CHAPTER 5

CHAPTER 5

FUZZY g·pre-CONTINUOUS MAPS IN FUZZY TOPOLOGICAL SPACE

In this chapter, fuzzy g·p-continuous maps, fuzzy g·p-irresolute maps, fuzzy g·p-closed maps, fuzzy g·p-maps, g·p-open maps and fuzzy g·p-homeomorphisms due to Benchalli et al. [15] are studied. Properties, characterization and implications of these mappings with other mappings are analyzed.

Section: 5.1

Preliminary Definitions and Results of Fuzzy g*pre- Continuous maps in Fuzzy Topological Spaces.

Definition: 5.1.1

Let X, Y be two fts. A function f: X → Y is called fg·-continuous function if $f^{-1}(A)$ is a g·-closed fuzzy set in X, for every closed fuzzy set A of Y.

Definition: 5.1.2

A fts (X, T) is called a fuzzy $T_{1/2^{\cdot}}$ space if every g·-closed fuzzy set is a closed fuzzy set.

Definition: 5.1.3

A fuzzy set A of a fts (X, T) is called a g·-pre-closed fuzzy set (g·p-closed) if pcl(A) ≤ U, whenever A ≤ U and U is a g-open fuzzy set in (X, T).

Definition: 5.1.4

A fuzzy set A of a fts (X, T) is called a g·-pre-open (g·-p-open) fuzzy set if its complement 1 – A is a g·p-closed fuzzy set.

Definition: 5.1.5

Let X and Y be fts. A map f: X → Y is said to be fuzzy g·p-continuous (fg·p-continuous) if the inverse image of every open fuzzy set in Y is a g·p-open fuzzy set in X.

Theorem: 5.1.6

A function f: X → Y is fg·p-continuous iff the inverse image of every closed fuzzy set in Y is g·p-closed fuzzy in X.

Theorem: 5.1.7

Every f-continuous function is fg·p-continuous function.

Proof

Let f: X → Y be a f-continuous function. Let V be an open fuzzy set in Y. Since f is f-continuous, $f^{-1}(V)$ is open fuzzy set in X. As every open fuzzy set is g·p -open fuzzy set, $f^{-1}(V)$ is g·p-open fuzzy set in X. Therefore, f is fg·p-continuous function. Hence the theorem.

The converse of the above theorem need not be true in general.

Example: 5.1.8

Let X = Y = {a, b, c} and the fuzzy set A and B defined as follows :

A = {(a, 1), (b, 0.9), (c, 0.8)},

B = {(a, 0.4), (b, 0.5), (c, 0.6)}.

Consider T = {0, 1, A} and σ = {0, 1, B}. Then (X, T) and (Y, σ) are fts. Let f: X → Y be the identity map. Then f is fg·p-continuous map but not fuzzy continuous.

Since B is open fuzzy set in Y, $f^{-1}(B)$ = is not closed fuzzy set in X but it is g·p-closed.

Theorem: 5.1.9

(i) Every fg·p-continuous function is a fgp-continuous function.

(ii) Every fg·p-continuous function is a fgsp-continuous function.

Example: 5.1.10

Let X = Y = {a, b, c} and the fuzzy sets A, B and C be defined as follows:

A = {(a, 1), (b, 0.9), (c, 0.8)}.

B = {(a, 0.4), (b, 0.6), (c, 0.8)} and

C = {(a, 0.6), (b, 0.8), (c, 0.4)}

Consider T = {0, 1, A} =σ. Then (X, T) and (Y,σ) are fts.

Let f: X $\rightarrow$ Y be such that f (a) = b, f (b) = c and f(c) = a. Then f is fgp-continuous and fgsp-continuous but not an fg·p-continuous map, for the closed set B in Y, f^{-1}(B) = C is not g·P-closed fuzzy set in X.

Theorem: 5.1.11

A function f: X $\rightarrow$ Y is fg·p-continuous and X is fuzzy $T_{P'}$-space. Then f is f-continuous.

Proof

Let F be closed fuzzy set in Y. Then f^{-1}(F) is a g·p-closed fuzzy set in X. Since X is fuzzy-$T_{P'}$-space, f^{-1}(F) is closed fuzzy set in X. Thus f is f-continuous function.

Hence the theorem.

Theorem: 5.1.12

A function f: X $\rightarrow$ Y is fgp-continuous and X is fuzzy-$T_{P'}$-space. Then f is fg·P-continuous.

