2

β^{**} Generalized Closed Sets in Intuitionistic Fuzzy Topological Spaces

2.1 Introduction

The concept of β open sets plays a significant role in general topology which was introduced by Abd El-Monsef et al. (1983). And many results were obtained by using the concepts of β closed sets. Kannan and Nagaveni (2012) have introduced $\hat{\beta}$ generalized closed sets and open sets in topological spaces. Yuksel and Noiri (1996) have defined the notion of β^* set and established a decomposition of continuity. Ali M. Mubarki et al. (2014) introduced β^{**} open sets and β^* continuity functions in topological spaces. Later the concept of intuitionistic fuzzy β-generalized closed sets is introduced by Saranya and Jayanthi (2016_a). Now we have introduced intuitionistic fuzzy β^{**} generalized closed sets and intuitionistic fuzzy β^{**} generalized open sets and analyzed many interesting propositions and results in this chapter.

2.2 Intuitionistic Fuzzy β^{**} Generalized Closed Sets

In this section we have introduced intuitionistic fuzzy β^{**} generalized closed set and investigated some of its properties.

Definition 2.2.1 : An IFS A of an IFTS (X, τ) is said to be an **intuitionistic fuzzy β^{**} generalized closed set** (IFβ^{**} GCS) if $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq U$ whenever $A \subseteq U$ and U is an IFOS in (X, τ).

Example 2.2.2 : Let X = {a, b} and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then (X, τ) is an IFTS. Here, the IFS $A = \langle x, (0.5_a, 0.6_b), (0.3_a, 0.4_b)\rangle$ is an IFβ^{**}GCS in X.

Proposition 2.2.3 : Every IFCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFCS in (X, τ). Let $A \subseteq U$ and U be an IFOS in (X, τ). Since A is an IFCS, cl(A) = A. Now $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq cl(A) \cap cl(A) = A \cap A = A \subseteq U$. Hence A is an IF$\beta^{**}$GCS in (X, τ).

Example 2.2.4 : Let X = {a, b} and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then, the IFS $A = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)\rangle$ is an IFβ^{**}GCS but not an IFCS in (X, τ), as $cl(A) = G_1^c \neq A$.

Proposition 2.2.5 : Every IFSCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFSCS in (X, τ). Let $A \subseteq U$ and U be an IFOS in (X, τ). Since A is an IFSCS, $int(cl(A)) \subseteq A$. Now $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq cl(A) \cap int(cl(A)) \subseteq int(cl(A)) \subseteq A \subseteq U$. Hence A is an IF$\beta^{**}$GCS in (X, τ).

Example 2.2.6 : Let X = {a, b} and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then, the IFS $A = \langle x, (0.3_a, 0.3_b), (0.7_a, 0.7_b)\rangle$ is an IFβ^{**}GCS but not an IFSCS in (X, τ), as $int(cl(A)) = G_1 \nsubseteq A$.

Proposition 2.2.7 : Every IFPCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFPCS in (X, τ). Let $A \subseteq U$ and U be an IFOS in (X, τ). Now $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq cl(A) \cap cl(int(A)) \subseteq cl(A) \cap A = A \subseteq U$, by hypothesis. Hence A is an IFβ^{**}GCS in (X, τ).

Example 2.2.8 : Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then, the IFS $A = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)\rangle$ is an IFβ^{**}GCS but not an IFPCS in (X, τ), as $cl(int(A)) = cl(G_1) = G_1^c \nsubseteq A$.

Proposition 2.2.9 : Every IFRCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFRCS in (X, τ). Since every IFRCS is an IFCS, by Proposition 2.2.3, A is an IFβ^{**}GCS in (X, τ).

Example 2.2.10 : Let $X = \{a, b\}$ and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then, the IFS $A = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)\rangle$ is an IFβ^{**}GCS but not an IFRCS in (X, τ), as $cl(int(A)) = G_1^c \neq A$.

Proposition 2.2.11 : Every IFαCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFαCS in (X, τ). Let $A \subseteq U$ and U be an IFOS in (X, τ). Now $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq A \cap int(cl(A)) \subseteq A \cap cl(A) = A \subseteq U$, by hypothesis. Hence A is an IFβ^{**}GCS in (X, τ).

