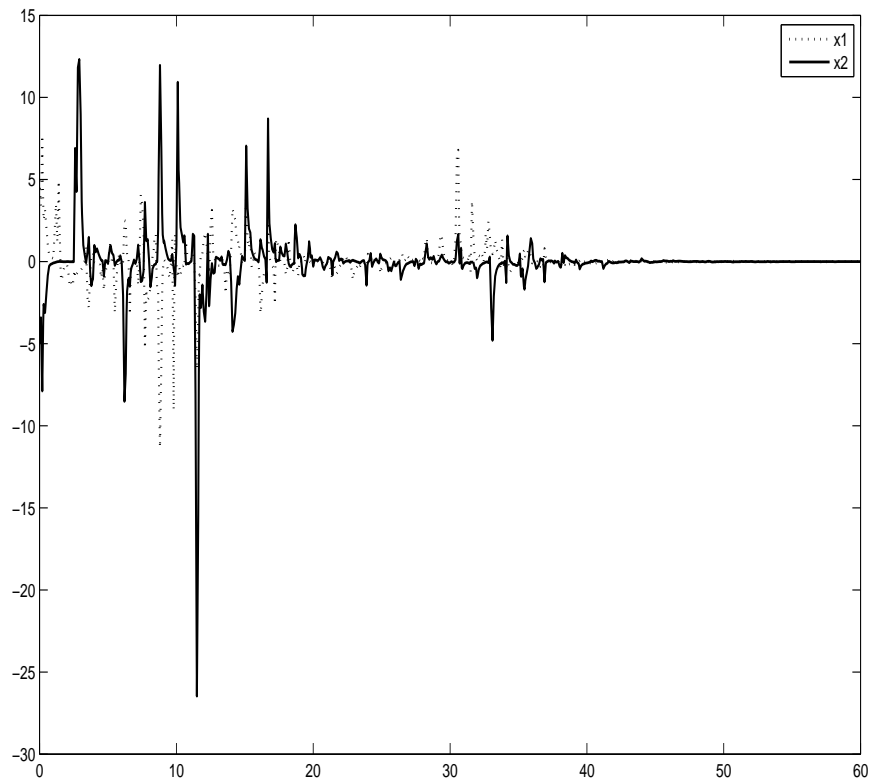


Figure 4.2: The state trajectories of Example 4.3 for $\bar{\tau} = 2.5$ with initial condition $(5, -5)$.

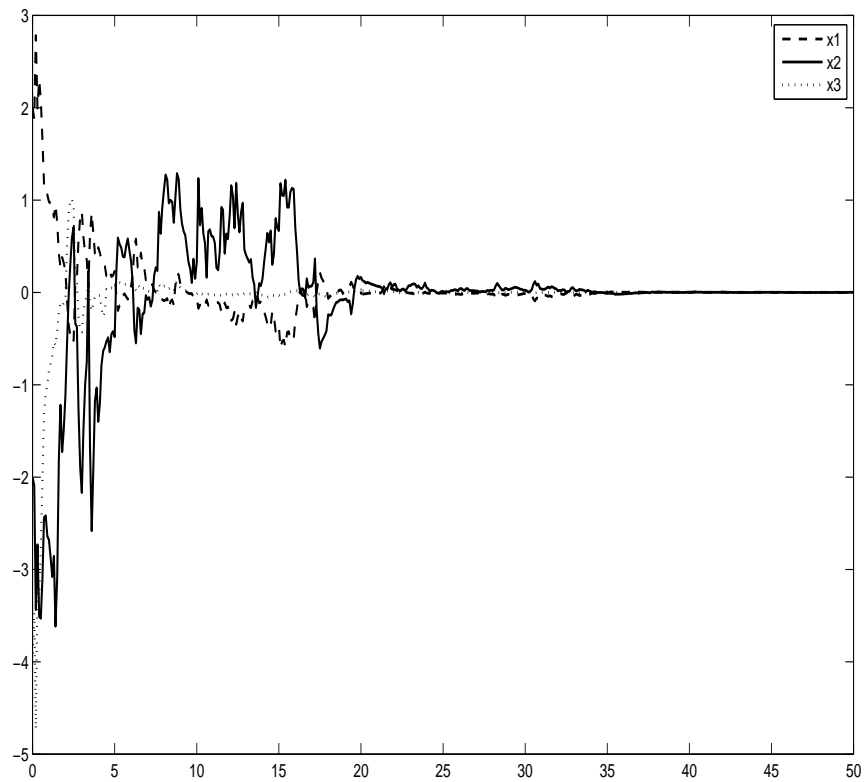


Figure 4.3: The state trajectories of Example 4.4 for $\bar{\tau} = 2$ with initial condition $(2, -2, -4)$.

