

TWO DIFFERENT APPROACHES TO FUZZY SEPARATION AXIOMS

BY
N. ANUSHEELA



A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE
FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN
(DEEMED UNIVERSITY) COIMBATORE-641 043 IN PARTIAL
FULFILMENT OF THE REQUIREMENTS FOR THE
DEGREE OF MASTER OF SCIENCE
IN MATHEMATICS

APRIL 1994

CERTIFICATE

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INTRODUCTION

INTRODUCTION

The fundamental concept of a fuzzy set, introduced by zadeh[14] in 1965 laid a foundation for building new branches of fuzzy mathematics. In 1968 Chang[1] introduced the notion of fuzzy topological spaces from then on, much research work has been carried out by generalising the concepts in Ordinary topological spaces to fuzzy topological spaces. Like other concepts separation axioms have been generalised to fuzzy topological spaces in various ways by different authors. Some of the contribution to fuzzy separation axioms are due to Gautner, Steinlage and Warren[3], Hutton and Reilly [5], Rodabaugh [11], Mirasarkar [8], Wuyts and Lowen [13], Ghanim, Kerre and Mashhour [4] Rekha srivastava, Lal and Arun. K. Srivastava [10] S.Ganguly, and S. Saha [2], M.Macho stadler and M.A. de Prada Vicente [1].

In this dissertation the fuzzy separation axioms in the following two papers are taken for discussion.

- (1) S. Ganguly, and S. Saha, On separation axioms and separations of connected sets in fuzzy topological spaces [2].
- (2) M. Macho Stadler and M.A. de Prada Vicente, Strong separations and strong. Countability in fuzzy topological spaces[7].

The approach in the first paper is based on the notion of quasi - coincidence and the approach in the second paper is based on level topologies.

P.M. Pu and Y.M.Liu have introduced the concept of quasio-coincidence between a fuzzy point and a fuzzy set and also between two fuzzy sets. S.Ganguly and S.Saha[2] have defined the following concepts using the notion of quasi-coincidence.

- (i) Weak and Strong separation of two fuzzy points.
- (ii) Weak and Strong separation of a fuzzy point and a fuzzy set.
- (iii) Weak and Strong separation of two fuzzy sets.

These concepts are used to define the fuzzy separation axioms T_0, T_1, T_2, T_3, T_4 and T_5 . Characterization of these spaces which are analogous to those in general topology are obtained. Chapter I is concluded by proving the following implications.

$$T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

In Chapter II we discuss the fuzzy separation axioms due to M.Macho Stadler and M.A. de Parada Vicente [7]. Here the fuzzy separation axioms are defined using level topologies. Given a fuzzy topological space (X, δ) the level topology $i_t(\delta)$ for each $t \in [0, 1)$ is defined by $i_t(\delta) = \{U^{-1}(t, 1], U \in \delta\}$. A fuzzy topological space (X, δ) is said to be

- (i) a t - T_i ($i = 1, 2$) f.t.s iff $(X, i_t(\delta))$ is a T_i -space.
- (ii) a F - T_i ($i = 1, 2$) f.t.s iff for each $t \in [0, 1)$ and $x \in X$, $t \cdot \chi_{\{x\}}$ is a fuzzy closed set.
- (iii) an Ultra- T_i ($i = 1, 2$) f.t.s. iff the topological space $(X, i(\delta))$ is T_i .
- (iv) a strong- T_i ($i = 1, 2$) f.t.s iff it is t - T_i for each $t \in [0, 1)$.

They [] have proved that

F - $T_i = \text{strong } T_i = \text{Ultra-}T_i$ for $i = 1, 2$ and constructed examples to indicate that the reciprocal implications are not true in general.

Similar definitions for regular and normal spaces are given. Equivalent conditions for $t-T_1$, $t-T_2$, t -regular and t -normal spaces have been obtained. Moreover all the four concepts, Strong- T_1 , Strong- T_2 , Strong-regular and Strong-normal are hereditary and productive.

PRELIMINARIES

PRELIMINARIES

The preliminary definitions and results needed for the discussion of this dissertation are given here

Defintion :

Let X be any set. A fuzzy set on X is a function from X to $I = [0,1]$
(i.e) it is a member of I^X .

Let U and V be two fuzzy sets then $U \subset V \implies U(x) \leq V(x)$ for all $x \in X$.

$$C = U \cup V \implies C(x) = (U \cup V)(x)$$

$$= \text{Max} \{U(x), V(x)\} \text{ for all } x \in X$$

$$C = U \cap V \implies C(x) = (U \cap V)(x)$$

$$= \text{Min}\{U(x), V(x)\} \text{ for all } x \in X$$

$$C = \cup U_{\alpha} \implies C(x) = (\cup U_{\alpha})(x)$$

$$= \sup \{U_{\alpha}(x) / \alpha \in \Lambda\} \text{ for all } x \in X$$

$$C = \cap U_{\alpha} \implies C(x) = (\cap U_{\alpha})(x) = \inf \{U_{\alpha}(x) / \alpha \in \Lambda\} \text{ for all } x \in X$$

The Complement of a fuzzy set U is given by $1 - U(x)$, for all $x \in X$ and it is denoted by U^c .

Definition:

The support of a fuzzy set U is given by $U_0 = \{x / U(x) > 0\}$

Definition :

Let f be a function from $X \rightarrow Y$. Let U be a fuzzy set in Y . Then the inverse of U is defined by $f^{-1}(U)(x) = U(f(x))$ for all $x \in X$

Definition :

Let U be a fuzzy set in X . Then the image of U is given by

$$f(U)(y) = \sup_{z \in f^{-1}(y)} U(z) \text{ if } f^{-1}(y) \text{ is not empty.}$$

$$= 0 \quad \text{otherwise}$$

$$\text{where } f^{-1}(y) = \{x / f(x) = y\}$$

Definition :

A fuzzy set $c_t \in I^X$ defined by $c_t(x) = t$, for every $x \in X$ is defined by $c_t(x) = t$, for every $x \in X$ is called a Constant fuzzy set ($t \in [0,1]$)

Definition :

An ordinary subset A of X can be considered as a fuzzy set indentifying it with its characteristic function, χ_A such fuzzy sets are called as Crisp sets.

Notation : The Crisp sets corresponding to ϕ and X denoted by c_0 and c_1
(i.e) $c_0(x) = 0$ and $c_1(x) = 1$

Defintion :

Let X be any set. A fuzzy topology on X is a family δ of fuzzy sets in X which satisfies the following conditions.

- (i) $c_0, c_1 \in \delta$
- (ii) for all $U, V \in \delta$, $U \cap V \in \delta$
- (iii) If $U_\alpha \in \delta$ for each $\alpha \in \Lambda$ then

$$\text{Sup}_{\alpha \in \Lambda} U_\alpha \in \delta$$

Then δ is called a fuzzy topology on X and every member of δ is called an open fuzzy set.

A fuzzy set is closed iff its complement is open.

Notation : A fuzzy topological space is written shortly by f.t.s

Definition : The interior of a fuzzy set U is defined by $\text{int } U = \sup \{V : V \subset U, V \in \delta\}$

Definition : The closure of a fuzzy set $U \in I^X$ is defined by $\text{cl } U = \inf \{V : U \subset V, V^c \in \delta\}$

Defintion : If (X, δ) is an fuzzy topological space. and $Y \subset X$, the collection

$$\delta_Y = \{U \cap \chi_Y : U \in \delta\}$$

is a fuzzy topology on Y . We refer to (Y, δ_Y) as fuzzy topological spaces.

Defintion :

Let $\tau(X)$ be the set of all topologies on X and $W(X)$ be the set of all fuzzy topologies on X . On R consider the topology $\tau_r = \{(\alpha, \infty) / \alpha \in R\} \cup \{\emptyset\}$ topologies on X . The topological space one obtains by giving I the induced topology is denoted by I_r .

Define $w : \tau(X) \rightarrow W(X) : \tau \rightarrow w(\tau)$ where

$w(\tau) = \mathcal{C}(X, I_r)$ the continuous functions from (X, τ) to I_r

Define the mapping $i : w(X) \rightarrow \delta(X) : d \rightarrow i(\delta)$

where $i(\delta)$ is the intial topology on X for the family of "functions" δ and the topological space I_r

Properties: (1) $i(w(\delta)) = \delta$

(2) $\delta \subset w(i(\delta))$

CHAPTER - I

I.

CHAPTER - I

FUZZY SEPARATION AXIOMS BASED ON QUASI - COINCIDENCE.

The concept of quasi-Coincidence between a fuzzy point and a fuzzy set, was introduced by P.M. Pu and Y.M. Liu [9] Using the concept S. Ganguly and S.Saha [2] have defined weak separation and strong separation of fuzzy points and fuzzy sets and have introduced the fuzzy separation axioms in fuzzy topological spaces. Characterisations analogous to those in general topology are obtained. Also they have proved that

$$T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

SECTION 1.1

Preliminaries:

In this chapter we follow the definition of a fuzzy point and its belongingness to a fuzzy set according to P.M. Pu and Y.M. Liu [9].

Definition : [9] 1.1.1

A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$, If its value at x is λ ($0 < \lambda \leq 1$), We denote this fuzzy point by x_λ , where the point x is called its support.

Definition [9] 1.1.2

The fuzzy point x_λ is said to be contained in a fuzzy set A, or to belong to A, denoted by $x_\lambda \in A$ iff $\lambda \leq A(x)$. Evidently, every fuzzy set A, can be expressed as the union of all the fuzzy points which belong to A.

Definition [9] 1.1.3

A fuzzy point x_λ is said to be quasi - Coincident (q - coincident in short) with A, $x_\lambda q A$ iff $\lambda > A^c(x)$ (i.e) $\lambda + A(x) > 1$.

Definition [9] 1.1.4

A fuzzy set A is said to be q - coincident with a fuzzy set B denoted by AqB , iff there exists $x \in X$, such that

$$A(x) > B^c(x) \quad (\text{i.e}) \quad A(x) + B(x) > 1.$$

Hence a fuzzy set A is not q-coincident with a fuzzy set B denoted by \overline{AqB} iff for every $x \in X$ (ie) $A(x) + B(x) \leq 1$.

Remark 1.1.5

$\overline{Aq}(\text{cl}A)^c$ for every fuzzy set A.

Theorem [9] 1.1.6

\overline{AqB} iff A and B^c are not q-Coincident particularly $x_\lambda \in A$ iff x_λ is not q-Coincident with A^c

Proof

Now $A \subset B$

$$\Leftrightarrow A(x) \leq B(x) \text{ for every } x \in X$$

$$\Leftrightarrow A(x) - B(x) + 1 \leq 1 \text{ for every } x \in X$$

$$\Leftrightarrow A(x) \leq 1 - B^c(x) \text{ for every } x \in X$$

$$\Leftrightarrow A \subseteq \overline{B^c} \text{ for every } x \in X.$$

Defintion 1.1.7

A fuzzy set A in a fuzzy topological space (X, δ) is said to be a neighbourhood of a fuzzy point x_λ iff there exists a fuzzy set $B \in \delta$, such that $x_\lambda \in B \subset A$.

Notation : Neighbourhood is shortly written by nbd. A nbd A of a fuzzy point x_λ is said to be open if $A \in \delta$.

A fuzzy set A in (X, δ) is called a q -nbd of x_λ iff there exists $B \in \delta$ such that $x_\lambda q B \subset A$.

Theorem 1.1.8

A fuzzy point $x_\lambda \in \text{cl} A$ iff each q - nbd of x_λ is q -coincident with A

Proof :

$$\text{cl} A = \bigcap \{F / A \subset F\}$$

$$\text{Now } x_\lambda \in \text{cl} A \Leftrightarrow x_\lambda \in \bigcap \{F / A \subset F, F^c \in \delta\}.$$

$\Leftrightarrow \lambda \leq (\cap F)(x)$ for every $F^c \in \delta$ with ACF

$\Leftrightarrow \lambda \leq F(x)$ for every ACF with $F^c \in \delta$

$\Leftrightarrow 1 - \lambda \geq 1 - F(x)$

$\Leftrightarrow 1 - \lambda \geq F^c(x)$ for every ACF with $F^c \in \delta$

$\Leftrightarrow 1 - \lambda \geq B(x)$ for every BCA^c with $B \in \delta$.