Example 2.2.12 : Let X = {a, b} and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then, the IFS $A = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)\rangle$ is an IFβ^{**}GCS but not an IFαCS in (X, τ), as $cl(int(cl(A))) = G_1^c \nsubseteq A$.

Proposition 2.2.13 : Every IFβCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFβCS in (X, τ). Let $A \subseteq U$ and U be an IFOS in (X, τ). Since A is an IFβCS, $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq cl(A) \cap A = A \subseteq U$, by hypothesis. Hence A is an IFβ^{**}GCS in (X, τ).

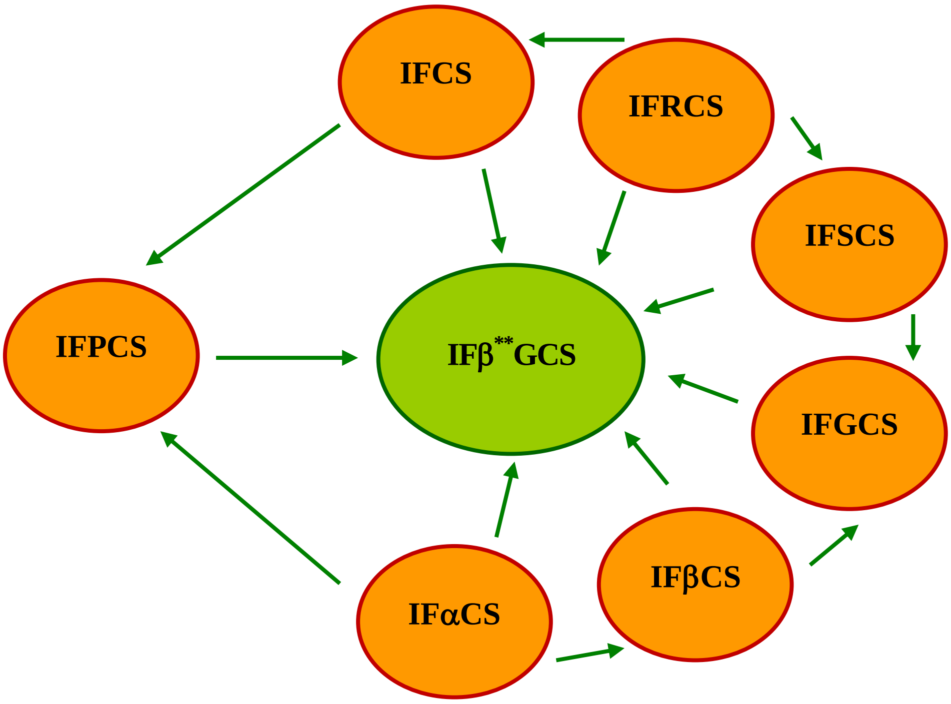
Example 2.2.14 : Let X = {a, b} and $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b)\rangle$. Then (X, τ) is an IFTS. Here the IFS $A = \langle x, (0.7_a, 0.8_b), (0.3_a, 0.2_b)\rangle$ is an IFβ^{**}GCS but not an IFβCS in (X, τ), as $int(cl(int(A))) = 1_\sim \nsubseteq A$.

Proposition 2.2.15 : Every IFGCS is an IFβ^{**}GCS in (X, τ) but not conversely in general.

Proof : Let A be an IFGCS in (X, τ). Let $A \subseteq U$ and U be an IFOS in (X, τ). Now $cl(int(cl(A))) \cap int(cl(int(A))) \subseteq cl(A) \cap cl(int(A)) = cl(A) \cap cl(A) = cl(A) \subseteq U$, by hypothesis. Hence A is an IFβ^{**}GCS in (X, τ).

Example 2.2.16 : Let X ={a, b} and $\tau = \{0_\sim, G, 1_\sim\}$ be an IFT on X, where $G = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$. Then the IFS $A = \langle x, (0.4_a, 0.4_b), (0.5_a, 0.6_b)\rangle$ is an IFβ^{**}GCS in X but not an IFGCS in (X, τ), as $cl(A) = G^c \nsubseteq G$, whereas $A \subseteq G$.

