Chapter - 3

CHAPTER 3

SOME PROPERTIES OF (r, s)-T_0 AND (r, s)- T_1 SPACES

In this chapter, we study some of the properties of product intuitionistic fuzzy topological spaces of (r, s)-T_0 and (r, s)-T_1 spaces introduced by S.E.Abbas and Biljana Krsteska [2]. Some important results and characterizations of product intuitionstic fuzzy topological spaces are studied. Some interesting properties of (r, s)-T_0 and (r, s)-T_1 spaces are discussed in detail.

SECTION: 3.1

PRELIMINARIES AND DEFINITIONS

Definition: 3.1.1

Let X be a non-empty set. An IFTS (X, τ, τ^*) is called stratified if $\tau(\underline{\alpha}) = 1$ and $\tau^*(\underline{\alpha}) = 0$ for each $\alpha \in I$.

Notation: 3.1.2

For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$.

Definition: 3.1.3

Let (X, τ, τ^*) be an ifts. Let the operators $C_{\tau,\tau^*}: I^X \times I_0 \times I_1 \to I^X$ be defined by $C_{\tau,\tau^*}(\lambda, r, s) = \wedge\{\mu \in I^X / \lambda \leq \mu, \tau(\tilde{1} - \mu) \geq r, \tau^*(\tilde{1} - \mu) \leq s\}$.

Theorem: 3.1.4

Let (X, τ, τ^*) be an ifts. Then for $\lambda, \mu \in I^X$ and $r, r_1 \in I_0$ and $s, s_1 \in I_1$, the operator C_{τ,τ^*} satisfies the following conditions:

(1) $C_{\tau,\tau^*}(\tilde{0}, r, s) = \tilde{0}$,

(2) $\lambda \le C_{\tau,\tau^*}(\lambda, r, s)$,

(3) $C_{\tau,\tau^*}(\lambda, r, s) \vee C_{\tau,\tau^*}(\mu, r, s) = C_{\tau,\tau^*}(\lambda \vee \mu, r, s)$,

(4) $C_{\tau,\tau^*}(\lambda, r, s) \le C_{\tau,\tau^*}(\lambda, r_1, s_1)$ if $r \le r_1$ and $s \ge s_1$,

(5) $C_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, r, s), r, s) = C_{\tau,\tau^*}(\lambda, r, s)$.

Definition: 3.1.5

Let X and Y be two non-empty set. A function $f: (X, \tau, \tau^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$

is

(1) **IF continuous** if $\tau(f^{-1}(\mu)) \ge \mathcal{U}(\mu)$ and $\tau^*(f^{-1}(\mu)) \le \mathcal{U}^*(\mu)$, for each $\mu \in I^Y$;

(2) **IF open** if $\tau(\mu) \le \mathcal{U}(f(\mu))$ and $\tau^*(\mu) \ge \mathcal{U}^*(f(\mu))$, for each $\mu \in I^X$;

(3) **IF homeomorphism** if and only if f is bijective and both f and f^{-1} are IF continuous.

Definition: 3.1.6

Let θ_X be a subset of I^X containing $\tilde{1}$ and $\tilde{0}$. A pair (β, β^*) of functions $\beta, \beta^*: \theta_X \to I$ is called **base** for an IGO on X if it satisfies the following conditions:

(B1) $\beta(\lambda)+\beta^*(\lambda) \leq 1, \forall\lambda\in\theta_X$,

(B2) $\beta(\tilde{1}) = 1$ and $\beta^*(\tilde{1}) = 0$,

(B3) $\beta(\lambda_1 \wedge \lambda_2) \geq \beta(\lambda_1) \wedge \beta(\lambda_2)$ and

$\beta^*(\lambda_1 \wedge \lambda_2) \leq \beta^*(\lambda_1) \vee \beta^*(\lambda_2), \forall\lambda_i \in\theta_X$, i = 1, 2.

Definition: 3.1.7

Let (τ, τ^*) be an intuitionistic fuzzy set on X. Then (β, β^*) is said to form a base of (τ, τ^*) if β and β^* are bases of τ and τ^*, respectively.

Definition: 3.1.8

Let (τ, τ^*) be an intuitionistic fuzzy set on X. Then (ξ, ξ^*) is said to form a subbase of (τ, τ^*) if ξ and ξ^* are subbases of τ and τ^*, respectively.