In other words for every open set B , satisfying $B(x) > 1 - \lambda$, B is not contained in A^c from theorem 1.1.6, B is not contained in A^c iff B is quasi-coincident with $(A^c)^c = A$. We have thus proved that x_λ is quasi-coincident with A , which is evidently equivalent to what we want to prove.

Section 1.2

Separations of fuzzy points and fuzzy sets.

In this section, weak and strong separations of fuzzy points and fuzzy sets are discussed.

Definitions 1.2.1

Two distinct fuzzy points x_λ and y_μ of X are said to be weakly separated iff when $x \neq y$ there is an open nbd of x_λ which is not q -coincident with y_μ and there is an open nbd of y_μ which is not q -coincident with x_λ when $x = y$ and $\lambda < \mu$, there is a q -nbd of y_μ which is not q -coincident with x_λ .

Definition : 1.2.2

Two distinct fuzzy points x_λ and y_μ of X are said to be strongly separated iff when $x \neq y$ x_λ and y_μ have open nbds, which are not q -coincident; when $x=y$ and $\lambda < \mu$, (say) x_λ has an open nbd, and y_μ has an open- q -nbd which are not q -coincident.

Theorem : 1.2.3

Strong separations of two distinct fuzzy points implies weak separations of those fuzzy points.

Proof :

Case (i) :

If $x \neq y$ then there exists open nbds A and B of x_λ and y_μ respectively such that $\bar{A}qB$.

\implies For every $z \in X$

$$A(z) + B(z) \leq 1 \longrightarrow (1)$$

$$y_\mu \in B \implies \mu \leq B(y)$$

To prove $y_\mu \bar{q} A$

From (1) we get

$$A(y) + B(y) \leq 1$$

$$B(y) \leq 1 - A(y)$$

$$\implies \mu \leq 1 - A(y)$$

$$\implies \mu \leq A^c(y)$$

$$\implies y_\mu \bar{q} A.$$

Similarly it can be shown $x_\lambda \bar{q} B$.

Case :(ii)

If $x=y$ then x_λ has an open nbd A and y_μ has an open q -nbd B which are not q -coincident (ie) $A \bar{q} B$.

Claim : $x_\lambda \bar{q} B$.

Since $A \bar{q} B$, $A(x) + B(x) \leq 1$

Therefore $A(x) \leq 1 - B(x)$

$$\implies \lambda \leq 1 - B(x)$$

$$\implies \lambda \leq B^c(x)$$

$$\implies x_\lambda \bar{q} B.$$

Hence in both cases condition for weak separation is satisfied.

Definition 1.2.4

A fuzzy point x_λ and a fuzzy set A in X such that $x_\lambda \in A$ are said to be weakly separated iff there are open fuzzy sets U and V such that $x_\lambda q U$; $A \bar{C} U$, $A \bar{q} U$ and $x_\lambda \notin V$

Definition 1.2.5

A fuzzy point x_λ and a fuzzy set A in X such that $x_\lambda \in A$ are said to be strongly separated iff there are open fuzzy sets U and V such that $x_\lambda q U$; $A \bar{C} V$, $U \bar{q} V$.

Theorem 1.2.6

Weak separation of a fuzzy point and a fuzzy set is implied by strong separation of that fuzzy point and that fuzzy set.

Proof :

Let x_λ be a fuzzy point and let A be any fuzzy set such that $x_\lambda \in A$. Then there exists open fuzzy sets U and V such that $x_\lambda q U$; $A \subset V, U \bar{q} V$.

To prove $A \bar{q} U$ and $x_\lambda \notin V$.

Since $U \bar{q} V$, for every $x \in X$

$$U(x) + V(x) \leq 1 \text{ -----(1)}$$

$A \subset V \implies A(x) \leq V(x)$ for every x .

therefore $A(x) + U(x) \leq V(x) + U(x) \leq 1$ (by 1)

Since $x_\lambda q U, \lambda > U^c(x)$

$$= 1 - U(x)$$

$$\geq V(x) \text{ -- by(1)}$$

therefore $x_\lambda \notin V$.

therefore x_λ and A are weakly separated.

Definition 1.2.7

Two fuzzy sets A and B of X are said to be weakly separated iff $ClA \bar{q} B$ and $ClB \bar{q} A$.

The condition in the above definition is equivalent to the following condition:

There are open fuzzy sets U and V such that ACU , BCV , $A\bar{q}V$ and $B\bar{q}U$.

Proof: $A\bar{q}V$

$$\implies ACV^c \text{ (by theorem 1.1.6)}$$

$$\implies clA CV^c$$

$$\implies clA CB^c \text{ (by theorem 1.1.6)}$$

$$\implies clA \bar{q} B$$

Similarly $cl B \bar{q} A$.

Conversely assume that the condition in the definition.

$$clA \bar{q} B$$

$$\implies \text{For every } x \in X$$

$$clA(x) + B(x) \leq 1$$

$$\text{(i.e) } clA(x) \leq 1 - B(x) \text{ for every } x$$

$$= B^c(x)$$

$$\therefore clA CB^c$$

$$\implies BC(clA)^c$$

$$\text{Similarly } clB \bar{q} A \implies AC(clB)^c$$

$$\text{Taking } U = (clB)^c \text{ and } V = (clA)^c$$

$$\text{Since } A\bar{q}(clA)^c$$

$$B\bar{q}(clB)^c$$

We get $A\bar{q}V$ and $B\bar{q}U$

Hence the proof.

Definition : 1.2.8

Two fuzzy sets A and B in X are said to be strongly separated iff there exists U and V in X such that ACU , BCV and UqV

Theorem : 1.2.9

Strong separation of two fuzzy sets implies weak separation of the same.

Proof :

Let A and B be any two fuzzy sets which are strongly separated, From the definition of strong separation, there exists open fuzzy sets U and V such that ACU , BCV and UqV

$$ACU \Rightarrow A(x) \leq U(x) \text{ -----} \rightarrow (1) \text{ for every } x \in X$$

$$BCV \Rightarrow B(x) \leq V(x) \text{ -----} \rightarrow (2) \text{ for every } x \in X$$

$$\text{Given } UqV \text{ for every } x \in X, U(x) + V(x) \leq 1$$

from (1) we get

$$A(x) + V(x) \leq 1 \text{ for every } x \in X$$

$$\Rightarrow Aq\bar{V}$$

From (2) we get

$$B(x) + U(x) \leq 1 \text{ for every } x \in X$$

$$\Rightarrow Bq\bar{U}$$

Hence A and B are weakly separated

Hence the Proof.

Section 1.3 :

Fuzzy Separation axioms, their characterizations and implications:

In this section we give the definitions and some characterisations of T_0, T_1, T_2 and normal spaces which are similar to those in ordinary topological spaces. Also it has been proved that $T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

Definition 1.3.1

A fuzzy topological space (X, δ) is said to be T_0 iff for any pair of distinct fuzzy points x_λ and y_μ of X either x_λ has a q -nbd which is not q -coincident with y_μ or y_μ has a q -nbd which is not q -coincident with x_λ .

Definition 1.3.2

A fuzzy topological space (X, δ) is said to be T_1 iff any pair of distinct fuzzy points in X are weakly separated.

Definition 1.3.3

A fuzzy topological space (X, δ) is said to be T_2 iff any pair of distinct fuzzy points in X are strongly separated.

Definition 1.3.4

A fuzzy topological space (X, δ) is said to be regular iff any fuzzy point x_λ and any closed fuzzy set A in X such that x_λ does not belong to A are strongly separated.

A T_1 - regular fuzzy topological space is said to be T_3 - Space

Definition : 1.3.5

A fuzzy topological space (X, δ) is said to be normal iff any two closed fuzzy sets in X which are not q-coincident are strongly separated.

A T_1 - normal fuzzy topological space is said to be a T_4 -space

Definition : 1.3.6

A fuzzy topological space (X, δ) is said to be completely normal iff any two weakly separated fuzzy sets are strongly separated.

A T_1 - completely normal fuzzy topological space is said to be a T_5 Space.

Theorems 1.1.6 and 1.1.8 are useful in obtaining the following characterizations.

Theorem 1.3.7

A fuzzy topological (X, δ) is said to be T_0 iff for any pair of distinct fuzzy point x_λ and y_μ of X either $x_\lambda \notin \text{Cl}y_\mu$ or $y_\mu \notin \text{Cl}x_\lambda$

Proof :

Assume X is T_0

Then x_λ has a q-nbd U which is not q-coincident with y_μ or y_μ has a q-nbd V which is not q-coincident with x_λ (i.e.) $y_\mu \bar{q} U$ or $x_\lambda \bar{q} V$

(i.e) $x_\lambda \notin Cl y_\mu$ or $y_\mu \notin Cl x_\lambda$

Conversely assume the given condition

Suppose $x_\lambda \notin Cl y_\mu$, Take $V = (Cl y_\mu)^c$

Claim :- $y_\mu \bar{q} V$

Consider for any z , $y_\mu(z) + V(z)$

$$\begin{aligned} \text{If } z=y \quad y_\mu(z) + V(z) & \\ &= \mu + 1 - Cl y_\mu(y) \\ &\leq \mu + 1 - \mu \\ &= 1. \end{aligned}$$

If $z \neq y$ $y_\mu(z) = 0$; $y_\mu(z) + V(z) \leq 1$

$$\therefore y_\mu \bar{q} V \longrightarrow (1)$$

Claim : $x_\lambda q V$

Consider

$$x_\lambda(x) + V(x) = \lambda + 1 - Cl y_\mu(x)$$

Since $x_\lambda \notin Cl y_\mu$

$$\lambda > Cl y_\mu(x)$$

$$(i.e) \lambda - Cl y_\mu(x) > 0$$

$$\therefore x_\lambda(x) + V(x) > 1$$

$$\therefore x_\lambda q V$$

(i.e) V is a q-nbd of x_λ \longrightarrow (2).

From (1) and (2), we get that there is a q-nbd V of x_λ which is not q-coincident with y_λ

Similarly if $y_\mu \notin Cl x_\lambda$ by taking $U = (Cl x_\lambda)^c$ We get a q-nbd of y_μ which is not q-coincident with x_λ .

Hence the space is To

Theorem 1.3.8

A fuzzy topological space (X, δ) is T_1 iff every fuzzy point x_λ is closed in X.

Proof :

Let X be T_1 , let x_λ be a fuzzy point in X and y_μ be another fuzzy point in X

Case :(i)

Let $x \neq y$ then there in open nbd V of y_1 such that $x_\lambda \bar{q} V$.

Since $y_1 \in V, 1 \leq V(y)$

$$(i.e) V(y) = 1$$

$$(i.e) V^c(y) = 0$$

Since $\mu > 0$ one gets that $\mu > V^c(y)$

$$(i.e) y_\mu q V$$

Hence V is an open q - nbd of y_μ .

Hence there is an open q-nbd of y_μ which is not q-coincident with x_λ .

\therefore By theorem 1.1.8 $y_\mu \notin \text{Cl}x_\lambda \longrightarrow (1)$

Case (ii)

Let $x=y$ and $\lambda < \mu$ then there exists a q-nbd of y_μ which is not q-coincident with x_λ

$\therefore y_\mu \notin \text{Cl}x_\lambda \text{ ---(2)}$

When $\mu < \lambda$

$$x_\mu < x_\lambda$$

$$(i.e) y_\mu < x_\lambda \implies y_\mu \in x_\lambda$$

So there is nothing to prove when $\mu < \lambda$

From 1 & 2 we get

$x_\lambda = \text{Cl}x_\lambda$, So that x_λ is fuzzy closed.