In the following diagram we have provided the relation between various types of intuitionistic fuzzy closedness. The reverse implications are not true in general in the below diagram.



Remark 2.2.17 : The union of any two IFβ^{**}GCSs need not be an IFβ^{**}GCS in (X, τ) in general.

Example 2.2.18 : Let X = {a, b}, $G_1 = \langle x, (0.6_a, 0.8_b), (0.4_a, 0.2_b)\rangle$ and $G_2 = \langle x, (0.5_a, 0.5_b), (0.4_a, 0.4_b)\rangle$. Then τ = {$0_\sim$, G_1, G_2, $1_\sim$} is an IFT on X. Here the IFSs $A = \langle x, (0.5_a, 0.4_b), (0.4_a, 0.5_b)\rangle$ and $B = \langle x, (0.4_a, 0.6_b), (0.5_a, 0.2_b)\rangle$ are IFβ^{**}GCSs in (X, τ) but $A \cup B = \langle x, (0.5_a, 0.6_b), (0.4_a, 0.2_b)\rangle$ is not an IFβ^{**}GCS in (X, τ), as $int(cl(int(A \cup B))) \cap cl(int(cl(A \cup B))) = 1_\sim \nsubseteq G_1$ whereas $A \cup B \subseteq G_1$.

Remark 2.2.19 : The intersection of any two IFβ^{**}GCSs need not be an IFβ^{**}GCS in (X, τ) in general.

Example 2.2.20 : Let X = {a, b}, $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b)\rangle$ and $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)\rangle$. Then τ = {$0_\sim$, G_1, G_2, $1_\sim$} is an IFT on X. Here the

IFSs $A = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.2_b)\rangle$ and $B = \langle x, (0.5_a, 0.6_b), (0.4_a, 0.4_b)\rangle$ are IFβ^{**}GCSs in (X, τ) but $A \cap B = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)\rangle$ is not an IFβ^{**}GCS in (X, τ), as $int(cl(int(A \cap B))) \cap cl(int(cl(A \cap B))) = 1_\sim \nsubseteq G_1, G_2$ whereas $A \cap B \subseteq G_1, G_2$.

Proposition 2.2.21 : If A is both an IFOS and an IFβ^{**}GCS in (X, τ), then A is an IFβCS in (X, τ).

Proof : Let A be an IFOS and an IFβ^{**}GCS in X. Then $int(cl(int(A))) \cap cl(int(cl(A))) \subseteq A$, as $A \subseteq A$. Now $int(cl(int(A))) = int(cl(int(A))) \cap cl(int(A)) \subseteq int(cl(int(A))) \cap cl(int(cl(A))) \subseteq A$, by hypothesis. Therefore $int(cl(int(A))) \subseteq A$ and hence A is an IFβCS in (X, τ).

Proposition 2.2.22 : If A is both an IFOS and an IFβ^{**}GCS in (X, τ), then A is an IFROS in (X, τ).

Proof : Let A be an IFOS and an IFβ^{**}GCS in X. Then $int(cl(int(A))) \cap cl(int(cl(A))) \subseteq A$, as $A \subseteq A$. Now $int(cl(A)) \subseteq int(cl(A)) \cap cl(A) \subseteq int(cl(int(A))) \cap cl(int(cl(A))) \subseteq A$ by hypothesis. Hence $int(cl(A)) \subseteq A$. Since A is an IFOS, it is IFPOS and $A \subseteq int(cl(A))$. Hence $A = int(cl(A))$ and A is an IFROS in (X, τ).

Proposition 2.2.23 : An IFS A of an IFTS (X, τ) is an IFβ^{**}GCS if and only if $A \bar{q} F \Rightarrow (int(cl(int(A))) \cap cl(int(cl(A)))) \bar{q} F$ for every IFCS F of X.

Proof : Necessity : Let F be an IFCS in X and $A \bar{q} F$, then $A \subseteq F^c$, where F^c is an IFOS, then $int(cl(int(A))) \cap cl(int(cl(A))) \subseteq F^c$, by hypothesis. Hence by Definition 1.1.14, $(int(cl(int(A))) \cap cl(int(cl(A)))) \bar{q} F$.