Definition: 3.1.9

Let (β, β^*) be an IF topological base for X. Define the function

$\tau_\beta, \tau^*_{\beta^*}$: $I^X \to I$ as follows:

For each $\mu \in I^X$,

$$\tau_\beta(\mu) = \begin{cases} \vee\left\{\underset{i\in J}{\wedge}\beta(\mu_i)\right\} & \text{, if } \mu = \underset{i\in J}{\vee}\mu_i, \mu_i \in \theta_X, \\ 1 & \text{, if } \mu = \tilde{0} \\ 0 & \text{, otherwise,} \end{cases}$$

where $\vee$ is taken over all families $\{\mu_i \in \theta_X / \mu = \underset{i\in J}{\vee}\mu_i\}$,

$$\tau^*_{\beta^*}(\mu) = \begin{cases} \wedge\left\{\underset{i\in J}{\vee}\beta^*(\mu_i)\right\} & \text{, if } \mu = \underset{i\in J}{\vee}\mu_i, \mu_i \in \theta_X, \\ 0 & \text{, if } \mu = \tilde{0} \\ 1 & \text{, otherwise,} \end{cases}$$

where $\wedge$ is taken over all families $\{\mu_i \in \theta_x / \mu = \underset{i\in J}{\vee}\mu_i\}$,

Then

(1) $\left(X, \tau_\beta, \tau^*_{\beta^*}\right)$ is an IFTS;

(2) A map f: $\left(Y, \tau, \tau^*\right) \to \left(X, \tau_\beta, \tau^*_{\beta^*}\right)$ is IF continuous if and only if

$\beta(\lambda) \le \tau(f^{-1}(\lambda))$ and $\beta^*(\lambda) \ge \tau^*(f^{-1}(\lambda))$ for all $\lambda \in \theta_X$.

Definition: 3.1.10

Let X be a product of the family $\{X_i / i\in\Gamma\}$ of sets, and for each $i\in\Gamma$ $\pi_i : X \to X_i$ a projection map. For each $\lambda\in I^X$, i, $j\in\Gamma$ and $\lambda_i \in I^{X_i}$, the following properties hold:

(1) $\pi_i(\pi_i^{-1}(\lambda_i) \wedge \lambda) = \lambda_i \wedge \pi_i(\lambda)$;

(2) if $\underset{x^i\in X_i}{\vee} \lambda_i(x^i) = \alpha_i$ for $i\in F$ with each finite index subset F of Γ-{j} and

put $\alpha = \underset{i\in F}{\wedge}\alpha_i$, then

a) $\underset{x\in X}{\vee}\left(\underset{i\in F}{\wedge}\pi_i^{-1}(\lambda_i)\right)(x) = \alpha$;

b) $\pi_i\left(\underset{i\in F}{\wedge}\pi_i^{-1}(\lambda_i)\right) = \underline{\alpha}$.

Definition: 3.1.11

Let (X, τ, τ^*) be an IFTS, $\mu\in I^X$, $x_t\in P_t(X)$, $r\in I_0$ and $s\in I_1$.

Then $Q(x_t, r, s) = \{\mu\in I^X / x_t q\, \mu, \tau(\mu) \ge r, \tau^*(\mu) \le s\}$.

A fuzzy set $\mu\in Q(x_t, r, s)$ is called **(r, s)-Q open neighbourhood** of x_t.

SECTION: 3.2

SOME PROPERTIES OF PRODUCT INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

Theorem: 3.2.1

Let $\{(X_i, \tau_i, \tau_i^*)\}_{i\in\Gamma}$ be a family of IFTSs, let X be a set and for each $i\in\Gamma$, $f_i : X \to X_i$ a map.

Let

$$\theta_X = \{\tilde{0} \neq \mu = \bigwedge_{j=1}^{n} f_{k_j}^{-1}(\gamma_{k_j}) / \tau_{k_j}(\gamma_{k_j}) > 0, \ \ \forall k_j \in K\},$$

for every finite set $K = \{k_1, \ldots\ldots, k_n\} \subset \Gamma$. Define the functions $\beta, \beta^*: \theta_X \to I$ on X by

$$\beta(\mu) \ = \ \vee\{\bigwedge_{j=1}^{n} \tau_{k_j}(\gamma_{k_j}) / \mu = \bigwedge_{j=1}^{n} f_{k_j}^{-1}(\gamma_{k_j})\},$$

$$\beta^*(\mu) \ = \ \wedge\{\bigvee_{j=1}^{n} \tau_{k_j}^*(\gamma_{k_j}) / \mu = \bigwedge_{j=1}^{n} f_{k_j}^{-1}(\gamma_{k_j})\},$$

where $\vee$ and $\wedge$ are taken over all finite subsets $K = \{k_1, \ldots\ldots, k_n\} \subset \Gamma$.

Then,

(1) (β, β^*) is an IF topological base on X;

(2) the IGO, $(\tau_\beta, \tau_{\beta^*}^*)$ generated by (β, β^*) is the coarsest IGO on X for which each $i\in\Gamma$, f_i is IF continuous;

(3) a map $f : (Y, \tau_1, \tau_1^*) \to (X, \tau_\beta, \tau_{\beta^*}^*)$ is IF continuous if and only if for each $i\in\Gamma$, $f_i \circ f$ is IF continuous.