Conversely, let every fuzzy point x_λ be fuzzy closed in X

Consider two distinct fuzzy points x_λ and y_μ in X

Conversely, let every fuzzy point x_λ be fuzzy closed in X .

Consider two distinct fuzzy points x_λ and y_μ in X .

Case (i)

If $x \neq y$

$$\begin{aligned} \text{Consider } (\text{Cl}x_\lambda)^c(y) &= 1 - \text{Cl}x_\lambda(y) \\ &= 1 - x_\lambda(y) \\ &= 1 \\ &> \mu = y_\mu(y) \end{aligned}$$

$$\therefore y_\mu \in (\text{Cl}x_\lambda)^c \longrightarrow (3)$$

$$\text{Similarly } x_\lambda \in (\text{Cl}y_\mu)^c \longrightarrow (4)$$

Let $V = (\text{Cl}x_\lambda)^c$ and $U = (\text{Cl}y_\mu)^c$

Then $x_\lambda \in U$ and $y_\mu \in V$ and as for any fuzzy set $A \bar{q}(CIA)^c$

We get from (3) and (4)

$$y_\mu \bar{q} U \text{ and } x_\lambda \bar{q} V$$

Case (ii) :

When $x=y$ and $\lambda < \mu$ (say)

Claim $y_\mu \notin x_\lambda$

Suppose $y_\mu \in x_\lambda$

Then $\mu \leq \lambda$ which is a contradiction

$$\therefore y_\mu \notin x_\lambda = (\text{Cl}x_\lambda)$$

Which implies y_μ has a q-nbd which is not q-coincident with x_λ

(by theorem 1.1.8) Hence (X, δ) is T_1 .

Theorem : 1.3.9

A fuzzy topological space (X, δ) is T_2 iff for every fuzzy point x_λ in X

$$x_\lambda = \bigcap \{ C \cap N : N \text{ is an open nbd of } x_\lambda \}$$

Proof :

$$\text{Let } N = \bigcap \{ C \cap N : N \text{ is an open nbd of } x_\lambda \}$$

Assume X is T_2

Let y_μ be a fuzzy point in X

Such that $y_\mu \notin x_\lambda$

Case (i) $x \neq y$

As the space is T_2 there are open fuzzy sets U and V such that $x_\lambda \in U, y_\mu \in V$ and

$$U \bar{q} V$$

$$\text{(i.e.) } V(y) = 1$$

$$\text{Hence } \mu > 0 = V^c(y)$$

$$\text{(i.e.) } y_\mu \bar{q} V$$

$$\text{(i.e.) } V \text{ is a q-nbd of } y_\mu \text{ such that } U \bar{q} V$$

Hence $y_\mu \notin C \cap U$

$$\text{(i.e.) } y_\mu \notin N$$

Case (ii)

$x=y$ and $\lambda < \mu$ (say)

Then as $y_\mu \notin x_\lambda$, $\lambda < \mu$

Hence as the space is T_2 there is an open nbd U of x_λ and open q-nbd V of y_μ

Such that $U \bar{q} V$.

Hence $y_\mu \notin \text{cl}V$

(i.e) $y_\mu \notin V$.

Hence $x_\lambda = N$

Conversely :

Assume $x_\lambda = N$, for every fuzzy point x_λ .

To Prove X is T_2

Let x_λ and y_μ be any two distinct fuzzy points in X .

Case (i) $x \neq y$

Then for any $\gamma > 0$, $y_\gamma \notin x_\lambda = N$

(i.e) $\gamma > \bigcap \{ \text{cl}N_x(y) \}$

By infimum property there is an open nbd U of x_λ such that $\text{cl}U(y) < \gamma$

(i.e.) $y_\gamma \notin \text{cl}U \longrightarrow (1)$

This is true for any δ .

Take γ with $\mu + \gamma < 1$, By (1) there is an open nbd U of x such

$$y_\gamma \notin \text{cl}U$$

Hence $\gamma > \text{cl}U(y)$

$$\text{(i.e.) } 1 - \gamma < (1 - \text{cl}U)(y)$$

$$\text{Let } V = (\text{cl}U)^c$$

$$\text{Then } V(y) > 1 - \gamma > \mu$$

$$\text{(i.e.) } y_\mu \in V$$

$$\text{Also as } U \bar{q}(\text{cl}U)^c, U \bar{q}V$$

Case (ii) $x = y$ and $\lambda < \mu$ (say)

$$\text{Then } y_\mu \notin x_\lambda = N$$

$$\text{(i.e.) } \mu > \bigcap (\text{cl } N_x)(y)$$

Hence there is an open nbd U of x_λ Such that $\text{cl } U(y) < \mu$

$$\text{(i.e.) } y_\mu \notin \text{cl}U \text{ and } y_\mu \notin (\text{cl}U)^c$$

Taking $V = (\text{cl}U)^c$, one gets that $y_\mu \in V$ and $U \bar{q} V$

Hence X is T_2

Theorem 1.3.10

A fuzzy topological space (X, δ) is regular iff for a fuzzy point x_λ and any open fuzzy set G in X such that $x_\lambda \in G$, there is an open fuzzy set U such that $x_\lambda \in U$ and $\text{cl}U \subset G$.

Proof:

Assume X is regular.

Let x_λ be a fuzzy point and G an open fuzzy set such that $x_\lambda \in G$

(i.e) $\lambda > G^c(x)$

Hence $x_\lambda \notin G^c$

By using regularity Condition there are open fuzzy sets U and V such that $x_\lambda \in U$

$G^c \subset V$ and $U \bar{q} V$

Now $U \bar{q} V \Rightarrow U(z) \leq 1 - V(z)$

$\Rightarrow U \subset V^c$

$\Rightarrow \text{cl}U \subset V^c$

$\Rightarrow \text{cl}U \subset G$ as $G^c \subset V \Rightarrow V^c \subset G$.

Conversely, assume the given condition Let $x_\lambda \notin A$ where A is fuzzy closed

$x_\lambda \notin A \Rightarrow x_\lambda \in A^c$

Let $G = A^c$

By assumption there is an open set U such that $x_\lambda \in U$ and $\text{cl}U \subset G$

Take $V = (\text{cl}U)^c$ then $A = G^c \subset V$

and as $U \subset V$, $U \subset V$

Hence X is regular.

The following theorem can be easily verified.

Theorem 1.3.11

A fuzzy topological space (X, δ) is regular iff for a subset A and closed fuzzy set B such that $x_\lambda \in B$ where ever $x_\lambda \notin A$, there are two open fuzzy sets U and V such that $B \subset V$, $U \subset A$ and $U \subset V$.

Theorem 1.3.12

A fuzzy topological space (X, δ) is normal iff for a fuzzy closed set F and an open fuzzy set U such that $F \subset U$, there exists an open fuzzy set V such that $F \subset V \subset U$.

Proof :

Let X be normal.

Let $F \subset U$ with F fuzzy closed and U fuzzy open.

$F \subset U \Rightarrow F(z) \leq U(z)$

$\Rightarrow F(z) - U(z) \leq 0$

$$\Rightarrow F(z) + U^c(z) \leq 1$$

$$\Rightarrow F\bar{q}U^c$$

Hence there are open fuzzy sets V_1 and V_2

$$V_1 \bar{q} V_2 \Rightarrow V_1 \mathbf{C}(V_2)^c$$

$$\Rightarrow (clV_1) \mathbf{C}(V_2)^c \mathbf{C} U$$

Thus $F \mathbf{C} V_1 \mathbf{C} clV_1 \mathbf{C} U$

Conversely

Let A and B be two closed fuzzy sets such that $A\bar{q}B$ (i.e) $A \mathbf{C} B^c$

By assumption there is an open fuzzy set U such that $A \mathbf{C} U \mathbf{C} clU \mathbf{C} B^c$

Hence $B \mathbf{C} (clU)^c$

Taking $V = (clU)^c$ we get $B \mathbf{C} V$ and $U \bar{q} V$

Hence X is normal.

This chapter is concluded proving the following implications.

Theorem 1.3.13

$$T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$$

Proof :

To Prove $T_1 \Rightarrow T_0$

Let x_λ and y_μ be any two distinct fuzzy points of X .

Case (i)

Let $x \neq y$ then there exists an open nbd V of x_1 such that $y_\mu \bar{q} V$

$$x_1 \in V \Rightarrow V(x) = 1$$

$$\text{since } \lambda > 0 = 1 - V(x)$$

$$\lambda > V^c(x)$$

$$\text{(i.e.) } x_\lambda q V$$

$$\text{(i.e.) } V \text{ is a } q\text{-nbd of } x_\lambda \text{ such that } y_\mu \bar{q} V$$

Case (ii)

when $x = y, \lambda < \mu$ (say)

By assumption there exists a q -nbd of y_μ which is not q -coincident with x_λ

Hence $T_1 \Rightarrow T_0$

To prove $T_2 \Rightarrow T_1$

Since strong separations of two fuzzy points implies weak separation of the same, to space is T_2 implies that it is T_1

To prove $T_3 \Rightarrow T_2$

Let x_λ, y_μ be any two distinct fuzzy points in X

Let $x \neq y$

$$x_\lambda(x) = \lambda > 0 = y_\mu(x) \text{ (since } x \neq y \text{)}$$

$$\text{(i.e.) } \lambda > y_\mu(x)$$

therefore $x_\lambda \notin y_\mu$

Since space is T_1 , y_μ is fuzzy closed. Hence there exists an open fuzzy set V such that $x_\lambda \in V$ and $y_\mu \bar{q} V$

$$x_\lambda \in V \implies \lambda < V(x)$$

$$1 - \lambda > 1 - V(x)$$

$$= V^c(x)$$

$$\implies x_{1-\lambda} \notin V^c$$

As the space is regular, there exists two open fuzzy sets U_1 and U_2 .

Such that $x_{1-\lambda} q U_1$ and $V^c \subset U_2$ and $U_1 \bar{q} U_2$

$$x_{1-\lambda} q U_1$$

$$\implies 1 - \lambda > U_1^c(x)$$

$$\lambda < 1 - U_1^c(x) \text{ for some } x \in X$$

$$= U_1(x) \text{ for some } x \in X$$

As $y_\mu \bar{q}V, y_\mu \in V^c \subset U_2$

Hence U_1 and U_2 satisfy the required condition.

In a similar way the case $x=y$ can be dealt with

To prove $T_4 \implies T_3$

Let x_λ be any fuzzy point and A be any closed fuzzy set such that $x_\lambda \notin A$

$$x_\lambda \notin A \implies \lambda > A(x)$$

Choose λ_0 such that $\lambda > \lambda_0 > A(x)$

Claim : $x_{1-\lambda_0} \bar{q}A$

Consider $x_{1-\lambda_0}(x) + A(x)$

$$= 1 - \lambda_0 + A(x)$$

$$< 1 - \lambda_0 + \lambda_0 = 1$$

Therefore $x_{1-\lambda_0}(x) + A(x) \leq 1$

For any $z \neq x$ as $A(z) \leq 1$

$$x_{1-\lambda_0}(z) + A(z) = 0 + A(z) \leq 1$$

Hence the claim.

Since the space is T_1 $x_{1-\lambda_0}$ is fuzzy closed.

Therefore by normality there exists open fuzzy sets U_1 and U_2 such that

$$x_{1-\lambda_0} \in U_1, A \subset U_2 \text{ and } U_1 \bar{q}U_2$$

Hence $1 - \lambda_0 \leq U_1(x)$

(i.e.) $\lambda_0 \geq U_1^c(x)$

Since $\lambda > \lambda_0 \geq U_1^c(x)$, $x_\lambda \notin U_1$

Already $A \subset U_2$ and $U_1 \bar{q} U_2$

Hence the space is T_3 .

To prove $T_3 \implies T_4$

Let A and B be two closed fuzzy sets, Such that $A \bar{q} B$.

Claim : A and B are weakly separated.