Sufficiency : Let U be an IFOS such that $A \subseteq U$. Then U^c is an IFCS and $A \subseteq (U^c)^c$. By hypothesis, $A \bar{q} U^c \Rightarrow (int(cl(int(A))) \cap cl(int(cl(A)))) \bar{q} U^c$. Hence $int(cl(int(A)) \cap$

cl(int(cl(A))) $\subseteq (U^c)^c$ = U. Therefore int(cl(int(A))) $\cap$ cl(int(cl(A))) $\subseteq$ U and A is an IFβ^{**}GCS in X.

Proposition 2.2.24 : For an IFOS A in (X, τ), the following conditions are equivalent:

(i) A is an IFCS,

(ii) A is an IFβ^{**}GCS and an IFQ-set.

Proof : (i) $\Rightarrow$ (ii) Since A is an IFCS, it is an IFβ^{**}GCS, by Proposition 2.2.3. Now int(cl(A)) = int(A) = A = cl(A) = cl(int(A)), by hypothesis. Hence A is an IFQ-set.

(ii) $\Rightarrow$ (i) Since A is both an IFOS and an IFβ^{**}GCS, by Proposition 2.2.22, A is an IFROS. Therefore A = int(cl(A)) = cl(int(A)) = cl(A), by hypothesis. Hence A is an IFCS in X.

Proposition 2.2.25 : An IFS A of X is an IFβ^{**}GCS if int(cl(int(A))) $\cap$ cl(int(cl(A))) $\subseteq$ ker(A).

Proof : Let U be any IFOS such that A $\subseteq$ U. By hypothesis, int(cl(int(A))) $\cap$ cl(int(cl(A))) $\subseteq$ ker(A) and since A $\subseteq$ U, ker(A) $\subseteq$ U. Therefore int(cl(int(A))) $\cap$ cl(int(cl(A))) $\subseteq$ U and hence A is an IFβ^{**}GCS in X.

Proposition 2.2.26 : For an IFβ^{**}GCS A in an IFTS (X, τ), the following conditions hold:

(i) If A is an IFROS then scl(A) is an IFβ^{**}GCS,

(ii) If A is an IFRCS then sint(A) is an IFβ^{**}GCS.

Proof : (i) Let A be an IFROS in (X, τ). Then int(cl(A)) = A. By Result 1.1.29, we have scl(A) = A $\cup$ int(cl(A)) = A. Since A is an IFβ^{**}GCS in X, scl(A) is an IFβ^{**}GCS in X.

(ii) Let A be an IFRCS in (X, τ). Then cl(int(A)) = A. By Result 1.1.29, we have sint(A) = A $\cap$ cl(int(A)) = A. Since A is an IFβ^{**}GCS in X, sint(A) is an IFβ^{**}GCS in X.

Proposition 2.2.27 : If an IFS A of an IFTS (X, τ) is intuitionistic fuzzy nowhere dense, then A is an IFβ^{**}GCS in X.

Proof : If A is an intuitionistic fuzzy nowhere dense subset, then by definition 1.1.18 we get, int(cl(A)) = $0_\sim$. Let A$\subseteq$ U where U is an IFOS in X. Then cl(int(cl(A))) $\cap$ int(cl(int(A))) $\subseteq$ cl(int(cl(A))) $\cap$ int(cl((A)) = cl(int(cl(A))) $\cap$ $0_\sim$ = $0_\sim$ $\subseteq$ U and hence A is an IFβ^{**}GCS in X.

Proposition 2.2.28 : If every IFS in (X, τ) is an IFβ^{**}GCS then IFO(X) $\subseteq$ IFγC(X).

Proof : Assume that every IFS in (X, τ) is an IFβ^{**}GCS. Let U $\in$ IFO(X). Then as U $\subseteq$ U, and by hypothesis, int(cl(U)) $\cap$ cl(int(U)) $\subseteq$ int(cl(int(U))) $\cap$ cl(int(cl(U))) $\subseteq$ U. Therefore U $\in$ IFγC(X) and hence IFO(X) $\subseteq$ IFγC(X).