Proof:

(1) (B_1) It is trivial.

(B_2) Since $\lambda = f_i^{-1}(\lambda)$ for each $\lambda \in \{\tilde{0}, \tilde{1}\}$

$\beta(\tilde{1}) = \beta(\tilde{0}) = 1$ and $\beta^*(\tilde{1}) = \beta^*(\tilde{0}) = 0$.

(B_3) For all finite subsets set $K = \{k_1, k_2, \ldots\ldots, k_p\}$ and

$J = \{j_1, j_2, \ldots\ldots, j_q\}$ of Γ such that

$$\lambda = \bigwedge_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}), \quad \mu = \bigwedge_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i})$$

we have

$$\lambda \wedge \mu = \left(\bigwedge_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i})\right) \wedge \left(\bigwedge_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i})\right).$$

Furthermore, we have for each $k \in K \cap J$,

$$f_k^{-1}(\lambda_k) \wedge f_k^{-1}(\mu_k) = f_k^{-1}(\lambda_k \wedge \mu_k).$$

Put $\lambda \wedge \mu = \bigwedge_{m_i \in K \cup J} f_{m_i}^{-1}(\mu_i)(\rho_{m_i})$,

$$\text{where } \rho_{m_i} = \begin{cases} \lambda_{m_i} & \text{if } m_i \in K - (K \cap J) \\ \mu_{m_i} & \text{if } m_i \in J - (K \cap J) \\ \lambda_{m_i} \wedge \mu_{m_i} & \text{if } m_i \in (K \cap J) \end{cases}.$$

We have

$$\beta(\lambda \wedge \mu) \geq \bigwedge_{j \in K \cup J} \tau_j(\rho_j)$$

$$\geq \left(\bigwedge_{i=1}^{P} \tau_{k_j}(\lambda_{k_j})\right) \wedge \left(\bigwedge_{i=1}^{q} \tau_{j_i}(\mu_{j_i})\right),$$

$$\beta^*(\lambda \wedge \mu) \leq \bigvee_{j \in K \cup J} \tau_j^*(\rho_j)$$

$$\leq \left(\bigvee_{i=1}^{P} \tau^*_{k_j}(\lambda_{k_j}) \right) \vee \left(\bigvee_{i=1}^{q} \tau^*_{j_i}(\mu_{j_i}) \right)$$

Then, $\beta(\lambda \wedge \mu) \geq \beta(\lambda) \wedge \beta(\mu)$ and $\beta^*(\lambda \wedge \mu) \leq \beta^*(\lambda) \vee \beta^*(\mu)$.

(2) For each $\lambda_i \in I^{X_i}$, one family $\{f_i^{-1}(\lambda_i)\}$, and $i \in \Gamma$, we have

$$\tau_\beta\left(f_i^{-1}(\lambda_i)\right) \geq \beta\left(f_i^{-1}(\lambda_i)\right) \geq \tau_i(\lambda_i),$$

$$\tau^*_{\beta^*}\left(f_i^{-1}(\lambda_i)\right) \leq \beta^*\left(f_i^{-1}(\lambda_i)\right) \leq \tau_i^*(\lambda_i).$$

Thus, for each $i \in \Gamma$, $f_i : (X, \tau_\beta, \tau^*_{\beta^*}) \to (X_i, \tau_i, \tau_i^*)$ is IF continuous.

Let $f_i : (X, \tau^0, \tau^{0*}) \to (X_i, \tau_i, \tau_i^*)$ be IF continuous, that is, for each $i \in \Gamma$ and $\lambda_i \in I^{X_i}$,

$$\tau^0\left(f_i^{-1}(\lambda_i)\right) \geq \tau_i(\lambda_i) \text{ and } \tau^{0*}\left(f_i^{-1}(\lambda_i)\right) \leq \tau_i^*(\lambda_i).$$

For all finite subsets $K = \{k_1, k_2, \ldots\ldots, k_p\}$ of Γ

such that $\lambda = \bigwedge_{i=1}^{P} f_{k_i}^{-1}(\lambda_{k_i})$, we have

$$\tau^0(\lambda) \geq \bigwedge_{i=1}^{P} \tau^0\left(f_{k_i}^{-1}(\lambda_{k_i})\right) \geq \bigwedge_{i=1}^{P} \tau_{k_i}(\lambda_{k_i}),$$

$$\tau^{0*}(\lambda) \leq \bigvee_{i=1}^{P} \tau^{0*}\left(f_{k_i}^{-1}(\lambda_{k_i})\right) \leq \bigvee_{i=1}^{P} \tau^*_{k_i}(\lambda_{k_i}).$$

It implies $\tau^0(\lambda) \geq \beta(\lambda)$ and $\tau^{0*}(\lambda) \leq \beta^*(\lambda)$ for each $\lambda \in I^X$.