Proof

Let F be a closed fuzzy set in Y. Then f^{-1}(F) is a gp-closed fuzzy set in X. Since X is fuzzy-$T_{P'}$-space, f^{-1}(F) is g·p-closed fuzzy set in X. Thus f is fg·p-continuous function.

Hence the theorem.

Theorem: 5.1.13

If f: X → Y is fg·p-continuous and g: Y → Z is fg·p-continuous and Y is fuzzy-$T_{P'}$-space, then g o f: X → Z is fg·p-continuous.

Proof

Let F be closed set in Z. Then $g^{-1}(F)$ is a g·p-closed fuzzy set in Y.

Since Y is fuzzy-$T_{P'}$-space, $g^{-1}(F)$ is closed fuzzy set in Y. Then $f^{-1}(g^{-1}(F))$ is g·p-closed in fuzzy X as f is fg·p-continuous. Thus g o f is fg·p-continuous function.

Hence the theorem.

Definition: 5.1.14

A function f: X → Y is said to be fuzzy g·p-irresolute (fg·p-irresolute) if the inverse image of every g·p-closed fuzzy set in Y is a g·p-closed fuzzy set in X.

Theorem: 5.1.15

A function f: X → Y is fg·p-irresolute function iff the inverse image of every g·p-open fuzzy set in Y is a g·p-open fuzzy set in X.

Proof

Let f: X → Y be fg·p-irresolute function.

To Prove: The inverse image of every g·p-open fuzzy set in Y is a g·p-open fuzzy set in X.

Let V be a g·p-open fuzzy set in Y. then 1 – V is a g·p-closed fuzzy set in Y. Since f is a fg·p-irresolute function, $f^{-1}(1 - V)$ is a g·p-closed fuzzy set in X. Since, $f^{-1}(1 - V) = 1 - f^{-1}(V)$, $f^{-1}(V)$ is a g·p-open fuzzy set in X.

Conversely,

Let V be a g·p-closed fuzzy set in Y. Then 1 – V is a g·p-open fuzzy set in Y. By hypothesis, $f^{-1}(1 - V)$ is g·p-open fuzzy set in X. (i.e.) $1 - f^{-1}(V)$ is g·p-open fuzzy set in X. Hence $f^{-1}(V)$ is a g·p-closed fuzzy set in X Hence f g·p is irresolute.

Hence the theorem.

Theorem: 5.1.16

Every fg·p-irresolute function is fg·p-continuous function.

Proof

Let f: X → Y be fg·p-irresolute function. Let F be a closed fuzzy set in Y. Then F is g·p-closed fuzzy set in Y. Since f is fg·p-irresolute, $f^{-1}(V)$ is a g·p-closed fuzzy set in X. Hence f is a fg·p-continuous function.

Hence the theorem.

Example: 5.1.17

Let X = Y = {a, b, c} and the fuzzy sets A, B, C, D and E be defined as follows:

A = {(a, 0.9), (b, 0.9), (c, 0.1)},

B = {(a, 0.8), (b, 0.5), (c, 0.6)},

C = {(a, 0.7), (b, 0.5), (c, 0.6)}

D = {(a, 0.5), (b, 0.2), (c, 0.3)},

E = {(a, 0.5), (b, 0.6), (c, 0.7)}.

Consider T = {0, 1, A, B, C, D} and σ = {0, 1, C}. Then (X, T) and (Y, T) are fts. Define f: X → Y by f(a) = c, f(b) = a and f(c) = b. Then f is fg·P-continuous but not fg·p-irresolute as the fuzzy set E is g·p-closed fuzzy set in Y, but $f^{-1}(E)$ = c is not g·p-closed fuzzy set in X.

Theorem: 5.1.18

Let f: X → Y, g: Y → Z be two functions. Then

(i) g o f : X → Z is fg·p-continuous if f is fg·p-continuous and g is f-continuous functions.

(ii) g o f : X → Z is fg·p-irresolute, if f and g are fg·p-irresolute functions.

(iii) $g \circ f : X \to Z$ is fg·p-continuous if f is fg·p-irresolute and g is fg·p-continuous.

Proof

(i) Let f: $X \to Y$ be fg·p-continuous and g: $Y \to Z$ be f-continuous.