2.3 Intuitionistic Fuzzy β^{**} Generalized Open Sets

In this section we have discussed and analyzed some of the properties of intuitionistic fuzzy β^{**} generalized open set and produced many interesting characterization theorems.

Definition 2.3.1: The complement A^c of an IFβ^{**}GCS A in an IFTS (X, τ) is called an **intuitionistic fuzzy β^{**} generalized open set** (IFβ^{**}GOS) in X.

The family of all IFβ^{**}GOSs of an IFTS (X, τ) is denoted by IFβ^{**}GO(X).

Example 2.3.2 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2,$1_\sim$} be an IFT on X, where G_1 = $\langle$x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)$\rangle$, G_2 = $\langle$x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)$\rangle$. Then (X, τ) is an IFTS. Here the IFS A = $\langle$x, (0.3_a, 0.4_b), (0.5_a, 0.6_b)$\rangle$ is an IFβ^{**}GOS in X.

Proposition 2.3.3 : Every IFOS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.4 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where G_1 = ⟨x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)⟩, G_2 = ⟨x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)⟩. Then, the IFS A = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ is an IFβ^{**}GOS but not an IFOS in (X, τ).

Proposition 2.3.5 : Every IFSOS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.6 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where G_1 = ⟨x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)⟩, G_2 = ⟨x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)⟩. Then, the IFS A = ⟨x, (0.7_a, 0.7_b), (0.3_a, 0.3_b)⟩ is an IFβ^{**}GOS but not an IFSOS in (X, τ).

Proposition 2.3.7 : Every IFPOS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.8 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where G_1 = ⟨x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)⟩, G_2 = ⟨x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)⟩. Then, the IFS A = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ is an IFβ^{**}GOS but not an IFPOS in (X, τ).

Proposition 2.3.9 : Every IFαOS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.10 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where G_1 = ⟨x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)⟩, G_2 = ⟨x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)⟩. Then, the IFS A = ⟨x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)⟩ is an IFβ^{**}GOS but not an IFαOS in (X, τ).

Proposition 2.3.11 : Every IFROS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.12 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.8_a, 0.6_b), (0.2_a, 0.4_b)\rangle$. Then, the IFS $A = \langle x, (0.5_a, 0.5_b), (0.5_a, 0.5_b)\rangle$ is an IFβ^{**}GOS but not an IFROS in (X, τ).

Proposition 2.3.13 : Every IFβOS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.14 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be as IFT on X, where $G_1 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.6_a, 0.7_b), (0.4_a, 0.3_b)\rangle$. Then (X, τ) is an IFTS. Here the IFS $A = \langle x, (0.3_a, 0.2_b), (0.7_a, 0.8_b)\rangle$ is an IFβ^{**}GOS but not an IFβOS in (X, τ).

Proposition 2.3.15 : Every IFGOS is an IFβ^{**}GOS in (X, τ) but not conversely in general.

Proof : Straightforward.

Example 2.3.16 : Let X ={a, b} and τ = {$0_\sim$, G, $1_\sim$} be an IFT on X, where $G = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$. Then the IFS $A = \langle x, (0.5_a, 0.6_b), (0.4_a, 0.4_b)\rangle$ is an IFβ^{**}GOS but not an IFGOS in (X, τ).

Remark 2.3.17 : The union of two IFβ^{**}GOSs need not be an IFβ^{**}GOS in (X, τ) in general.

Example 2.3.18 : Let X = {a, b}, $G_1 = \langle x, (0.5_a, 0.7_b), (0.5_a, 0.3_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)\rangle$. Then τ = {$0_\sim$, G_1, G_2, $1_\sim$} is an IFT on X. Here the

IFSs $A = \langle x, (0.5_a, 0.2_b), (0.5_a, 0.6_b)\rangle$ and $B = \langle x, (0.4_a, 0.4_b), (0.5_a, 0.6_b)\rangle$ are IFβ^{**}GOSs in X but $A \cup B = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)\rangle$ is not an IFβ^{**}GOS in X.