Take $U = B^c$ and $V = A^c$

Since $A \bar{q} B$ for every $x \in X$

$$A(x) + B(x) \leq 1$$

$$B(x) \leq 1 - A(x) = A^c(x)$$

Therefore $B \subset A^c$. Similarly $A \subset B^c$

Since $A \bar{q} (C \setminus A)^c$ and $B \bar{q} (C \setminus B)^c$. Since A and B are closed sets $A \bar{q} A^c$ and $B \bar{q} B^c$

(i.e) $A \bar{q} U$ and $B \bar{q} V$

A and B are weakly separated

Since the space is T_5 , A and B are strongly separated.

Therefore the space is T_4 .

CHAPTER - II

CHAPTER -II

FUZZY SEPARATION AXIOMS BASED ON LEVEL TOPOLOGIES

This chapter is devoted to the study of fuzzy separation axioms due to M.Macho Stadler and M.A. de Prada Vicente[7]. These authors have developed t-level theory in fuzzy topology, in their work on t-prefilter theory [6], using this concept of t-Ti (i=1,2), F-Ti(i=1,2), strong -Ti (i=1,2), Ultra-Ti (i=1,2) fuzzy topological spaces have been defined. Similar fuzzy topological spaces have been defined corresponding to regularity and normality notions. It has been proved that $FT_i \implies \text{Strong } -Ti \implies \text{Ultra-Ti}$. Examples are given to show that the reciprocal implications donot hold. Equivalent conditions for a space to be t-T₁, t-T₂, t-regular and t-normal have been obtained and it has b shown that the concepts of strong -T₁, Strong T₂, Strong- regular and strong normal are hereditary and productive.

Now, we give the preliminaries needed for this chapter.

Section 2.1

Preliminaries

In this chapter we follow the definition of a fuzzy point and its belongingness to a fuzzy set according to C.K. Wong [12].

Definition : [12] 2.1.1

A fuzzy point x_t is a fuzzy set, whose support is the single point x and its value is $t \in (0,1)$.

Notation : The collection of all fuzzy point in x is denoted by $\beta_f(x)$.

Definition : [12] : 2.1.2

The fuzzy point $p=x_t$ is set to belong to U (written as $p \in U$) if $t < U(x)$.

Definition : 2.1.3

Let (X, δ) be a f.t.s., for each $t \in [0, 1)$, the level topology $i_t(\delta)$ is defined by, $i_t(\delta) = \{ U^{-1}(t, 1]; U \in \delta \}$

Definition: 2.1.4

If (X, τ) is a topological space and iff for all $t \in [0, 1)$, we put $w_t(\tau) = \{ U \in I^X; U^{-1}(t, 1] \in \tau \}$,

then $w_t(\tau)$ is a fuzzy topology on X which will be called t -topologically generated fuzzy topology.

Definition 2.1.5

A filter on a set X is a collection \mathcal{F} of nonempty subset of X satisfying the following conditions :

- (i) $\mathcal{F} \neq \phi$
- (ii) $A \cap B \in \mathcal{F}$ for every $A, B \in \mathcal{F}$
- (iii) If $A \subset B$ and if $A \in \mathcal{F}$ then $B \in \mathcal{F}$

Notation : We denote the collection of all filters on X by $\mathcal{F}(X)$

Defintion 2.1.6

A **prefilter** on a set X is a collection \mathcal{F} of nonzero fuzzy sets on X satisfying the following conditions:

- (i) $\mathcal{F} \neq \phi$
- (ii) $A \cap B \in \mathcal{F}$ for every pair of fuzzy sets $A, B \in \mathcal{F}$
- (iii) If $A \subset B$ and if $A \in \mathcal{F}$ then $B \in \mathcal{F}$

Notation : (i) We denote the collection of all pre-filters on X by $P(X)$

(ii) $P^t(X)$ will denote the set of all prefilters on X which exclude the constant functions C_t (i.e $\mathcal{F} \in P^t(X)$ iff $\mathcal{F} \in P(X)$ and $C_t \notin \mathcal{F}$)

The elements of $P^t(X)$ are called **t-Prefilters**

Definition : 2.1.7

Associated with each t- prefilter \mathcal{F} we define a filter on X by

$$i_t(\mathcal{F}) = \{U^{-1}(t, 1) : U \in \mathcal{F}\}$$

Associated with each filter $\overline{\mathcal{F}}$ on X we define a t-prefilter for each $t \in (0, 1)$ as follows

$$w_t(\overline{\mathcal{F}}) = \{U \in \Gamma^X : U^{-1}(t, 1) \in \overline{\mathcal{F}}\}$$

Definition : 2.1.7

Associated with each t -prefilter \mathcal{F} we define a filter on X by

$$i_t(\mathcal{F}) = \{U^{-1}(t,1) : U \in \mathcal{F}\}$$

Associated with each filter \mathcal{F} on X we define a t -prefilter for each $t \in (0,1)$ as follows

$$w_t(\mathcal{F}) = \{U \in I^X : U^{-1}(t,1) \in \mathcal{F}\}$$

Notation : The following set containing $\mathcal{B}_F(X)$ is denoted by

$$\mathcal{B}_F^*(X) = \{x_t : t \in (0,1) \text{ and } x \in X\}$$

Definition : 2.1.8

If (X, δ) is a f.t.s. $\mathcal{F} \in \mathcal{P}(X)$ and $x_t \in \mathcal{B}_F^*(X)$, we say \mathcal{F} t -converges to $p=x_t$

(noted as $\mathcal{F} \xrightarrow{t} p$) if :

- (i) $\mathcal{F} \in \mathcal{P}^t(X)$
- (ii) The filter $i_t(\mathcal{F})$ on X converges to x in the following topological space $(X, i_t(\delta))$

Theorem : 2.1.9

Let (X, δ) be an f.t.s. and $Y \subset X$, then $i_t(\delta_Y) = (i_t(\delta))_Y$

Proof :

$$\begin{aligned} i_t(\delta_Y) &= \{ U_Y^{-1}(t, 1] / U_Y \in \delta_Y \} \\ &= \{ U^{-1}(t, 1] \cap Y / U \in \delta \} \\ &= (i_t(\delta))_Y. \end{aligned}$$

Theorem 2.1.10

Let X be a set and $\{(Y_j, \gamma_j); j \in J\}$

a family of f.t.s. 's. For all $j \in J$, let f_j be a function from X to Y_j . Then

$$i_t \left(\text{Sup}_{j \in J} f_j^{-1}(v_j) \right) = \text{Sup}_{j \in J} f_j^{-1}(i_t(v_j)).$$

Remark : 2.1.11

The Union $\cup \{f_j^{-1}(v_j) / j \in J\}$ is a sub base for a fuzzy topology on X making every f_j fuzzy continuous and is the smallest fuzzy topology with this property.

It is denoted by $\text{Sup}_{j \in J} f_j^{-1}(v_j)$

Definition : 2.1.12

If $X = \prod_{j \in J} X_j$ and $P_j : X \rightarrow X_j$ is the j th Projection the Product fuzzy topology

$\delta = \pi \delta_j$ over X is defined as the initial fuzzy topology for collection

$\{ p_j / j \in J \}$ and $\{ \delta_j / j \in J \}$

(i.e) $\delta = \text{Sup}_{j \in J} \{ P_j^{-1}(\delta_j) \}$.

Corollary : 2.1.13

Let $\{ (X_j, \delta_j) / j \in J \}$ be a family of f.t.s 's then $i_t(\pi \delta_j) = \pi i_t(\delta_j)$

Proof :

By theorem (2.1.10)

$$i_t(\delta) = \text{Sup}_{j \in J} \{ P_j^{-1}(i_t(\delta_j)) \} = \pi i_t(\delta_j).$$

Section 2.2

T_1 fuzzy topological spaces

Definition 2.2.1

(X, δ) is said to be a $t-T_1$ f.t.s. iff $(X, i_t(\delta))$ is a T_1 space

Definition 2.2.2

(X, δ) is said to be an $E-T_1$ f.t.s. iff for each $t \in (0, 1)$ and $x \in X$, $t\chi_{\{x\}}$ is a fuzzy closed set.

Definition 2.2.3

(X, δ) is a ultra- T_1 f.t.s iff the topological space $(X, i(\delta))$ is T_1

Definition 2.2.4

(X, δ) is a strong- T_1 f.t.s. iff it is $t-T_1$ for each $t \in [0, 1)$

Notation : If (X, δ) is an f.t.s. we shall denote $w_t o_i(\delta) = w_t(i_t(\delta))$ for each $t \in [0, 1)$

The following theorem gives a set of equivalent conditions for a space to be $t-T_1$.

Theorem 2.2.5

Let (X, δ) be an f.t.s. we have the following equivalent conditions.

- (i) (X, δ) is t - T_1
- (ii) for each $x \in X$, there is $U_x \in \delta$ such that $U_x(x)$ and $U_x(y) > t$ for $y \neq x$
- (iii) for each $x \in X$ we have $\chi_{X - \{x\}} \in w_{t, t}(\delta)$
- (iv) $(X, w_{t, t}(\delta))$ is $F - T_1$

Proof :

To Prove (i) \Rightarrow (ii)

Assume (X, δ) is $t - T_1$

Let $x \in X$, Since $(X, i_t(\delta))$ is T_1 , $\{x\}$ is closed.

therefore $X - \{x\} \in i_t(\delta)$

This implies that there exists $U_x \in \delta$ such that

$$X - \{x\} = U_x^{-1}(t, 1]$$

$$\Rightarrow x \notin U_x^{-1}(t, 1]$$

$$\Rightarrow U_x(x) \leq t.$$

$$\text{If } y \neq x \text{ then } y \in X - \{x\} = U_x^{-1}(t, 1].$$

which implies that $U_x(y) > t$.

Hence (ii)

To Prove (ii) \Rightarrow (iii)

Let $x \in X$, then there exists $U_x \in \delta$ such that $U_x(x) \leq t$ and $U_x(y) > t$ for $y \neq x$

This implies that $X - \{x\} = U_x^{-1}(t, 1] \in \mathcal{I}_t(\delta)$

Let $V = \chi_{X - \{x\}}$

Take any $z \in V^{-1}(t, 1]$

Then $V(z) > t$

$\Rightarrow V(z) = 1$

$\Rightarrow z \in X - \{x\}$

$\Rightarrow z \neq x$

$\Rightarrow U_x(z) > t$

$\Rightarrow z \in U_x^{-1}(t, 1]$

Hence $V^{-1}(t, 1] \subset U_x^{-1}(t, 1] \dots \dots (1)$

Let $z \in U_x^{-1}(t, 1]$

$\Rightarrow U_x(z) > t$ and $z \neq x$

$\Rightarrow z \in X - \{x\}$

$$\Rightarrow V(z) = 1$$

$$\Rightarrow z \in V^{-1}(t, 1]$$

$$\text{Hence } U_x^{-1}(t, 1] \subset V^{-1}(t, 1] \dots \dots \dots (2)$$

Therefore we get

$$\chi_{X - \{x\}}^{-1}(t, 1] = U_x^{-1}(t, 1] \in i_t(\delta)$$

therefore By definition $\chi_{X - \{x\}} \in w_t(i_t(\delta))$

To Prove (iii) \Rightarrow (iv)

Assume $\chi_{X - \{x\}} \in w_t(o_i(\delta))$

Take any $s \in (0, 1]$.