Let V be an open fuzzy set in Z. Since g is f-continuous, $f^{-1}(V)$ is open fuzzy in Y. Since f is fg·p-continuous, $g^{-1}(f^{-1}(V))$ is g·p-open fuzzy set is X. Hence $g \circ f : X \to Z$ is a fg·p-continuous map.

(ii) Let f: $X \to Y$ and g: $Y \to Z$ be fg·p-irresolute functions.

Ler V be a g·p-closed fuzzy set in Z. Then $g^{-1}(V)$ is g·p-closed fuzzy set in Y as g is fg·p-irresolute.

Since, f is fg·p-irresolute, $f^{-1}(g^{-1}(V))$ is g·p-closed fuzzy set in X. Hence $g \circ f$: $X \to Z$ is fg·p-irresolute. Hence the theorem.

Theorem: 5.1.19

If f: $X \to Y$, g: $X \to Z$ be two fuzzy functions. If f is fg·p-continuous and g is fg·p-irresolute and Y is fuzzy-T_{P}·-space, then $g \circ f$: $X \to Z$ is fg·p-irresolute function.

Proof

Let F be a g·p-closed fuzzy set in Z. Then $g^{-1}(F)$ is g·p-closed fuzzy set in Y as g is fg·p-irresolute. Since Y is fuzzy-T_{P}·-space, $g^{-1}(F)$ is closed fuzzy set in Y. Again since f is fg·p-continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is g·p-closed fuzzy set in X. Hence $g \circ f$ is fg·p-irresolute function.

Hence the theorem.

Theorem: 5.1.20

Let f: $X \to Y$ be a fg·p-continuous function and Y is fuzzy-T_{P}·-space, then f is fg·p-irresolute function.

Proof

Let F be g·p-closed fuzzy set in Y. Then F is closed fuzzy set in Y as Y is fuzzy- $T_{P'}$-space. Since f is fg·p-continuous, we have $f^{-1}(F)$ is g·p-closed in X and hence f is fg·p-irresolute function.

Hence the theorem.

Theorem: 5.1.21

Let f: X → Y be an onto, fg·p-irresolute and a closed map. If X is a fuzzy-$T_{P'}$-space, then Y is also fuzzy-$T_{P'}$-space.

Proof

Let F be g·p-closed fuzzy set in Y. Then $f^{-1}(F)$ is g·p-closed in X as f is fg·p-irresolute. Since X is fuzzy-$T_{P'}$-space, $f^{-1}(F)$ is closed in X. And also $f(f^{-1}(F))$ = F is closed in Y, as f is closed and onto function. Hence Y is fuzzy-T_p^*-space.

Hence the theorem.

Definition: 5.1.22

A function f: X → Y is said to be fuzzy g·p-open (fg·p-open) map if the image of every open fuzzy set in X is a g·p-open fuzzy set in Y.

Definition: 5.1.23

A function f: X → Y is said to be fuzzy g·p-closed (fg·p-closed) map if the image of every closed fuzzy set in X is g·p-closed fuzzy set in Y.

Theorem: 5.1.24

If f: X → Y is a fuzzy g·p-open map and Y is fuzzy-$T_{P'}$-space, then f is f-open map.

Proof

Let V be an open fuzzy set in X. Since f is fg·p-open map f (V) is a g·p-open fuzzy set in Y. As Y is fuzzy-$T_{P'}$-space, f (V) is a open fuzzy set in Y. Hence f: X → Y be a f-open map. Hence the theorem.

Theorem: 5.1.25

If f: X → Y be a fg·p-closed map and Y is fuzzy-$T_{P'}$-space, then f is f-closed map.

Proof

Let V be a closed set of X and f be fg·p-closed maps. Then f (V) is g·p-closed fuzzy set in Y. Since Y is fuzzy-$T_{P'}$-space, f (V) is closed fuzzy set in Y and hence f is f-closed map.

Hence the theorem.

Theorem: 5.1.26

A map f : X → Y is fg·p-closed iff for each fuzzy set S of Y and for each open fuzzy set U such that $f^{-1}(S) \subseteq U$, there is a g·p-open fuzzy set V of Y such that S $\leq$ V and $f^{-1}(V) \subseteq U$.

Proof

Suppose f is fg·P-closed map. Let S be a fuzzy set of Y, and U be an open fuzzy set of X, such that $f^{-1}(S) \subseteq U$. Then V = Y – f(X – U) is a g·p-open in Y such that S $\leq$ V and $f^{-1}(V) \subseteq U$.