By theorem 3.1.9 (2) $\tau^0 \geq \tau_\beta$ and $\tau^{0*} \leq \tau^*_{\beta^*}$.

(3) ($\Rightarrow$) Let $f : (Y, \tau_1, \tau_1^*) \to (X, \tau_\beta, \tau^*_{\beta^*})$ be an IF continuous.

For each $i \in \Gamma$ and $\lambda_i \in I^{X_i}$, we have

$$\tau_1\big((f_i \circ f)^{-1}(\lambda_i)\big) = \tau_1\big((f^{-1}(f_i^{-1}(\lambda_i))\big) \geq \tau_\beta\big(f_i^{-1}(\lambda_i)\big)$$

$$\geq \tau_i(\lambda_i)$$

$$\tau_1^*\big((f_i \circ f)^{-1}(\lambda_i)\big) = \tau_1^*\big((f^{-1}(f_i^{-1}(\lambda_i))\big) \leq \tau^*_{\beta^*}\big(f_i^{-1}(\lambda_i)\big)$$

$$\leq \tau_i^*(\lambda_i)$$

Hence $f_i \circ f : (Y, \tau_1, \tau_1^*) \to (X_i, \tau_i, \tau^*_{i^*})$ is IF continuous.

($\Leftarrow$) For all finite subsets $K = \{k_1, k_2, \ldots\ldots, k_p\}$ of Γ

such that $\lambda = \overset{P}{\underset{i=1}{\wedge}} f_{k_i}^{-1}(\lambda_{k_i})$,

since $f_{k_i} \circ f : (Y, \tau_1, \tau_1^*) \to (X_{k_i}, \tau_{k_i}, \tau^*_{k_i})$ is IF continuous,

$$\tau_1\big(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\big) \geq \tau_{k_i}(\lambda_{k_i}) \qquad \text{(A)}$$

$$\tau_1^*\big(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\big) \leq \tau^*_{k_i}(\lambda_{k_i}) \qquad \text{(B)}$$

Hence, we have

$$\tau_1(f^{-1}(\lambda)) = \tau_1\left(f^{-1}(\overset{P}{\underset{i=1}{\wedge}} f_{k_i}^{-1}(\lambda_{k_i}))\right)$$

$$= \tau_1\left(\overset{P}{\underset{i=1}{\wedge}} f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\right)$$

$$\geq \overset{P}{\underset{i=1}{\wedge}} \tau_1\big(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\big)$$

$$\geq \bigwedge_{i=1}^{P} \tau_{k_i}(\lambda_{k_i}))$$ (by (A))

$$\tau_1^*(f^{-1}(\lambda)) = \tau_1^*\left(f^{-1}(\bigwedge_{i=1}^{P} f_{k_i}^{-1}(\lambda_{k_i}))\right)$$

$$= \tau_1^*\left(\bigwedge_{i=1}^{P} f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\right)$$

$$\leq \bigvee_{i=1}^{P} \tau_1^*\left(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\right)$$

$$\leq \bigvee_{i=1}^{P} \tau_{k_i}^*(\lambda_{k_i}))$$ (by (B))

It implies $\tau_1(f^{-1}(\lambda)) \geq \beta(\lambda)$ and $\tau_1^*(f^{-1}(\lambda)) \leq \beta^*(\lambda)$ for all $\lambda \in I^X$.

By theorem 3.1.9 (2) f : (Y, τ_1, τ_1^*) $\rightarrow$ (X, $\tau_\beta, \tau_{\beta^*}^*$) is IF continuous.

Definition: 3.2.2

Let (X, τ, τ^*) be an IFTS and $A \subset X$. The triple (A, τ/A, τ^*/A) is said to be a **subspace** of (X, τ, τ^*) if (τ/A, τ^*/A) is the coarsest IGO on A for which the inclusion map i is IF continuous.

Definition: 3.2.3

Let X be the product $\prod_{i \in \Gamma} X_i$ of the family {(X_i, τ_i, τ_i^*) / $i \in \Gamma$} of IFTSs. The coarsest IGO, $(\tau, \tau^*) = (\otimes\tau_i, \otimes\tau_i^*)$ on X for which each the projections $\pi_i : X \rightarrow X_i$ is IF continuous, is called the **product IGO** of {(τ_i, τ_i^*) / $i \in \Gamma$} and (X, τ, τ^*) is called the **product IFTS**.