Consider $V_x = 1 - s \chi_{\{x\}}$

To prove $V_x \in w_t(i_t(\delta))$

i.e To prove $V_x^{-1}(t, 1] \in i_t(\delta)$

Now $y \in V_x^{-1}(t, 1] \Rightarrow V_x(y) > t$

Case (i)

$$s \geq 1 - t$$

Let $y = x$. Then $1 - s \chi_{\{x\}}(y) = 1 - s \leq 1 - 1 + t = t$

(i.e) $V_x(x) \leq t$

Hence $x \notin V_x^{-1}(t, 1]$

For every $y \neq x$ $V_x(y) = 1 > t$

(i.e) $y \in V_x^{-1}(t, 1]$

Therefore $V_x^{-1}(t, 1] = X - \{x\}$

Case ii

$s < 1 - t$

when $y = x$

$V_x(y) = 1 - s > t$

when $y \neq x$

$V_x(y) = 1 - 0 > t$

Hence $V_x^{-1}(t, 1] = X - \{x\}$ if $s \geq 1 - t$

$= X$ if $s < 1 - t$

By (iii) $\chi_{X - \{x\}}^{-1}(t, 1] \in i_t(\delta)$

(i.e) $X - \{x\} \in i_t(\delta)$

Also $X \in i_t(\delta)$

Hence $V_x^{-1}(t, 1] \in i_t(\delta)$

therefore $\forall x \in w_t(i_1(\delta))$

(i.e) $s \cdot \chi_{\{x\}}$ is fuzzy closed for every $s \in (0,1]$

To prove (iv) \Rightarrow (i)

Assume $(X, w_t o i_1(\delta))$ is $F-T_1$

to prove $X - \{x\} \in i_1(\delta)$

By assumption for every $t \in (0,1]$ and every $x \in X$

$t \cdot \chi_{\{x\}}$ is fuzzy closed in $w_t(i_1(\delta))$

(i.e) for every $t \in (0,1]$

$(1-t) \chi_{\{x\}}$ is fuzzy closed.

$\Rightarrow ((1-t) \chi_{\{x\}})^c \in w_t(i_1(\delta))$

$\Rightarrow 1 - (1-t) \chi_{\{x\}} \in w_t(i_1(\delta))$

$\Rightarrow (1 - (1-t) \chi_{\{x\}})^{-1}(t,1] \in i_1(\delta) \dots \dots \dots (1)$

$1 - (1-t) \chi_{\{x\}}(y) = 1$ if $y \neq x$

$= t$ if $y = x$

When ever $y \neq x, y \in (1 - (1-t) \chi_{\{x\}})^{-1}(t,1]$

and if $y = x, y \notin (1 - (1-t) \chi_{\{x\}})^{-1}(t,1]$

therefore $(1 - (1 - t)) \chi_{\{x\}}^{-1}(t, 1] = X - \{x\}$

Hence by (1), $X - \{x\} \in i_t(\delta)$.

Corollary : 2.2.6

If (X, δ) is a t -topologically generated f.t.s, then (X, δ) is $t - T_1$ if it is $F - T_1$

Proof

(X, δ) is t -topologically generated

\Rightarrow there exists a topology t such that $\delta = w_t(\tau)$.

By theorem 2.2.5 (X, δ) is $t - T_1$.

$\Leftrightarrow (X, w_t, i_t(\delta))$ is $F - T_1$.

$\Leftrightarrow (X, w_t(\tau))$ is $F - T_1$.

Since $i_t(w_t(\tau)) = \tau$

$\Leftrightarrow (X, \delta)$ is $F - T_1$.

$F - T_1 \Rightarrow$ strong $-T_1 \Rightarrow$ Ultra $- T_1$ But if the space is topologically generated then these three concepts are equivalent The following Theorem proves these facts.

Theorem 2.2.7

Consider the following properties.

- (i) (X, δ) is $F - T_1$;
- (ii) (X, δ) is strong $-T_1$
- (iii) (X, δ) is Ultra $-T_1$

Then (i) \Rightarrow (ii) \Rightarrow (iii)

Proof :

To prove (i) \Rightarrow (ii)

Assume (X, δ) is $F - T_1$

since $\delta \in w_t(i_t(\delta))$ for each $t \in [0, 1)$

$(X, w_t(i_t(\delta)))$ is $F - T_1$ for each $t \in [0, 1)$

$\Rightarrow (X, \delta)$ is $t - T_1$ by Theorem (2.2.5)

$\Rightarrow (X, \delta)$ is strong T_1 .

to prove (ii) \Rightarrow (iii)

Assume (X, δ) is strong $-T_1$

(X, δ) is strong T_1

$\Rightarrow (X, \delta)$ is $t - T_1$ for every $t \in [0, 1)$

$\Rightarrow (X, i_t(\delta))$ is T_1 for every $t \in [0,1)$

$\Rightarrow (X, i(\delta))$ is T_1 , since $i(\delta)$ is the topology generated by $\{i_t(\delta) / t \in [0,1)\}$

Hence (X, δ) is Ultra T_1 .

Corollary 2.2.8

Let (X, δ) be a topological generated f.t.s. Then the three concepts in the above theorem 2.2.7

Let (X, δ) be a topologically generated f.t.s. Then the three concept in the above theorem (2.2.7) are equivalent.

Proof:

It is enough to prove (iii) \Rightarrow (i)

Assume (X, δ) is ultra - T_1

Since δ is topologically generated

$\delta = w(\tau)$ for same topology τ

$\therefore (X, \delta)$ is Ultra - T_1

$\Rightarrow (X, i(w(\tau)))$ is T_1

$\Rightarrow (X, \tau)$ is T_1

$\Rightarrow \{x\}$ is closed

$\Rightarrow X - \{x\} \in \tau$

Now to prove $1-t \chi_{\{x\}} \varepsilon \delta = w(\tau)$

for every $t \in [0,1)$

(i.e) to prove for every $t \in [0,1)$

$A = (1-t \chi_{\{x\}})^{-1} (\varepsilon, 1] \varepsilon \tau$ for every $\varepsilon > 0$

$$\begin{aligned} 1-t \chi_{\{x\}}(y) &= 1 && \text{if } y \neq x \\ &= 1-t && \text{if } y = x \end{aligned}$$

Case (i)

$$t \geq 1 - \varepsilon$$

$$\implies 1-t \leq \varepsilon$$

In this case $x \notin A$

and for every $y \neq x$ $y \in A$

$$A = X - \{x\} \varepsilon \tau$$

Case (ii)

$$t < 1 - \varepsilon$$

$$1-t > \varepsilon$$

\therefore for every y in X belongs to A

$$A = X \varepsilon \tau \quad A \varepsilon \tau$$

$\therefore \tau \chi_{\{x\}}$ is fuzzy closed

Hence (X, δ) is $F-T_1$

The reciprocal implications in theorem 2.2.7 are not generally true. The following example illustrate these.

Example 2.2.9 Ultra- T_1 need no imply strong - T_1

Let X be any set and $\delta = \{c_0, c_1\} \cup \{1/2, \chi_A / ACX\}$

by definition we get

$$i_t(\delta) = \{U^{-1}(t, 1] / U \varepsilon \delta\}$$

Case i

when $t < 1/2$

$$c_0^{-1}(t, 1] = \phi \text{ and } c_1^{-1}(t, 1] = X$$

Given A is a proper subset of X

$$\text{Let } B = 1/2 \chi_A^{-1}(t, 1]$$

take any $y \in B$

$$y \in B \implies 1/2 \chi_A(y) \varepsilon (t, 1]$$

$$\begin{aligned} 1/2 \chi_A(y) &= 0 \text{ if } y \notin A \\ &= 1/2 \text{ if } y \in A \end{aligned}$$

$$1/2 \chi_A(y) \leq 1/2 \text{ for every } y \in X$$

If $t < 1/2$ every y in A will belong to B and every y not in A will not belong to B

$$\therefore (1/2 \chi_A)^{-1}(t, 1] = A$$

\therefore Every $A \subset X$ is in $i_t(\delta)$

$$\therefore i_t(\delta) = 2^X \text{ if } t < 1/2$$

Case ii.

If $t \geq 1/2$,

$$(1/2 \chi_A)^{-1}(t, 1] = \phi$$

$$\therefore i_t(\delta) = \{X, \phi\}$$

Since in discrete topology is not T_1

$(X, i_t(\delta))$ is not T_1 for $t \geq 1/2$

$\therefore (X, \delta)$ is not strong - T_1

Since $i(\delta)$ is the discrete topology

$\{x\}$ is closed for every $x \in X$

$\therefore (X, i(\delta))$ is T_1

hence (X, δ) is Ultra - T_1

Example 2.2.10 Strong- T_1 need not imply F- T_1

Let $\delta = \{ \chi_A : A \subset X \}$

let $D = 1 - 1/2 \chi_{\{x\}}$

$$\begin{aligned} \therefore D(y) &= 1 && \text{if } y \neq x \\ &= 1/2 && \text{if } y = x \end{aligned}$$

$\therefore 1 - 1/2 \chi_{\{x\}} \neq \chi_A$ for any $A \subset X$ and does not belong to δ

$\therefore 1/2 \chi_{\{x\}}$ is not fuzzy closed.

$\therefore (X, \delta)$ is not F- T_1

By definition

$$i_t(\delta) = \{ \chi_A^{-1}(t, 1] / A \subset X \}$$

since $\chi_A^{-1}(t, 1] = A$,

$i_t(\delta) = 2^X$ which is a discrete topology

$\therefore (X, i_t(\delta))$ is T_1

True for every $t \in [0, 1)$

Hence (X, δ) is strong - T_1

The following two theorems give generalisation of the following two results in Ordinary topological spaces :

- (i) Every sub space of a T_1 space is T_1
- (ii) Product of T_1 space is T_1

Theorem 2.2.11

Any fuzzy sub space of a strong - T_1 f.t.s. is a strong - T_1 f.t.s.

Proof :

Let (X, δ) be a f.t.s., Consider $Y \subset X$, then

(Y, δ_Y) is sub space of (X, δ)

To prove $(Y, i_t(\delta_Y))$ is T_1

Now, to prove $\{y\}$ is closed in $i_t(\delta_Y)$

Since $\{y\}$ is closed in (X, δ)

$$X - \{y\} = \bigcup_{U \in \delta} U^{-1}(t, 1), \quad U \in \delta$$

$$\text{since } (X - \{y\}) \cap Y = Y - \{y\}$$

$$Y - \{y\} = \bigcup_{U \in \delta} U^{-1}(t, 1) \cap Y$$

hence $\{y\}$ is closed in $(i_t(\delta))_Y$

$$\text{Since } (i_t(\delta))_Y = i_t(\delta_Y)$$

by theorem 2.1.9

we get $\{y\}$ is closed in $i_1(\delta_Y)$

(i.e) $(Y, i_1(\delta_Y))$ is T_1

hence (Y, δ_Y) is strong - T_1

Theorem 2.2.12

Any non-empty product spaces is strong - T_1 iff each factors space is strong - T_1

Proof :

Let $\{(X_j, \delta_j)\}$ be a family of f.t.s than the product topology δ on $X = \prod_j X_j$ is given by $\delta = \text{Sup}_{j \in J} \{P_j^{-1}(\delta_j)\}$

By corollary 2.1.13

we get $i_1(\delta) = \prod_j i_1(\delta_j)$

Assume each factor space is strong- T_1

To prove (X, δ) is strong - T_1

Consider any point $x \in X$

let $P_j(x) = x_j$ for every j

$P_j(x) \in x_j$

since (X_j, δ_j) is strong - T_1 $P_j(x)$ is closed in $i_t(\delta_j)$, which implies $P_j^{-1}(P_j(x))$ is closed in $i_t(\delta)$

Since intersection of closed sets is closed.

We get $\cap P_j(P_j(x))$ is closed in $i_t(\delta)$

Take any $y \in \cap P_j^{-1}(P_j(x))$

$$\implies P_j(y) = P_j(x) \quad \text{for every } j$$

$$= x_j \quad \text{for every } j$$

$$\implies y=x$$

$\therefore \cap P_j^{-1}(P_j(x)) = \{x\}$ is closed in $i_t(\delta)$

$\therefore (X, i_t(\delta))$ is T_1

which is true for every t

hence (x, δ) is strong - T_1

Conversely ,

Consider (X_j, δ_j) , Take $x_j \in X_j$

Consider x in X such that $P_j(x) = x_j$

We know x is closed $\therefore \{P_j(x)\}$ is closed

(i.e) $\{x_j\}$ closed.