Conversely, suppose that F is a closed fuzzy set of X. Then $f^{-1}(Y - f(F) \leq X - F$ and X – F is a open fuzzy set. By hypothesis, there is a g·p-open fuzzy set V of Y such that Y – f(X – V=U) $\leq$ V and $f^{-1}(V) \leq X - F$. Therefore, $F \leq X - f^{-1}(V)$. Hence $Y - V \leq f(V) \leq f(X - f^{-1}(V)) \leq Y - V \Rightarrow f(F) = Y - V$. Since Y – V is g·p-closed fuzzy set. f (F) is g·p-closed fuzzy set and thus f is a fg·p-closed fuzzy map.

Hence the theorem.

Theorem: 5.1.27

If f: X → Y is a f-closed map and g: Y → Z is fg·p-closed maps, then g o f: X → Z is fg·p-closed map.

Proof

Let V be a closed fuzzy set in X. Let f: X → Y be a f-closed map and g: Y → Z be fg·p-closed map. Then f (V) is closed fuzzy set in Y and g (f (V)) is g·p-closed fuzzy set in Z. Hence g o f is g·p-closed map.

Hence the theorem.

Theorem: 5.1.28

If f : X → Y and g : Y → Z are fg·p-closed maps and Y is fuzzy-T_{P^*}-space, then g o f : X → Z is fg·p-closed maps.

Proof

Let V be a closed fuzzy set in X. Let f: X → Y and g: Y → Z be fg·p-closed maps and Y be a fuzzy-T_p^*-space. Then f (V) is a g·p-closed fuzzy set in Y. As Y is fuzzy-T_p^*-space, f (V) is a closed fuzzy set in Y. Hence g (f (V) is a g·p-closed fuzzy set in Z. Hence g o f: X → Z is a fuzzy g·p-closed maps.

Hence the theorem.

Theorem: 5.1.29

Let f: X → Y, g: Y → Z be two maps such that g o f: X → Z is fg·p-closed map.

(i) If f is fuzzy continuous and surjective, then g is fg·p-closed map.

(ii) If g is fg·p-irresolute and injective, then f is fg·p-closed map.

Proof

(i) Let G be a closed fuzzy set of Y. Then $f^{-1}(G)$ is closed fuzzy set in X as f is fuzzy continuous. Since g o f is fg·p-closed map, (g o f) $f^{-1}(G)$) = g(G) is g·p-closed fuzzy set in Z. Hence g: Y → Z fg·p-closed map.

(ii) Let F be a closed fuzzy set in X. Then (g o f) (F) is g·p-closed fuzzy set in Z, and so g^{-1}(g o f) (F) = f (F) is g·p-closed fuzzy set in Y. Since g is fg·p-irresolute and injective. Hence f is a fg·p-closed map.

Hence the theorem.

Theorem: 5.1.30

If A is a g·p-closed fuzzy set in X and f: X → Y is bijective, f-continuous and fg·p-closed, then f (A) is g·p-closed fuzzy set in Y.

Proof

Let f (A) ≤ O where O is an open fuzzy set in Y. Since f is f-continuous, f^{-1}(O) is an open fuzzy set containing A. Hence Pcl(A) ≤ f^{-1}(O) as A is g·p-closed fuzzy set. Since f is fg·p-closed, f (Pcl(A)) is g·p-closed fuzzy set contained in the open fuzzy set O. So f(A) is fg·p-closed fuzzy set in Y.

Hence the theorem.

Section: 5.2

Fuzzy g·pre-Homeomorphisms in Fuzzy Topological Spaces

In this section, fuzzy g·pre-homeomorphism in fuzzy topological spaces is analyzed. Properties and characterizations of these maps are discussed.

Definition: 5.2.1

A function f: X → Y is called fuzzy g·p-homeomorphism (fg·p-homeomorphism) if f and f^{-1} are fg·p-continuous.

Theorem: 5.2.2

Every f-homeomorphism is fg·p-homeomorphism.

Proof

Let f: X → Y be fuzzy homeomorphism. Then f and f^{-1} are f-continuous. Therefore, f and f^{-1} are fg·p-continuous. Hence f is fg·p-homeomorphism. Hence the theorem.