Lemma: 3.2.4

Let $(Y, \mathcal{U}, \mathcal{U}^*)$ be an IFTS and (β, β^*) be an IF topological base on X. If $f : (X, \beta, \beta^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is a function such that $\beta(\lambda) \leq \mathcal{U}(f(\lambda))$ and $\beta^*(\lambda) \geq \mathcal{U}^*(f(\lambda))$ for all $\lambda \in \theta_X$, then $f : \{(X, \tau_\beta, \tau^*_{\beta^*}) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is IF open.

Proof:

Let $(Y, \mathcal{U}, \mathcal{U}^*)$ be an IFTS and (β, β^*) be an IF topological base on X.

Let $f : (X, \beta, \beta^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ be a function such that $\beta(\lambda) \leq \mathcal{U}(f(\lambda))$ and $\beta^*(\lambda) \geq \mathcal{U}^*(f(\lambda))$ for all $\lambda \in \theta_X$.

To prove: $f : \{(X, \tau_\beta, \tau^*_{\beta^*}) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is IF open.

Suppose there exists $\mu \in I^X$ such that $\tau_\beta(\mu) > \mathcal{U}(f(\mu))$ or $\tau^*_{\beta^*}(\mu) < \mathcal{U}^*(f(\mu))$,

then there exists a family $\{\lambda_i \in \theta_X / \mu = \vee_{i \in \Gamma} \lambda_i\}$

such that $\tau_\beta(\mu) \geq \wedge_{i \in \Gamma} \beta(\lambda_i) > \mathcal{U}(f(\mu))$ (or) $\tau^*_{\beta^*}(\mu) \leq \vee_{i \in \Gamma} \beta^*(\lambda_i) < \mathcal{U}^*(f(\mu))$.

On the otherway,

since $\beta(\lambda) \leq \mathcal{U}(f(\lambda))$ and $\beta^*(\lambda) \geq \mathcal{U}^*(f(\lambda)) \forall \lambda \in \theta_X$, Then we have

$$\underset{i\in\Gamma}{\wedge}\beta(\lambda_i) \leq \underset{i\in\Gamma}{\wedge}\mathcal{U}(f(\lambda_i)) \leq \mathcal{U}[\underset{i\in\Gamma}{\vee}(f(\lambda_i))] = \mathcal{U}[f(\underset{i\in\Gamma}{\vee}(\lambda_i))] = \mathcal{U}(f(\mu)),$$

$$\underset{i\in\Gamma}{\vee}\beta^*(\lambda_i) \geq \underset{i\in\Gamma}{\vee}\mathcal{U}^*(f(\lambda_i)) \geq \mathcal{U}^*[\underset{i\in\Gamma}{\vee}(f(\lambda_i))] = \mathcal{U}^*[f(\underset{i\in\Gamma}{\vee}(\lambda_i))] = \mathcal{U}^*(f(\mu)).$$

It is a contradiction.

Hence f is IF open.

Theorem: 3.2.5

Let $(X, \tau_\beta, \tau^*_{\beta^*})$ be a product space of a family $\{(X_i, \tau_i, \tau_i^*) \ / \ i\in\Gamma\}$ of IFTS's. Then the following statements are equivalent:

1) a projection map $\pi_j : (X, \tau_\beta, \tau^*_{\beta^*}) \to (X_j, \tau_j, \tau_j^*)$ is IF open;

2) for every $\mu = \underset{i\in\Gamma}{\wedge}\pi_i^{-1}(\lambda_i)$ such that $\underset{x\in X_i}{\vee}\lambda_i(x) = \alpha_i$ for each $\alpha_i\in I$ and $i\in\Gamma_0$ such that a finite index subset Γ_0 of Γ-{j} and $\tau_i(\lambda_i) > 0$, then $\underset{i\in\Gamma_0}{\wedge}\tau_i(\lambda_i) \leq \tau_j(\underline{\alpha})$ and $\underset{i\in\Gamma_0}{\vee}\tau_i^*(\lambda_i) \geq \tau_j^*(\underline{\alpha})$, where $\alpha = \underset{i\in\Gamma_0}{\wedge}\alpha_i$.

Proof:

(1) $\Rightarrow$ (2)

For every $\mu = \underset{i\in\Gamma}{\wedge}\pi_i^{-1}(\lambda_i)$ such that $\underset{x\in X_i}{\vee}\lambda_i(x) = \alpha_i$ for each $\alpha_i\in I$ and $i\in\Gamma_0$ such that a finite index subset Γ_0 of Γ-{j}.

By lemma 3.1.10 (b), we have,

for $\alpha = \underset{i\in\Gamma_0}{\wedge}\alpha_i$, $\pi_j(\mu) = \pi_j\left(\underset{i\in\Gamma_0}{\wedge}\pi_i^{-1}(\lambda_i)\right) = \underline{\alpha}$.

