Chapter I

CHAPTER I

PRELIMINARY DEFINITIONS AND NOTATIONS

Throughout the thesis, we assume that $m \leq n$. The matrix I_n is the $(n \times n)$-identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and write I, J, and O, respectively. The matrix $E_{i,j}$, called a cell, denotes the matrix with exactly one nonzero entry, that being a one in the (i,j)-entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denotes the matrix whose j^{th} column is all ones and is zero elsewhere. We denote by |A| the number of nonzero entries in the matrix A. We denote by A[i,j][k,l] the (2×2)-submatrix of A which lies in the intersection of the i^{th} and j^{th} rows with the k^{th} and l^{th} columns.

Definition :1.1

For a matrix A, if $a_{ij}=0$ for all i and j provided that $i \neq j$ then A is called a **diagonal matrix** and denoted by diag(a_{11} , a_{12} , a_{13} ,……, a_{nm}).

Definition :1.2

A matrix P is called a **permutation matrix** if and only if one entry of its every row and every column is 1 and the other entries are 0.

Definition :1.3

A matrix A is called **idempotent** if $A^2 = A$.

Definition :1.4

A matrix is said to be **right invertible (left invertible)** if AB = I_n (BA = I_n) for some matrix B. The matrix B is called a **right inverse (left inverse)** of A . If A is both right and left invertible, then it is called **invertible matrix.**

Definition :1.5

A matrix A is said to **dominate** matrix B = [b_{ij}] if $a_{ij} = 0$ implies that $b_{ij} = 0$ and we write $A \geq B$. If $A \geq B$ and there is some pair (i,j) such that $a_{ij} \neq 0$ but $b_{ij} = 0$, then we write A > B.

Definition :1.6

If A = (α_{ij}) then the **transpose** of A, written as A' or A^{tr} or A^T or A^t is the matrix $A' = (\gamma_{ij})$ where $\gamma_{ij} = \alpha_{ji}$ for each i and j.

The transpose of A is the matrix obtained by interchanging the rows and columns of A .

Definition :1.7

For $\alpha \in F$, the matrices

$$\alpha I = \begin{pmatrix} \alpha & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \alpha \end{pmatrix}$$

(blank spaces indicate only 0 entries) are called **scalar matrices.**

Definition :1.8

If the elements a_{rs} of the matrix A are complex numbers, the matrix formed by the conjugates of a_{rs} which are denoted by $\overline{a_{rs}}$ is called the **conjugate** of the matrix and is denoted by $\overline{A}$. Hence $\overline{A} = [\overline{a_{rs}}]$.

Definition :1.9

If A is a square matrix and A is the transpose of its conjugate i.e., if A^T=A, such a matrix is called a **Hermitian matrix** and if A^T=-A, then such a matrix is called **Skew-Hermitian matrix**.

Definition :1.10

A matrix A is called **monomial** if it has exactly one nonzero element in each row and column.

Definition :1.11

A **line** of a matrix A is a row or a column of the matrix A.

Definition :1.12

If A and B are matrices and $A \geq B$, we denote by A\B the matrix C with entries

$$c_{i,j} = \begin{cases} 0 & \text{if } \ b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise} \end{cases}$$

Notation : 1.13

Let $\Delta_{m,n} = \{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n\}$, and E_{ij} be the $m \times n$ matrix whose $(i,j)^{th}$ entry is 1 and whose other entries are all 0, and

$$E_{m,n} = \{E_{i,j} \mid (i,j) \in \Delta_{m,n}\}.$$

Definition : 1.14

A **semiring** consists of a set **S** and two binary operations on **S**, addition(+) and multiplication(.) such that :

(i) (**S**,+) is an abelian monoid (identity denoted by 0)

(ii) (**S**, .) is a monoid (identity denoted by 1)

(iii) multiplication distributes over addition

(iv) s.0 = 0.s =0 for all $s \in$ **S** and

(v) $1 \neq 0$.

Note : 1.15

Usually **S** denotes both the semiring and the set. Thus all rings with identity are semirings

Definition : 1.16

A semiring **S** is **commutative** if (**S**, **.**) is abelian.

Definition : 1.17

A semiring **S** is **antinegative** if 0 is the only element to have an additive inverse.

Note : 1.18

From the definition no ring is antinegative except {0}. Algebra terms such as **unit** and **zero-divisor** are defined for semirings as for rings.

Definition : 1.19

Let $\mathbb{C}$ be any chain with lower bound 0 and upper bound 1, then ($\mathbb{C}$,+, .) = ($\mathbb{C}$,max,min) is a **chain semiring**.

Definition : 1.20

If $\mathbb{F}$ is the real interval [0,1], then ($\mathbb{F}$,max,min) is a semiring, the **fuzzy semiring**.

Definition : 1.21

A semiring **S** is called an **antiring** if a+b=0 implies that a=b=0 for any $a,b \in$ **S**.

Definition : 1.22

A semiring **S** is called **Boolean** if **S** is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M.

Definition : 1.23

Let $\mathbb{B} = \{0,1\}$, then ($\mathbb{B}$,+, .) is a **Boolean algebra** if 0+0=0.0=0.1=1.0=0 and 1+1=1.1=1.

Hence Boolean algebra is a semiring.

Definition : 1.24

The **general linear group** of degree n over a field F (such as R or C), written as GL(n,F), is the group of $n \times n$ invertible matrices with entries from F, with the group operation that of ordinary matrix multiplication. (This is indeed a group because the product of two invertible matrices is again invertible, as in the inverse of one). If the field is clear from context we sometimes write GL(n), or GL_n .

Note : 1.25

Algebraic terms such as units and zero divisors are defined for semirings as rings, 0 and 1 denote the additive identity and the multiplicative identity respectively.

Notation : 1.26

Let **S** be a semiring and $a \in$ **S**. We denote by a^k the k^{th} power of a and by ka the sum a+a+...........+a (k times) for any positive integer k.

Notation : 1.27

Let $\mathbf{M}_{m,n}(\mathbf{S})$ denote the set of all $m \times n$ matrices with entries in a semiring **S**. If m=n, we use the notation $\mathbf{M}_n(\mathbf{S})$ instead $\mathbf{M}_{n,n}(\mathbf{S})$.

Definition : 1.28

A mapping T : $\mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ is called a **linear operator** if $T(\alpha A+\beta B)=\alpha T(A)+\beta T(B)$ for all $A,B \in \mathbf{M}_{m,n}(\mathbf{S})$ and for all $\alpha,\beta \in \mathbf{S}$.

Definition : 1.29

For any vector $\mathbf{u} \in \mathbf{M}_{m,1}(\mathbf{S})$, we define $|\mathbf{u}|$ to be the number of nonzero entries in **u.**