Hence each factor space is strong - T_1

Section 2.3

T₂ Fuzzy Topological Spaces

Definition : 2.3.1

(X, δ) is said to be t-T₂ f.t.s. iff $(X, i_t(\delta))$ is T₂

Definition 2.3.2

(X, δ) is said to be F-T₂ iff for any two fuzzy points with different supports $p = x_t$ and $q = y_s$ there exists U, V such that $U \cap V = c_0$, $p \in U$ and $q \in V$

Definition : 2.3.3.

(X, δ) is said to be a strong - T₂ f.t.s iff it is t-T₂ for each $t \in [0, 1)$

Definition : 2.3.4

(X, δ) is said to be an ultra-T₂ f.t.s iff $(X, i(\delta))$ is T₂

Theorem 2.3.5

If (X, δ) is strong-T₂ then it is strong-T₁

Proof

Assume (X, δ) is strong - T₂

$\implies (X, \delta)$ is t-T₂ for each $t \in [0, 1)$

$\implies (X, i_t(\delta))$ is T₂ for each $t \in [0, 1)$

$\implies (X, i_t(\delta))$ is T_1 for each t (since $T_2 \implies T_1$)

$\implies (X, \delta)$ is $t-T_1$ for each t

$\implies (X, \delta)$ is strong T_1

Now we obtain a set of equivalent conditions for a $t-T_2$ space

Theorem 2.3.6

Let (X, δ) be an f.t.s, we have the following conditions :

- (i) (X, δ) is $t-T_2$;
- (ii) Let $x \neq y$ be points in X ; then there exists $U, V \in \delta$ such that $U \cap V \leq c_t$
 $U(x) > t$ and $V(y) > t$;
- (iii) $(X, w_t \circ i_t(\delta))$ is $F-T_2$
- (iv) if $\mathcal{F} \in p_t(x)$, there is at most one point $x \in X$ such that $\mathcal{F} \xrightarrow{t} x$

Proof :

To prove (i) \implies (ii)

assume (X, δ) is $t-T_2$

Then $(X, i_t(\delta))$ is T_2

If $x \neq y$ there exists $U, V \in \delta$ such that $U^{-1}(t, 1] \cap V^{-1}(t, 1] = \emptyset \implies (1)$
 and $x \in U^{-1}(t, 1]$ and $y \in V^{-1}(t, 1]$

Now, $x \in U^{-1}(t, 1] \implies U(x) > t$

and $y \in V^{-1}(t, 1] \implies V(y) > t$.

Claim :

Either $U(x) \leq t$ or $V(x) \leq t$

Suppose not then $U(x) > t$ and $V(x) > t$

$\implies x \in U^{-1}(t, 1]$ and $x \in V^{-1}(t, 1]$

$\implies x \in U^{-1}(t, 1] \cap V^{-1}(t, 1]$

Which is a contradiction to (1)

\therefore either $U(x) \leq t$ or $V(x) \leq t$

$\therefore (U \cap V)(x) \leq t = c_t(x)$ for every x

$\therefore U \cap V \leq c_t$

To prove (ii) \implies (iii)

Take any two fuzzy points $p = x_t$ and $q = y_s$ with different supports. Then $x \neq y$

By (ii) there exists $U, V \in \delta$ such that $U \cap V \leq c_t$;

$U(x) > t$ and $V(y) > t$

Since $U, V \in \delta$, $A = U^{-1}(t, 1]$ and $B = V^{-1}(t, 1] \in i_t(\delta)$

Let $U^* = \chi_A$ and $V^* = \chi_B$

$U^*, V^* \in w_t(i_t(\delta))$

$$\begin{aligned} U^* \cap V^* &= \chi_{U(t,1]} \cap \chi_{V(t,1]} \\ &= \chi_\phi = 0 = c_0 \end{aligned}$$

Claim: $p \in U^*$

Since $U(x) > t$, $x \in U^{-1}(t,1]$

$$\Rightarrow U^*(x) = 1$$

Since $t < 1$, $p \in U^*$

Similarly $q \in V^*$

Hence $(X, w_t(i_t(\delta)))$ is $F-T_2$

To prove (iii) \Rightarrow (iv)

Suppose there exist x, y with $x \neq y$ such that $\mathcal{F} \dashrightarrow x$, $\mathcal{F} \dashrightarrow y$

choose r, s such that $t < r < 1$ and $t < s < 1$

Consider the two fuzzy points x_r and y_s

Let $p = x_r$ and $q = y_s$

Take $U, V \in w_t(i_t(\delta))$ such that

$p \in U, q \in V$ and $U \cap V = c_0$

Then $r < U(x)$ and $s < V(y)$

$\therefore t < U(x)$ and $t < V(y)$

(i.e) $x \in U^{-1}(t,1]$ and $y \in V^{-1}(t,1]$

Since $\mathcal{F} \rightarrow x_t$ and $\mathcal{F} \rightarrow y_t$

$i_t(\mathcal{F}) \rightarrow x$ and $i_t(\mathcal{F}) \rightarrow y$

Hence $U^{-1}(t,1]$ and $V^{-1}(t,1] \in i_t(\mathcal{F})$

$\therefore U^{-1}(t,1] \cap V^{-1}(t,1] \neq \emptyset$

$\Rightarrow (U \cap V) \neq c_0$ which is contradiction

Hence (iv)

To prove (iv) \Rightarrow (i)

Assume (iv)

Let \mathcal{F} be a filter on X then

$w(\mathcal{F}) \in P^t(X)$, By (iv) there is atmost one $x \in X$ such that $w_t(\mathcal{F}) \rightarrow x_t$

(i.e) $i_t(w_t(\mathcal{F})) \rightarrow x$

(i.e) $\mathcal{F} \rightarrow x$

(i.e) there is atmost one $x \in X$ such that $\mathcal{F} \rightarrow x$

$\Rightarrow (X, i_t(d))$ is T_2 [6] $\Rightarrow (X, \delta)$ is $t - T_2$

Hence the proof.

Corollary 2.3.7

Let (X, δ) be a t - topologically generated f.t.s (X, δ) is $t - T_2$ iff it is $F - T_2$

Proof:

Similar to T_1 case.

As in the case of fuzzy T_1 spaces, here also the following implications hold good.

$$FT_2 \implies \text{strong-} T_2 \implies \text{Ultra-} T_2$$

These concepts are equivalent when the space is topologically generated.

Theorem 2.3.8

Let (X, δ) be a f.t.s consider the following properties.

- (1) (X, δ) is $F - T_2$
- (2) (X, δ) is Strong - T_2
- (3) (X, δ) is Ultra - T_2

Then (i) \implies (ii) \implies (iii)

Proof :

(i) \implies (ii)

Assume (X, δ) is $F - T_2$

Then as $\delta \in w_t(i_t(\delta))$ for every t ,

$(X, w_t(i_t(\delta)))$ is $F - T_2$

by theorem 2.3.6 (X, δ) is $t - T_2$ for every t which implies that (X, δ) is strong T_2 .

(ii) \implies (iii)

Assume (X, δ) is strong $- T_2$

which implies (X, δ) is $t - T_2$ for every $t \in [0, 1)$. By definition $(X, i_t(\delta))$ is T_2 for every $t \in [0, 1)$ which implies $(X, i(\delta))$ is T_2 , since $i(\delta)$ is the topology generated by $i_t(\delta)$

Hence (X, δ) is Ultra $- T_2$

Corollary 2.3.9

Let (X, δ) be a topologically generated f.t.s. Then the three concepts in theorem 2.3.8 are equivalent.

Proof:

To prove (iii) \implies (i)

$\delta = w(\tau)$

for some topology τ on X

(X, δ) is Ultra $- T_2$

$\implies (X, i(\delta))$ is T_2

Since $i(\delta) = i(w(\tau)) = \tau$

We get (X, τ) is T_2

Consider two fuzzy points $p = x_t$ and $q = y_s$ with different supports

$\therefore x \neq y$ There exists $A, B \in \tau$ such that $x \in A$, $y \in B$ and $A \cap B = \phi$

Take $U = \chi_A$ and $V = \chi_B$, $U, V \in w(\tau)$

Then $t < U(x)$, $s < V(y)$

and $U \cap V = c_0$

$\therefore (X, w(\tau))$ is $F - T_2$

(i.e) (X, δ) is $F - T_2$

The reciprocal implication in theorem 2.3.8 are not generally true. The following examples illustrate these

Example 2.3.10 Ultra - T_2 need not imply strong - T_2

Let X be a set and the fuzzy topology

$$\delta = \{c_0, c_1\} \cup \{1/2\chi_A : A \subset X\}$$

$$i_t(\delta) = \begin{cases} (X, \phi) & \text{if } t \geq 1/2 \\ 2^X & \text{if } t < 1/2 \end{cases}$$

It is shown in example 2.2.9 that (X, δ) is not strong - T_2 . Hence as Strong - T_2 implies strong - T_1

(X, δ) is not Strong - T_2

Since $i(\delta)$ is discrete topology on X , it is T_2 . Hence (X, δ) is a ultra - T_2 .

Example 2.3.11

Strong - T_2 need not imply F - T_2 Let X be a set and let σ be collection defined, as follows.

$$\sigma = \{U_{x,t} : x \in X \text{ and } t \leq 1/2\} \cup \{U \geq c_{1/2}\}$$

$$\text{where } U_{x,t} = t \cdot \chi_{\{x\}} \cup 1/2 \chi_{X - \{x\}}$$

Then σ is a subbase for a fuzzy topology δ , in X . (X, δ) is not F - T_2 as $x_{3/4}$ and $y_{3/4}$ cannot be separated by disjoint open fuzzy sets.

Claim : (X, δ) is strong - T_2

Case i

when $t < 1/2$ and $x \neq y$

$$\text{Let } U = 1/2 \chi_{\{x\}}$$

$$U(z) = 0 \text{ if } z \neq x$$

$$= 1/2 \text{ if } z = x$$

which implies that $U \leq 1/2 \therefore U \in \delta$

Similarly $V = 1/2 \chi_{\{y\}} \implies V \leq 1/2 \implies V \in \delta$

$$\text{so } (U \cap V)(x) = \text{Min} \{U(x), V(x)\}$$

$$= \text{Min} \{U(x), 0\}$$

$$= 0$$

$$(U \cap V)(y) = \text{Min} \{U(y), V(y)\}$$

$$= \text{Min} \{0, V(y)\}$$

$$= 0$$

The case for which $z \neq x$ and $z \neq y$

$$(U \cap V)(z) = \text{Min} \{U(z), V(z)\}$$

$$= \text{Min} \{0, 0\} = 0$$

Hence $U \cap V = c_0$

Since $U(x) = 1/2 > t$ and $V(y) = 1/2 > t$,

$x \in U^{-1}(t, 1]$ and $y \in V^{-1}(t, 1]$

Also $U^{-1}(t, 1]$ and $V^{-1}(t, 1] \in i_t(\delta)$

Since $U \cap V = c_0$, $U^{-1}(t, 1] \cap V^{-1}(t, 1] = \emptyset$

Case : ii

When $t \geq 1/2$ and $x \neq y$

take some $\varepsilon > 0$ such that $t + \varepsilon < 1$

$$\varepsilon < 1 - t \leq t$$

Consider $U = U_{x, t + \varepsilon}$ and $V = U_{y, t + \varepsilon}$

$$\text{then } U \cap V = \{(t + \varepsilon) \chi_{\{x\}} \cup ((t + \varepsilon)/2) \chi_{X - \{x\}}\} \cap \{(t + \varepsilon) \chi_{\{y\}} \cup ((t + \varepsilon)/2) \chi_{X - \{y\}}\}$$

$$\begin{aligned} (U \cap V)(z) &= (t + \varepsilon)/2 \text{ for every } z \\ &= c_{(t + \varepsilon)/2} < c_t \end{aligned}$$

Also $U(x) = t + \varepsilon > t$ and $V(y) = t + \varepsilon > t$

$\therefore x \notin U^{-1}(t, 1]$ and $y \notin V^{-1}(t, 1]$ and since $U \cap V < c_t$, $U^{-1}(t, 1] \cap V^{-1}(t, 1] = \phi$

Thus the space $(X, i_t(\delta))$ is T_2 for every t .

hence (X, δ) is strong- T_2

By proceeding in a similar way as in the case of strong- T_1 spaces, the following two results can be easily seen to be true.