Since $\mu \in \theta_X$, by Theorem 3.2.1, we have

$$\bigwedge_{i \in \Gamma_0} \tau_i(\lambda_i) \le \beta(\mu) \le \tau_\beta(\mu),$$

$$\bigvee_{i \in \Gamma_0} \tau_i^*(\lambda_i) \ge \beta^*(\mu) \ge \tau^*_{\beta^*}(\mu).$$

Since π_j is IF open, we have

$$\tau_\beta(\mu) \le \tau_j(\pi_j(\mu)) = \tau_j(\underline{\alpha}),$$

$$\tau^*_{\beta^*}(\mu) \ge \tau_j^*(\pi_j(\mu)) = \tau_j^*(\underline{\alpha}).$$

Hence $\bigwedge_{i \in \Gamma_0} \tau_i(\lambda_i) \le \tau_j(\underline{\alpha})$ and $\bigvee_{i \in \Gamma_0} \tau_i^*(\lambda_i) \ge \tau_j^*(\underline{\alpha})$.

(2) $\Rightarrow$ (1):

From Lemma 3.2.4, it is enough to show that

$\beta(\mu) \le \tau_j(\pi_j(\lambda))$ and $\beta^*(\mu) \ge \tau_j^*(\pi_j(\lambda))$ for all $\lambda \in \theta_X$.

Suppose that there exists $\gamma \in \theta_X$ such that

$$\beta(\gamma) > \tau_j(\pi_j(\gamma)) \text{ or } \beta^*(\gamma) < \tau_j^*(\pi_j(\gamma)).$$

Then there exists a finite index subset Γ_0 of Γ-{j} with

$\gamma = \pi_j^{-1}(\lambda_j) \wedge \left[\bigwedge_{i \in \Gamma_0} \pi_i^{-1}(\lambda_i) \right]$ (take $\lambda_j = \tilde{1}$) such that

$$\beta(\gamma) \ge \tau_j(\lambda_j) \wedge \left[\bigwedge_{i \in \Gamma_0} \tau_i(\lambda_i) \right] > \tau_j(\pi_j(\gamma)),$$

$$\beta^*(\gamma) \le \tau_j^*(\lambda_j) \vee \left[\bigvee_{i \in \Gamma_0} \tau_i^*(\lambda_i) \right] < \tau_j^*(\pi_j(\gamma)).$$

On the otherway, by lemma 3.1.10 (2), we have

$$\pi_j(\gamma) = \pi_j\left[\pi_j^{-1}(\lambda_j) \wedge \left[\bigwedge_{i\in\Gamma_0} \pi_i^{-1}(\lambda_i)\right]\right]$$

$$= \lambda_j \wedge \pi_j\left[\bigwedge_{i\in\Gamma_0} \pi_i^{-1}(\lambda_i)\right]$$

$$= \lambda_j \wedge \underline{\alpha}$$

where $\bigvee_{x\in X_i} \lambda_i(x) = \alpha_i$ and $\alpha = \bigwedge_{i\in\Gamma_0} \alpha_i$.

Since $\bigwedge_{i\in\Gamma_0} \tau_i(\lambda_i) \leq \tau_j(\underline{\alpha})$ and $\bigvee_{i\in\Gamma_0} \tau_i^*(\lambda_i) \geq \tau_j^*(\underline{\alpha})$.

We have

$$\tau_j(\pi_j(\gamma)) = \tau_j(\lambda_j \wedge \underline{\alpha})$$

$$\geq \tau_j(\lambda_j) \wedge \tau_j(\underline{\alpha})$$

$$\geq \tau_j(\lambda_j) \wedge (\bigwedge_{i\in\Gamma_0} \tau_i(\lambda_i))$$

$$\tau_j^*(\pi_j(\gamma)) = \tau_j^*(\lambda_j \wedge \underline{\alpha})$$

$$\leq \tau_j^*(\lambda_j) \wedge \tau_j^*(\underline{\alpha})$$

$$\leq \tau_j^*(\lambda_j) \vee (\bigvee_{i\in\Gamma_0} \tau_j^*(\lambda_i)).$$

It is a contradiction.

Hence $\beta(\mu) \leq \tau_j(\pi_j(\lambda))$ and $\beta^*(\mu) \geq \tau_j^*(\pi_j(\lambda))$ for all $\lambda\in\theta_X$.

Hence $\pi_j : (X, \tau_\beta, \tau^*_{\beta^*}) \rightarrow (X_j, \tau_j, \tau_j^*)$ is IF open.

Theorem: 3.2.6

Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ of IFTSs and (X_j, τ_j, τ_j^*) be stratified. Then the following properties hold:

1) (X, τ, τ^*) is stratified;

2) a projection map $\pi_j : X \to X_j$ is IF open.