Remark 2.3.19 : The intersection of two IFβ^{**}GOSs need not be an IFβ^{**}GOS in (X, τ) in general.

Example 2.3.20 : Let $X = \{a, b\}$, $G_1 = \langle x, (0.6_a, 0.8_b), (0.4_a, 0.2_b)\rangle$ and $G_2 = \langle x, (0.5_a, 0.5_b), (0.4_a, 0.4_b)\rangle$. Then $\tau = \{0_\sim, G_1, G_2, 1_\sim\}$ is an IFT on X. Here the IFSs $A = \langle x, (0.4_a, 0.5_b), (0.5_a, 0.4_b)\rangle$ and $B = \langle x, (0.5_a, 0.2_b), (0.4_a, 0.6_b)\rangle$ are IFβ^{**}GOSs in X but $A \cap B = \langle x, (0.4_a, 0.2_b), (0.5_a, 0.6_b)\rangle$ is not an IFβ^{**}GOS in X.

Proposition 2.3.21 : An IFS A of an IFTS (X, τ) is an IFβ^{**}GOS if and only if $F \subseteq cl(int(cl(A))) \cup int(cl(int(A)))$ whenever F is an IFCS and $F \subseteq A$.

Proof : Necessity : Suppose A is an IFβ^{**}GOS in X. Let F be an IFCS, such that $F \subseteq A$. Then F^c is an IFOS and $A^c \subseteq F^c$, by hypothesis A^c is an IFβ^{**}GCS. We have $int(cl(int(A^c))) \cap cl(int(cl(A^c))) \subseteq F^c$. Therefore $F \subseteq cl(int(cl(A))) \cup int(cl(int(A)))$.

Sufficiency : Let U be an IFOS, such that $A^c \subseteq U$ and $U^c \subseteq A$ then $U^c \subseteq cl(int(cl(A))) \cup int(cl(int(A)))$, by hypothesis. Therefore $int(cl(int(A^c))) \cap cl(int(cl(A^c))) \subseteq U$ and A^c is an IFβ^{**}GCS. Hence A is an IFβ^{**}GOS in X.

Proposition 2.3.22 : Let (X, τ) be an IFTS. Then for every $A \in IFS(X)$ and for every $B \in IFRO(X)$, $B \subseteq A \subseteq cl(int(cl(B))) \cap int(cl(int(B)))$ implies $A \in IF\beta^{**}GO(X)$.

Proof : Let B be an IFROS in X. Then $B = int(cl(B))$. By hypothesis, $A \subseteq cl(int(cl(B))) \cap int(cl(int(B))) \subseteq cl(B) \cap int(cl(B)) \subseteq int(cl(B)) \subseteq int(cl(A))$ as $B \subseteq A$. Therefore A is an IFPOS and by Proposition 2.3.7, A is an IFβ^{**}GOS. Hence $A \in IF\beta^{**}GO(X)$.

2.4 Theoretical Applications of Intuitionistic Fuzzy β^{**} Generalized Closed Sets

In this section we have investigated some theoretical applications of intuitionistic fuzzy β^{**} generalized closed sets by defining new spaces and obtained many interesting propositions.

Definition 2.4.1 : An IFTS (X, τ) is an **intuitionistic fuzzy $\beta^{**}pT_{1/2}$** (IF$\beta^{**}pT_{1/2}$) **space** if every IFβ^{**}GCS is an IFPCS in X.

Example 2.4.2 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.3_b), (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.3_b)\rangle$. Then, IFPC(X) = {$0_\sim$, $1_\sim$, $\mu_a \in [0, 1]$, $\mu_b \in [0, 1]$, $\nu_a \in [0, 1]$, $\nu_b \in [0, 1]$ / $0 \le \mu_a + \nu_a \le 1$ and $0 \le \mu_a + \nu_a \le 1$} and IF$\beta^{**}$GC(X) ={$0_\sim$,$1_\sim$, $\mu_a \in [0, 1]$, $\mu_b \in [0, 1]$, $\nu_a \in [0, 1]$, $\nu_b \in [0, 1]$ / $0 \le \mu_a + \mu_b \le 1$, $0 \le \mu_a + \mu_b \le 1$}. Therefore the space (X, τ) is an intuitionistic fuzzy $\beta^{**}pT_{1/2}$ space, as every IFβ^{**}GCS is an IFPCS in this (X, τ).