Theorem : 2.3.12

Every subspace of a strong- T_2 space is strong- T_2 .

Theorem 2.3.13

A non empty product space is strong T_2 iff each factor space is strong- T_2

SECTION 2.4

Regular fuzzy topological spaces

Definition : 2.4.1

(X, δ) is said to be t-regular iff $(X, i_t(\delta))$ is regular.

Definition : 2.4.2

(X, δ) is said to be a strong regular f.t.s. iff (X, δ) is t-regular for each $t \in [0, 1)$

Definition : 2.4.3

(X, δ) is said to be F-regular iff for each fuzzy point p and $U \in \delta$ such that $p \in U$ there is $V \in \delta$ such that $p \in V \subset V^c \mid V^c \subset U$.

Definition : 2.4.4

(X, δ) is said to be a ultra-regular f.t.s. iff $(X, i(\delta))$ is regular.

Now, we obtain a set of equivalent condition for a space to be t-regular

Theorem : 2.4.5

Let (X, δ) be a f.t.s. , we have the following equivalent conditions.

- (i) (X, δ) is t-regular

(ii) If $x \in X$, $U_0 \in \delta$ such that $U_0(x) > t$ there are $U, V \in \delta$ such that $U \cap V \in c_t$, $U(x) > t$ and $U_0 \cup V > c_t$.

(iii) If $x \in X$, $U_0 \in w_{t, i_1}(\delta)$ such that $U_0(x) > t$ there are $U, V \in w_{t, i_1}(\delta)$ such that $U \cap V \in c_0$, $U(x) > t$, $U_0 \cup V > t$

(iv) if $\mathcal{F} \in \text{Pt}(x)$ and $\mathcal{F}^t \rightarrow x_t$ then

$$w_t(i_1(\mathcal{F})) \xrightarrow{t} x_t$$

Proof :

To prove (i) \implies (ii)

Assume (X, δ) is t -regular

$\implies (X, i_1(\delta))$ is regular

Take $x \in X$, $U_0 \in \delta$ such that $U_0(x) > t$

Which implies that $x \in U_0^{-1}(t, 1] \in i_1(\delta)$

Let $A = U_0^{-1}(t, 1]$ and let $F = A^c$

$\therefore x \notin F$ and F is closed in $i_1(\delta)$

Since $(X, i_1(\delta))$ is regular there exists open sets B_1 and $B_2 \in i_1(\delta)$ such that $x \in B_1$ and $F \subset B_2$ such that $B_1 \cap B_2 = \emptyset$

Then $B_1 = U^{-1}(t, 1]$ and $B_2 = V^{-1}(t, 1]$ Where $U, V \in \delta$

Take any $z \in X$

If $z \in B_1, z \notin B_2$

$$\therefore V(z) \leq t$$

$$\therefore (U \cap V)(z) \leq t$$

If $z \notin B_1, U(z) \leq t, (U \cap V)(z) \leq t \implies (U \cap V)z \leq c_t(z)$ for every $z \in X$.

$$\implies U \cap V \leq c_t$$

Since $x \in B_1, U(x) > t$

Take any $z \in X$.

If $z \in A, U_0(z) > t$

$$\therefore (U_0 \cup V)(z) > t$$

If $z \notin A, z \in F$

$$\implies z \in B_2$$

$$\implies V(z) > t$$

$$(U_0 \cup V)(z) > t$$

hence $U_0 \cup V > c_t$

hence (ii) To prove (ii) \implies (iii)

Take $x \in X$ and $U_0 \in w_t(i_t(\delta))$ such that $U_0(x) > t$

Now $U_0 \in w_t(i_t(\delta))$

\implies there is a $V_0 \in \delta$ such that $U_0^{-1}(t,1) = V_0^{-1}(t,1)$

By hypothesis there exists $E, F \in \delta$

Such that $E \cap F \leq c_t$, $E(x) > t$, $V_0 \cup F > 4$

Let $U = E \cap \chi E^{-1}(t,1)$ and $V = F \cap \chi F^{-1}(t,1)$

$E \in \delta \implies E^{-1}(t,1) \in i_t(\delta)$

$\implies \chi_{E^{-1}(t,1)} \in w_t(i_t(\delta))$

Also as $\delta \mathbf{C} w_t(i_t(\delta))$

$E \mathbf{C} w_t(i_t(\delta))$

$\therefore U \in w_t(i_t(\delta))$

Similarly $V \in w_t(i_t(\delta))$

Take $z \in X$

If $z \in E^{-1}(t,1)$ then $z \notin F^{-1}(t,1)$ as $E \cap F \leq c_t$.

Hence $\chi_{F^{-1}(t,1)}(z) = 0 \implies 1$

$\implies (U \cap V)(z) = 0$

If $z \notin E^{-1}(t,1)$ $\chi_{E^{-1}(t,1)}(z) = 0$

$$\therefore (U \cap V)(z) = 0 \rightarrow 2$$

hence from (1) and (2) we get

$$U \cap V = c_0$$

Since $E(x) > t \implies x \in E^{-1}(t, 1]$, $U(x) > t$

Suppose $U_0(z) \leq t$ and $V(z) \leq t$

then $V_0(z) \leq t$ and $F(z) \leq t$

which implies $(V_0 \cup F)(z) \leq t$

which is a contradiction to $V_0 \cup F > ct$

hence $U_0 \cup V > c_t$

to prove (iii) \implies (iv)

let $\mathcal{J} \rightarrow t \rightarrow x_t$

Claim: $i_1(\delta)$ is regular.

Let $x \in X$ and F be a closed set in $i_1(\delta)$ with $x \notin F$

this implies $x \in F^c \in i_1(\delta)$

There exists $U_0 \in \delta$ such that

$F^c = U_0^{-1}(t, 1]$, then $U_0(x) > t$

$$\therefore x \in U_0^{-1}(t, 1]$$

As $\delta \in w_t(i_t(\delta))$

$$U_0 \in w_t(i_t(\delta))$$

By (iii) there are $U, V \in w_t \circ i_t(\delta)$

Such that $U \cap V = c_0$

Let $A = U_0^{-1}(t, 1]$ and $B = V^{-1}(t, 1]$

Then $A, B \in i_t(\delta)$ and as $U_0 \cup V > c_t$

$$A \cap B = \phi \text{ also } x \in A$$

Now to prove FCB

If $z \notin B$ then $V(z) \leq t$

$$\implies U_0(z) > t$$

$$\implies z \in F^c$$

$$\implies z \notin F$$

$\therefore FCB$

Hence the claim.

since $i_t(\delta)$ is regular and $i_t(\mathcal{F}) \rightarrow x$

$\overline{w_t(i_t(\mathcal{F}))}$ converges to x_t [6]

To prove (iv) \implies (i)

suppose $\overline{\mathcal{F}} \rightarrow x$ in $(X, i_t(\delta))$

$$\implies w_t(\overline{\mathcal{F}}) \xrightarrow{-t-} x_t$$

\therefore by (iv) we get

$$\overline{w_t(i_t(w_t(\mathcal{F})))} \xrightarrow{-t-} x_t$$

$$\implies \overline{w_t(\overline{\mathcal{F}})} \xrightarrow{-t-} x_t$$

$$\implies \overline{\mathcal{F}} \rightarrow x \text{ in } (X, i_t(\delta))$$

$$\implies (X, i_t(\delta)) \text{ is regular}$$

$$\implies (X, \delta) \text{ is } t\text{-regular}$$

Theorem 2.4.6

Let (X, δ) be a strong - regular space, Then (X, δ) is ultra-regular

Proof

Assume (X, δ) to be strong - regular

(i.e) to prove $(X, i(\delta))$ is regular

Take $\overline{\mathcal{F}} \in F(x)$ such that $\overline{\mathcal{F}} \rightarrow x$ in $(X, i(\delta))$

$\implies \bar{\tau} \rightarrow x$ in $(X, i_1(\delta))$ for every $t \in [0,1)$

As $(X, i_1(\delta))$ is regular $\bar{\tau} \rightarrow x$ in $(X, i_1(\delta))$ for each $t \in [0,1)$ [6]

which implies $\bar{\tau} \rightarrow x$ in $(X, i(\delta))$

$\therefore (X, i(\delta))$ is regular.

Hence (X, δ) is Ultra regular.

The converse of the above result need not be true as the following example illustrates this

Example 2.4.7 Ultra - regular need not imply strong - regular

Let X be a set and consider the fuzzy topology $\delta = \langle \{c_0, c_1, 1/2\}$
 $\chi_{\{x\}} \cup \{1/4\chi_A : x_0 \notin A\}$

$$\begin{cases} (X, \phi) & \text{if } t \geq 1/2 \\ \{X, \phi, \{x_0\}\} & \text{if } t \in [1/4, 1/2) \\ 2^X & \text{if } t < 1/4 \end{cases}$$

For $t \geq 1/2$ as $i_1(\delta)$ is $\{X, \phi\}$

$(X, i_1(\delta))$ is not regular.

So (X, δ) is not strong regular.

As $i(\delta)$ is the discrete topology

$(X, i(\delta))$ is regular.

There is no relation between the concepts of F - regular and strong - regular as is seen from the following two examples.

Example 2.4.8 F - regular but not strong - regular.

Consider a set X and

$$\delta = \{U \in I^X : U(x_0) = 1/2\} \cup \{c_0, c_1\}$$

for x_0 fixed in X. (X, δ) is F - regular

But it is not strong - regular as but it is not strong - regular as $i_0(\delta) = \{A : x_0 \in A\}$ is not regular.

Example 2.4.9 Strong regular need not imply F - regular.

$$\text{Let } \delta = \{1/4 \chi_{\{x_0\}}, 1/4 \cdot \chi_{X - \{x_0\}}, c_{1/4}, c_0, c_1\}$$

(X, δ) is strong regular as

$$i_t(\delta) = \begin{cases} X, \phi, \{x_0\}, X - \{x_0\} & \text{if } t < 1/4 \\ \{X, \phi\} & \text{if } t \geq 1/4 \end{cases}$$

But it is not F - regular. Let $x_0 = y$ Then the fuzzy point $p = y_{1/8} \in 1/4 \cdot \chi_{\{x_0\}} = U \in \delta$

The Unique $E \in \delta$ such that $E \subset U$ and $p \in E$ is $E = U$. But $c_{1/4} \in U$ is not

included in U.

Definition : 2.4.10

(X, δ) is said to be strong - T_3 f.t.s iff it is strong regular and strong - T_1 .

Theorem 2.4.11

If (X, δ) is strong - T_3 then it is strong - T_2 .

Proof :

Assume (X, δ) is strong T_3

$\implies (X, \delta)$ is strong - regular and strong - T_1 (by definition)

$\implies (X, i_1(\delta))$ is regular and $(X, i_1(\delta))$ is T_1

$\implies (X, i_1(\delta))$ is T_2 .

$\implies (X, \delta)$ is strong - T_2 .

Hence the result.

By proceeding in similar way as in the case of strong - T_1 spaces the following two results can be easily seen to be true.

Theorem : 2.4.12

Every subspace of a strong - regular (strong - T_3) f.t.s is a strong - regular f.t.s (respectively - strong - T_3)

Theorem : 2.4.13

A non empty fuzzy product space is strong regular (strong - T_3) iff each factor space is strong regular (strong - T_3)

SECTION 2.5

Normal fuzzy topological spaces :

Defintion 2.5.1

(X, δ) is said to be t - normal iff $(X, i_t(\delta))$ is normal

Definition : 2.5.2

(X, δ) is said to be strong normal iff it is strong normal for each $t \in [0,1)$

Theorem : 2.5.3

In an f.t.s we have the following equivalent conditions.