Proof:

Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ of IFTSs and (X_j, τ_j, τ_j^*) be stratified.

(1) To prove: (X, τ, τ^*) is stratified.

It is clear that,

for all $\alpha \in I$,

$$\tau(\underline{\alpha}) \geq \beta(\underline{\alpha}) = \vee\{\underset{i\in\Gamma_0}{\wedge} \tau_i(\lambda_i) / \underline{\alpha} = \underset{i\in\Gamma_0}{\wedge} \pi_i^{-1}(\lambda_i)\} \geq \tau_j(\underline{\alpha}) = 1,$$

$$\tau^*(\underline{\alpha}) \leq \beta^*(\underline{\alpha}) = \wedge\{\underset{i\in\Gamma_0}{\vee} \tau_i^*(\lambda_i) / \underline{\alpha} = \underset{i\in\Gamma_0}{\wedge} \pi_i^{-1}(\lambda_i)\} \leq \tau_j^*(\underline{\alpha}) = 0.$$

Hence $\tau(\underline{\alpha}) = 1$ and $\tau^*(\underline{\alpha}) = 0$ for all $\alpha \in I$.

$\therefore$ (X, τ, τ^*) is stratified.

(2) To prove: A projection map $\pi_j : X \to X_j$ is IF open.

Since $\tau_j(\underline{\alpha}) = 1$ and $\tau_j^*(\underline{\alpha}) = 0$ for all $\alpha \in I$,

It satisfies the condition of theorem 3.2.5 (2).

Therefore, by theorem 3.2.5, we get,

a projection map $\pi_j : X \to X_j$ is IF open.

Hence the proof.

Theorem: 3.2.7

Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ of IFTSs and let (X_j, τ_j, τ_j^*) be stratified. Then for each $\tilde{X}_j = X_j \times \Pi\{y^i / i \neq j\}$ in X parallel to X_j, $\pi_j \mid \tilde{X}_j : \tilde{X}_j \to X_j$ is an IF homeomorphism.

Proof:

Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ of IFTSs and let (X_j, τ_j, τ_j^*) be stratified.

To prove: $\pi_j \mid \tilde{X}_j : \tilde{X}_j \to X_j$ is an IF homeomorphism.

Let $\tilde{X}_j = X_j \times \Pi\{y^i / i \neq j\}$.

Since $i : \tilde{X}_j \to \tilde{X}_j$ and $\pi_j : \tilde{X}_j \to X_j$ are IF continuous,

$\pi_j \circ i = \pi_j \mid \tilde{X}_j$ is IF continuous. Also $\pi_j \mid \tilde{X}_j$ is bijective.

Now it is enough to show that $\pi_j \mid \tilde{X}_j$ is IF open.

Suppose there exists $\mu \in I^{\tilde{X}_j}$ such that

$\tau \mid \tilde{X}_j(\mu) > \tau_j(\pi_j \mid \tilde{X}_j(\mu))$ (or)

$\tau^* \mid \tilde{X}_j(\mu) < \tau_j^*(\pi_j \mid \tilde{X}_j(\mu))$.

Then there exists $\gamma \in I^X$ with $\mu = i^{-1}(\gamma)$ such that

$$\tau \mid \tilde{X}_j(\mu) \geq \tau(\gamma) > \tau_j(\pi_j \mid \tilde{X}_j(\mu))$$

(or) $\tau^* \mid \tilde{X}_j(\mu) \leq \tau^*(\gamma) < \tau_j^*(\pi_j \mid \tilde{X}_j(\mu))$.

From the definition of (τ, τ^*), there exists a family $\{\gamma_k \in \theta_X \,/\, \gamma = \vee_{k\in K} \gamma_k\}$ such that

$$\tau(\gamma) \geq \wedge_{k\in K} \beta(\gamma_k) > \tau_j(\pi_j \mid \tilde{X}_j(\mu))$$

(or) $\tau^*(\gamma) \leq \vee_{k\in K} \beta^*(\gamma_k) < \tau_j^*(\pi_j \mid \tilde{X}_j(\mu))$ (C)

On the otherway, since each $\gamma_k \in \theta_X$, there exists a finite index F_k of $\Gamma - \{j\}$ with $\gamma_k = \pi_j^{-1}\left(\lambda_{k_j}\right) \wedge \left(\wedge_{i\in F_k} \pi_i^{-1}\left(\lambda_i\right)\right)$.

Since $\pi_i^{-1}(\lambda_i)(x) = y^i$ for $i \neq j$, then for each $x \in \tilde{X}_j$,

$$\wedge_{i\in F_k} \pi_i^{-1}(\lambda_i)(x) = \left(\wedge_{i\in F_k} \lambda_i\,(y^i) \right).$$

Put $\alpha_k = \left(\wedge_{i\in F_k} \lambda_i\,(y^i) \right)$.