Table 4.1: Maximum allowable upper bound of $\bar{\tau}$ with different μ for fixed $\tau_0 = 0.6$

<i>Methods</i>		$\mu_1 = 0.97$	$\mu_1 = 1$	$\mu_1 = 1.5$	$\mu_1 = 2$	<i>unknown</i>
Huang and Feng (2007)		-	-	-	-	0.419
Huang and Feng (2007)		0.785	0.779	0.779	0.779	0.779
Chen and Lu (2008)		0.771	0.746	0.746	0.746	0.746
Li et al.(2008 a)	$\alpha_0 = 0.2$	1.294	1.294	1.292	1.291	1.279
Theorem 4.3.3	$\alpha_0 = 0.2$	2.7482	2.7481	2.7456	2.7429	2.7339
Fu et al.(2009)	$\alpha_0 = 0.4$	1.338	1.337	1.324	1.299	1.281
Theorem 4.3.3	$\alpha_0 = 0.4$	3.1819	3.1809	3.1642	3.1567	3.1567
Fu et al.(2009)	$\alpha_0 = 0.6$	1.430	1.426	1.303	1.292	1.292
Theorem 4.3.3	$\alpha_0 = 0.6$	3.9210	3.9177	3.8990	3.8990	3.8990
Fu et al.(2009)	$\alpha_0 = 0.8$	1.615	1.579	1.323	1.323	1.323
Theorem 4.3.3	$\alpha_0 = 0.8$	5.6591	5.6591	5.6591	5.6591	5.6591

Table 4.2: Maximum allowable upper bound of $\bar{\tau}$ with different μ for fixed $\tau_0 = 0.4$

<i>Methods</i>		$\mu_1 = 0.2$	$\mu_1 = 0.6$	$\mu_1 = 1$	$\mu_1 = 1.5$	$\mu_1 = 2$	$\mu_1 = 2.5$
Fu et al. (2009)	$\alpha_0 = 0.2$	0.972	0.972	0.971	0.970	0.968	0.967
Theorem 4.3.3	$\alpha_0 = 0.2$	2.2317	2.2297	2.2275	2.2245	2.2213	2.2179
Fu et al. (2009)	$\alpha_0 = 0.5$	1.092	1.083	1.071	1.044	1.024	1.024
Theorem 4.3.3	$\alpha_0 = 0.5$	2.9122	2.8929	2.8706	2.8586	2.8586	2.8586
Fu et al. (2009)	$\alpha_0 = 0.8$	1.545	1.490	1.342	1.242	1.242	1.242
Theorem 4.3.3	$\alpha_0 = 0.8$	4.9575	4.8599	4.8160	4.8160	4.8160	4.8160
Fu et al. (2009)	$\alpha_0 = 0.99$	5.523	5.181	3.529	3.243	3.243	3.243
Theorem 4.3.3	$\alpha_0 = 0.99$	24.9910	24.0252	23.9266	23.9266	23.9266	23.9266

Table 4.3: Maximum allowable upper bound of $\bar{\tau}$ with different μ for fixed $\tau_0 = 0.9$

<i>Methods</i>		$\mu_1 = 0.7$	$\mu_1 = 0.85$	$\mu_1 = 1$	$\mu_1 = 2$	$\mu_1 = 3$	<i>unknown</i>
Chen and Lu (2008)		-	7.7377	-	-	-	2.3514
Zhang et al (2009)		-	9.6876	-	-	-	2.3879
Theorem 4.3.3	$\alpha_0 = 0.7$	9.7325	9.7325	9.7325	9.7325	9.7325	9.7325
Theorem 4.3.3	$\alpha_0 = 0.8$	10.0598	10.0598	10.0598	10.0598	10.0598	10.0598