(i) (X, δ) is t - normal

(ii) Let $U, V \in \delta$ such that $U \cup V > c_t$. There are $U_0, V_0 \in \delta$ with $U_0 \cap V_0 \leq c_t$,

$U_0 \cup U > c_t$ and $V_0 \cup V > c_t$.

(iii) Let $U, V \in w_t \text{oi}_t(\delta)$ such that $U \cup V = c_1$. There are $U_0, V_0 \in w_t \cdot (i_t(\delta))$ such that $U_0 \cap V_0 = c_0$, $U_0 \cup U > c_t$ and $V_0 \cup V > c_t$

Proof :

Assume (i) (X, δ) is t - normal

to prove (i) \implies (ii)

Let $U, V \in \delta$ and $U \cup V > c_t$

$U^{-1}(t, 1]$ and $V^{-1}(t, 1] \in i_t(\delta)$

Take any $z \in X$

If $z \notin U^{-1}(t, 1]$ This $\implies U(z) \leq t$

As $U \cup V > c_t$, $V(z) > t$

$\therefore z \in V^{-1}(t, 1]$

$\therefore X = U^{-1}(t, 1] \cup V^{-1}(t, 1]$

Consider the two closed set F_1 and F_2 given by $F_1 = (U^{-1}(t, 1])^c$ and

$F_2 = (V^{-1}(t, 1])^c$

$z \in F_1 \implies z \notin U^{-1}(t, 1]$

$\implies U(z) \leq t$

$\implies V(z) > t$

$\implies z \in V^{-1}(t, 1]$

$$\implies z \notin F_2$$

$$\text{Similarly } z \in F_2 \implies z \notin F_1$$

$$\therefore F_1 \cap F_2 = \phi$$

since by (i) we have $(X, i_t(\delta))$ is normal. Therefore there exists two sets A and B

$$\text{such that } A = U_0^{-1}(t,1] \text{ and } B = V_0^{-1}(t,1]$$

$$\text{with } U_0, V_0 \in \delta \text{ with } F_1 \subset U_0^{-1}(t,1] \text{ and } F_2 \subset V_0^{-1}(t,1] \text{ and } A \cap B = \phi$$

$$\therefore (U_0 \cap V_0)(z) = \text{Min} \{ U_0(z), V_0(z) \}$$

$$z \notin U^{-1}(t,1]$$

$$\implies z \notin U^{-1}(t,1]$$

$$\implies z \in F_1$$

$$\therefore z \in U_0^{-1}(t,1]$$

$$\therefore X = U_0^{-1}(t,1] \cup U^{-1}(t,1]$$

$$\text{Similarly } X = V_0^{-1}(t,1] \cup V^{-1}(t,1]$$

$$\text{Also } U_0 \cap V_0 \geq c_t, U_0 \cup U > c_t \text{ and } V_0 \cup V > c_t$$

$$(ii) \implies (iii)$$

$$\text{Let } U, V, \varepsilon w_t(i_t(\delta)) \text{ such that } U \cup V = c_t$$

There exist $U^*, V^* \in \delta$ such that $U^{-1}(t,1] = (U^*)^{-1}(t,1]$

and $V^{-1}(t,1] = (V^*)^{-1}(t,1]$ Also $U^* \cup V^* > c_t$

By (ii) there are $U_0^*, V_0^* \in \delta$ such that $U_0^* \cap V_0^* \leq c_t$

$U_0^* \cup U^* > c_t, V_0^* \cup V^* > c_t$ Let $U_0 = U_0^* \cap \chi_{(U_0^*)^{-1}(t,1]}$

and $V_0 = V_0^* \cap \chi_{(V_0^*)^{-1}(t,1]}$

Then $U_0, V_0 \in w_t(i_t(\delta))$ and $U_0 \cap V_0 = c_0$

$U_0 \cup U > c_t$ and $V_0 \cup V > c_t$

To prove (iii) \implies (i)

To Prove $(X, i_t(\delta))$ is normal

Take F_1, F_2 be closed sets in $i_t(\delta)$ with $F_1 \cap F_2 = \emptyset$

$\therefore F_1^c, F_2^c \in i_t(\delta)$

\therefore there exists $U, V \in \delta$ such that

$F_1^c = U^{-1}(t,1]$ and $F_2^c = V^{-1}(t,1]$

If $z \notin U^{-1}(t,1], U(z) \geq t$

$\implies z \notin F_1^c$

$\implies z \in F_1$

$$\cup z z \Leftarrow$$

$$[U]^{-1} \cup z z \Leftarrow$$

$$1 < (z) \cup \Leftarrow$$

$$U^*(z) = \mathcal{G}$$

If $z \in F$, $\Leftarrow z \in U^*(z)$ which implies

$$\text{clearly } U \cap V = \emptyset$$

Then $U, V \in \mathcal{G}$

$$\text{Let } U = U^*(z), V = U^*(z)$$

$$U \cap V \subset U^*(z)$$

$U \cap V \subset U^*(z)$ Hence there are $U^*(z) \cap V^*(z) = U^*(z) \cap V^*(z)$ such that $U^*(z) \cap V^*(z) = U^*(z) \cap V^*(z)$

$U^*(z) \cap V^*(z) = U^*(z) \cap V^*(z)$, Also clearly

Take $U^*(z) = U^*(z)$ and $V^*(z) = U^*(z)$

$$\therefore X = U^*(z) \cap V^*(z)$$

$$[U]^{-1} \cup z z \Leftarrow$$

$$\Leftarrow V(z) > 1$$

$$\Leftarrow z \in F$$

$$\Leftarrow z \in F$$

$\therefore F_1 \subset U$ Similarly $F_2 \subset V$

Hence the space is normal

Definition 2.5.4

(X, δ) is F-normal iff for each $V \in \delta$ and $U \in \delta$ such that $V \subset U$ there is $E \in \delta$ such that $V \subset E \subset U$

Definition 2.5.5.

(X, δ) is strong - T_4 iff it is strong - normal is strong - T_1

Theorem : 2.5.6

If (X, δ) is strong - T_4 , then it is strong - T_3

Proof :

Similar to theorem 2.5.3

There are no relations among the concept of normality in fuzzy topological spaces. The following examples illustrates this.

Example : 2.5.7 A space which is F - normal but neither strong - normal nor Ultra - normal.

Let (X, τ) be topological space which is not normal Let $x_0 \notin X$ and Consider

$$X^* = X \cup \{x_0\}$$

Let the fuzzy topology be

$$\delta = \{U \in I^X : U|_X \varepsilon w(\tau) \text{ and } U(x_0) = 1/4\} \cup \{c_0, c_1\}$$

Take V which is fuzzy closed.

$$\text{Then } V^c \varepsilon \delta \implies V^c(x_0) = 1/4$$

$$1 - V(x_0) = 1/4$$

$$V(x_0) = 1 - 1/4 = 3/4$$

\therefore the only open fuzzy set which

Contains V is c_1

$$\text{Hence } V < c_1 = \overline{c_1} = c_1.$$

$\therefore (X^*, \delta)$ is F -normal

$$i_1(\delta) = \begin{cases} \{t U\{X^* \text{ if } t \geq 1/4 \\ \{A \cup \{x_0\} / A \varepsilon \tau\} \text{ if } t < 1/4 \end{cases}$$

Since $t \geq 1/4$ $(X^*, i_1(\delta))$ is not - normal

$\therefore (X^*, \delta)$ is not strong - normal

$$i(\delta) = \{A \cup \{x_0\}, A \varepsilon \tau\} \cup \{\tau\}$$

Let F be closed set in $i(\delta)$ which implies x_0 does not belong to F

$\therefore F_1, F_2$ is closed in $i(\delta)$

implies F_1, F_2 are closed in τ

Since (X, τ) is not τ -normal

(X, δ) is normal

Then (X, δ) is not Ultra-normal

Example : 2.5.8 A space which is both strong τ -normal and Ultra-normal but not τ -normal.

Let X be a set and $\delta = \{U : U \geq c_{3/4}\}$ Take $V = 1/4 \chi_{\{x\}}$ be a fuzzy closed set and

$$U = \chi_{\{x\}} \cup 3/4 \chi_{X - \{x\}}$$

therefore $U(x) \geq 3/4$ for every $x \in X \therefore U \in \delta$

Since $V(x) \leq 1/4$ and $U(x) \geq 3/4 \quad \forall x \in X$

If $E \in \delta$ then $E(x) \geq 3/4$ for every $x \in X$

$$c \in E = \bigcap \{ B \in \delta, E \subseteq B \}$$

$$E \subseteq B \implies B(x) \geq 3/4 \implies B \in \delta$$

As $B \in \delta$, B is closed

$\therefore B$ is both Closed and open

$$\therefore B = c_1$$

$\therefore c \in E = c_1$ not contained in U

\therefore there exists no open set $E \in \delta$ such that $\overline{VCECECU}$

therefore (X, δ) is not F -normal.

$$i(\delta) = \{U^{-1}(t, 1) / U \in \delta \text{ and } U(x) > 3/4\} = 2^X$$

[If $A \subset X$

Define a mapping $f : X \rightarrow I$ by

$$\begin{aligned} U(x) &= 3/4 && \text{if } x \in A \\ &= 0 && \text{if } x \notin A \end{aligned}$$

And $\therefore (X, i(\delta))$ is normal

$$i_t(\delta) = \begin{cases} (X, f) & \text{if } t < 3/4 \\ 2^X & \text{if } t \geq 3/4 \end{cases}$$

Hence (X, δ) is strong-normal and Ultra normal.

Example 2.5.9

A space which is ultra normal but not strong normal

Let X be a set a consider a fuzzy topology $\delta = \{[c_0, c_1] \cup [1/2, \chi_A / (x_0 \in A)] \cup \{1/4 \chi_A / A \subset X\}$

where $x_0 \in X$, then we have

$$i_t(\delta) = \begin{cases} \{2^x & \text{if } t < 1/4 \\ \{A: x_0 \in A\} & \text{if } t \in [1/4, 1/2) \\ \{X, \phi\} & \text{if } t \geq 1/2 \end{cases}$$

Then $i(\delta)$ is the discrete topology which is normal, but $\{ACX; x_0 \in A\}$ is not normal so (X, δ) is Ultra - normal and not strong - normal.

Example : 2.5.10 A space which is strong - normal but not Ultra - normal.

Consider the following fuzzy sets on I

$$U(x) = \begin{cases} 1/2 & \text{if } x \in (0, 1/2) \cup \{1\} \\ t & \text{if } x \in \{0\} \cup [1/2, 1) \end{cases}$$

$$V(x) = \begin{cases} 1/2 & \text{if } x \in (0, 1/2) \cup \{1\} \\ t & \text{if } x \in [1/2, 1) \\ 1 & \text{if } x = 0 \end{cases}$$

Consider the fuzzy topology $\delta = \{c_0, c_1, U, V\}$ we have

$$i_t(\delta) = \begin{cases} \{\phi, I, (0, 1]\} & \text{if } t < 1/2 \\ \{\phi, I, (t, 1) \cup \{0\}\} & \text{if } t \geq 1/2 \end{cases}$$

Each $(X, \mathcal{I}_t(\delta))$ is normal, for they are not disjoint closed sets, Then (X, δ) is strong normal. we have

$$\mathcal{I}(\delta) = \{\emptyset, I, (0,1], (t,1), (t,1) \cup \{0\} : t \in [1/2,1)\}$$

then $\{0\}$ and $(0,1/2] \cup \{1\}$ are disjoint closed set and there are not disjoint open sets containing them. Then $\mathcal{I}(\delta)$ is not normal.

$\therefore (X, \delta)$ is not Ultra-normal.

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