Definition 2.4.3 : An IFTS (X, τ) is an **intuitionistic fuzzy $\beta^{**}gT_{1/2}$** (IF$\beta^{**}gT_{1/2}$) **space** if every IFβ^{**}GCS is an IFGCS in X.

Example 2.4.4 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.3_b), (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.3_b)\rangle$. IFGC(X) = {$0_\sim$, $1_\sim$, $\mu_a \in [0, 1]$, $\mu_b \in [0, 1]$, $\nu_a \in [0, 1]$, $\nu_b \in [0, 1]$ / $0 \le \mu_a + \nu_a \le 1$ and $0 \le \mu_a + \nu_a \le 1$}. IF$\beta^{**}$GC(X) = {$0_\sim$, $1_\sim$, $\mu_a \in [0, 1]$, $\mu_b \in [0, 1]$, $\nu_a \in [0, 1]$, $\nu_b \in [0, 1]$ / $0 \le \mu_a + \mu_b \le 1$, $0 \le \mu_a + \mu_b \le 1$}. The space (X, τ) is an intuitionistic fuzzy $\beta^{**}gT_{1/2}$ space, as every IFβ^{**}GCS is an IFGCS in this (X, τ).

Remark 2.4.5 : Not every IF$\beta^{**}pT_{1/2}$ space is an IF$T_{1/2}$ space. This can be seen easily from the following example.

Example 2.4.6 : Let X = {a, b} and τ = {$0_\sim$, G_1, G_2, $1_\sim$} be an IFT on X, where $G_1 = \langle x, (0.5_a, 0.3_b), (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.4_b), (0.5_a, 0.3_b)\rangle$. Then, IFPC(X) = {$0_\sim$, $1_\sim$, $\mu_a \in [0, 1]$, $\mu_b \in [0, 1]$, $\nu_a \in [0, 1]$, $\nu_b \in [0, 1]$ / $0 \le \mu_a + \nu_a \le 1$ and

$0 \leq \mu_a + \nu_a \leq 1\}$ and IFβ^{**}GC(X) $=\{0_\sim, 1_\sim, \mu_a \in [0, 1], \mu_b \in [0, 1], \nu_a \in [0, 1], \nu_b \in [0, 1] \,/\, 0 \leq \mu_a + \mu_b \leq 1, 0 \leq \mu_a + \mu_b \leq 1\}$. Therefore the space (X, τ) is an IFβ^{**}p$T_{1/2}$ space, but not an IF$T_{1/2}$ space, since the IFS A = ⟨x, $(0.6_a, 0.7_b)$, $(0.4_a, 0.3_b)$⟩ is an IFGCS, but not an IFCS, as cl(A) = $1_\sim \neq$ A.

Proposition 2.4.7 : An IFTS (X, τ) is an IFβ^{**}g$T_{1/2}$ space if and only if IFGO(X) = IFβ**GO(X).

Proof : Necessity : Let A be an IFβ^{**}GOS in (X, τ), then A^c is an IFβ^{**}GCS in (X, τ). By hypothesis, A^c is an IFGCS in (X, τ). Hence A is an IFGOS in (X, τ). Thus IFGO(X) = IFβ^{**}GO(X).

Sufficiency : Let A be an IFβ^{**}GCS in (X, τ). Then A^c is an IFβ^{**}GOS in (X, τ). By hypothesis, A^c is an IFGOS in (X, τ). Therefore A is an IFGCS in (X, τ). Hence (X, τ) is an IFβ^{**}g$T_{1/2}$ space.

Proposition 2.4.8 : Let X be an IFβ^{**}p$T_{1/2}$ space. Then for an IFS A the following conditions are equivalent :

(i) A ∈ IFβ^{**}GO(X),

(ii) A ⊆ int(cl(A)),

(iii) There exists an IFOS G such that G ⊆ A ⊆ int(cl(A)).