The converse of the above theorem need not be true in general.

Example: 5.2.3

Let X = Y = {a, b, c} and the fuzzy set A, B and C be defined as follows:

A = {(a, 1), (b, 0.8), (c, 0.8)}

B = {(a, 0.3), (b, 0.6), (c, 0.8)}

C = {(a, 0.4), (b, 0.6), (c, 0.8)}

Consider T = {0, 1, A} and σ = {0, 1, B}. Then (X, T) and (Y,σ) are fts.

Define f: X → Y by f (a) = a, f (b) = c and f(c) = b. then f is fg·p-homeomorphism but not f-homeomorphism as A is open in X.

f (A) = A is not open in Y. Therefore, f^{-1}: Y → X is not f-continuous.

Theorem: 5.2.4

Let f: X → Y be a bijective function. Then the following are equivalent.

a) f is fg·p-homeomorphism

b) f is fg·p-continuous and fg·p-open maps

c) f is fg·p-continuous and fg·p-closed maps.

Proof

(a) ⇒ (b)

Let f be fg·p-homeomorphism. Then f and f^{-1} are fg·p-continuous.
To prove that f is fg·p-open map.

Let U be an open fuzzy set in X. Since f^{-1}: Y → X is fg·p-continuous, $(f^{-1})^{-1}(U)$ =f (U) is fg·p-open in Y. Therefore, f(U) is fg·p-open in Y. Hence f is fg·p-open map.

(b) ⇒ (a)

Let f be fg·p-open and fg·p-continuous map.

To prove that f^{-1}: Y → X is fg·p-continuous.

Let U be an open fuzzy set in Y. Then, f(U) is a fg·p-open set in Y since f is fg·p-open map. Now $(f^{-1})^{-1}(U)$ =f (U) is fg·p-open set in Y. Therefore f^{-1}: Y → X is fg·p-continuous. Hence f is fg·p-homeomorphism.

(b) ⇒ (c)

Let f be fg·p-continuous and fg·p-open map.

To prove that, f is fg·p-closed map.

Let F be a closed fuzzy set in X. Then 1 – F is an open fuzzy set in X.

Since f is fg·p-open map, f (1 – F) is a g·p-open fuzzy set in Y. Now, f (1 – F) = 1 – f (F). Therefore, f (F) is fg·p-closed in Y. Hence f is a fg·p-closed map.

(c) ⇒ (b)

Let f be fg·p-continuous and fg·p-closed map.

To prove that, f is fg·p-open map. Let Y be an open fuzzy set in X. Then1 – U is a closed fuzzy set in X. Since f is fg·p-closed map, f(1 – U) is fg·p-closed in Y. Now, f (1 – U) = 1 – f (U). Therefore, f (U) is fg·p-open in Y. Hence f is fg·p-open map. Hence the theorem.

Theorem: 5.2.5

If f : X → Y fg·P-homeomorphism and g : Y → Z is fg·p-homeomorphism and Y is fuzzy-$T_{P^{\cdot}}$-space, then g o f : X → Z is fg·p-homeomorphism.

Proof

To show that, g o f and $(g \circ f)^{-1}$ are fuzzy g·semi-continuous.

Let U be an open fuzzy set in Z. Since g: Y → Z is fg·p-continuous, $g^{-1}(U)$ is fg·p-open in Y. Then $g^{-1}(U)$ is open fuzzy set in Y as Y is fuzzy-$T_{P^{\cdot}}$-space. Also since f: X → Y is fg·p-continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is g·p-open in X. Therefore, g o f is fg·p-continuous. Again, let U be an open fuzzy set in X. Since f^{-1}: Y → X is fg·p-continuous, $(f^{-1})^{-1}(U) = f(U)$ is g·p-open fuzzy set in Y. And so f (U) is open fuzzy set in Y as Y is fuzzy-$T_{P^{\cdot}}$-space. Also since g^{-1}: Z → Y is fg·p-continuous, $(g^{-1})^{-1}(f(U)) = g(f(U)) = (g \circ f)(U)$ is g·p-open fuzzy set in Z. Therefore, $((g \circ f)^{-1})^{-1}(U) = (g \circ f)(U)$ is g·p-open fuzzy set in X. Hence $(g \circ f)^{-1}$ is fg·p-continuous. Thus, g o f is fg·p-homeomorphism.

Hence the theorem.