Proof : (i) ⇒ (ii) Let A ∈ IFβ^{**}GO(X). This implies A is an IFPOS in X, since X is an IFβ^{**}p$T_{1/2}$ space. Then A^c is an IFPCS in X. Therefore cl(int(A^c)) ⊆ A^c. This implies A ⊆ int(cl(A)).

(ii) ⇒ (iii) Let A ⊆ int(cl(A)). Hence int(int(A)) ⊆ A ⊆ int(cl(A)). Then there exists IFOS G in X such that G ⊆ A ⊆ int(cl(A)), where G = int(A).

(iii) ⇒ (i) Suppose that there exists IFOS G such that G ⊆ A ⊆ int(cl(A)). It is clear that $(\text{int}(\text{cl}(A)))^c \subseteq A^c$. This implies cl(int($A^c$)) ⊆ A^c .That is A^c is an IFPCS in X. This implies A is an IFPOS in X. Hence A ∈ IFβ^{**}GO(X).

Proposition 2.4.9 : Let (X, τ) be an $IF\beta^{**}pT_{1/2}$ space. Then

(i) Any union of $IF\beta^{**}$GCS is an $IF\beta^{**}$GCS,

(ii) Any intersection of $IF\beta^{**}$GOS is an $IF\beta^{**}$GOS.

Proof : (i) Let $\{A_i\}_{i\in I}$ be a collection of $IF\beta^{**}$GCSs. Since (X, τ) is an $IF\beta^{**}pT_{1/2}$ space, every $IF\beta^{**}$GCS is an IFPCS. As any union of IFPCS is an IFPCS, $\bigcup_{i\in I} A_i$ is an IFPCS. Since every IFPCS is an $IF\beta^{**}$GCS, $\bigcup_{i\in I} A_i$ is an $IF\beta^{**}$GCS.

(ii) can be proved easily by taking complement in (i).

Proposition 2.4.10 : An IFTS (X, τ) is an intuitionistic fuzzy $\beta^{**}pT_{1/2}$ space if and only if $IF\beta^{**}GO(X) = IFPO(X)$.

Proof : Necessity : Let A be an $IF\beta^{**}$GOS in (X, τ), then A^c is an $IF\beta^{**}$GCS in (X, τ). By hypothesis, A^c is an IFPCS in (X, τ). Hence A is an IFPOS in X. Thus $IF\beta^{**}GO(X) = IFPO(X)$.

Sufficiency : Let A be an $IF\beta^{**}$GCS in (X, τ). Then A^c is an $IF\beta^{**}$GOS in (X, τ). By hypothesis, A^c is an IFPOS in (X, τ) and hence A is an IFPCS (X, τ). Therefore (X, τ) is an intuitionistic fuzzy $\beta^{**}pT_{1/2}$ space.

Proposition 2.4.11 : For any IFS A in (X, τ) where X is an $IF\beta^{**}pT_{1/2}$ space, $A \in IF\beta^{**}GO(X)$ if and only if for every IFP $p_{(\alpha,\beta)} \in A$, there exists an $IF\beta^{**}$GOS B in X such that $p_{(\alpha,\beta)} \in B \subseteq A$.

Proof : Necessity : If $A \in IF\beta^{**}GO(X)$, then we can take B = A so that $p_{(\alpha,\beta)} \in B \subseteq A$ for every IFP $p_{(\alpha,\beta)} \in A$.

Sufficiency : Let A be an IFS in (X, τ) and assume that there exists $B \in IF\beta^{**}GO(X)$ such that $p_{(\alpha,\beta)} \in B \subseteq A$. Since X is an $IF\beta^{**}pT_{1/2}$ space, B is an IFPOS. Then $A = \bigcup_{p_{(\alpha,\beta)} \in A} \{p_{(\alpha,\beta)}\} \subseteq \bigcup_{p_{(\alpha,\beta)} \in A} B \subseteq A$. Therefore $A = \bigcup_{p_{(\alpha,\beta)} \in A} B$, which is an IFPOS. Hence A is an $IF\beta^{**}$GOS in X.