Let $\mu_k = i^{-1}(\gamma_k)$ for each $k \in K$.

Then,

$$\begin{aligned}
\pi_j \mid \tilde{X}_j(\mu_k)\,(x^j) &= \vee\{\mu_k(x) \,/\, x \in \tilde{X}_j,\ \pi_j \mid \tilde{X}_j(x) = x^j\} \\
&= \vee\{i^{-1}(\gamma_k)(x) \,/\, x \in \tilde{X}_j,\ \pi_j(x) = x^j\ (\mu_k = i^{-1}(\gamma_k))\} \\
&= \vee\{\gamma_k(x) \,/\, x \in \tilde{X}_j,\ \pi_j(x) = x^j\}
\end{aligned}$$

$$= \vee\{\pi_j^{-1}(\lambda_{k_j})(x) \wedge (\bigwedge_{i\in F_k} \pi_i^{-1}(\lambda_i)(x) \,/\, x \in \tilde{X}_j,\ \pi_j(x) = x^j)\}$$

$$= \vee\{\lambda_{k_j}(\pi_j(x)) \wedge (\bigwedge_{i\in F_k} \lambda_i)(\pi_i(x)) \,/\, x \in \tilde{X}_j,\ \pi_j(x) = x^j\}$$

$$= \lambda_{k_j}(x^j) \wedge (\bigwedge_{i\in F_k} \lambda_i)y^i$$

$$= \lambda_{k_j}(x^j) \wedge \alpha_k$$

$$= (\lambda_{k_j} \wedge \underline{\alpha}_k)(x^j).$$

Hence $\pi_j \mid \tilde{X}_j(\mu_k) = \lambda_{k_j} \wedge \underline{\alpha}_k$.

Thus,

$$\begin{aligned}
\tau_j(\pi_j \mid \tilde{X}_j(\mu_k)) &= \tau_j(\lambda_{k_j} \wedge \underline{\alpha}_k) \\
&\geq \tau_j(\lambda_{k_j}) \wedge \tau_j(\underline{\alpha}_k) \\
&= \tau_j(\lambda_{k_j}) \\
&\geq \tau_j(\lambda_{k_j}) \wedge (\bigwedge_{i\in F_k} \lambda_i)
\end{aligned}$$

$$\begin{aligned}
\tau_j^*(\pi_j \mid \tilde{X}_j(\mu_k)) &= \tau_j^*(\lambda_{k_j} \wedge \underline{\alpha}_k) \\
&\leq \tau_j^*(\lambda_{k_j}) \vee \tau_j^*(\underline{\alpha}_k) \\
&= \tau_j^*(\lambda_{k_j}) \\
&\leq \tau_j^*(\lambda_{k_j}) \vee (\bigwedge_{i\in F_k} \lambda_i)
\end{aligned}$$

From the definition of (β, β^*), it implies

$$\tau_j(\pi_j \mid \widetilde{X}_j(\mu_k)) \geq \beta(\gamma_k),$$

$$\tau_j^*(\pi_j \mid \widetilde{X}_j(\mu_k)) \leq \beta^*(\gamma_k).$$

Thus,

$$\tau_j(\pi_j \mid \widetilde{X}_j(\mu)) \geq \bigwedge_{k\in K} \tau_j(\pi_j \mid \widetilde{X}_j(\mu_k)) \geq \bigwedge_{k\in K} \beta(\gamma_k),$$

$$\tau_j^*(\pi_j \mid \widetilde{X}_j(\mu)) \leq \bigvee_{k\in K} \tau_j^*(\pi_j \mid \widetilde{X}_j(\mu_k)) \leq \bigvee_{k\in K} \beta^*(\gamma_k).$$

It is a contradiction to (c).

$\therefore$ $\pi_j \mid \widetilde{X}_j$ is an IF open.

Hence $\pi_j \mid \widetilde{X}_j$ is an IF homeomorphism.

Example: 3.2.8

Let $X = \{x^1, x^2, x^3\}$, $Y = \{y^1, y^2\}$ and $Z = \{z^1, z^2\}$ be sets $W = X \times Y \times Z$ a product set.

Let $\pi_1 : W \to X$, $\pi_2 : W \to Y$ and $\pi_3 : W \to Z$ be the projection maps.

Define $\tau_1, \tau_1^* : I^X \to I$ by

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 1 & \text{otherwise,} \end{cases}$$

where $\lambda_1(x^1) = 0.5$, $\lambda_1(x^2) = 0.2$ and $\lambda_1(x^3) = 0.3$.

Also, $\widetilde{X}_j = \{(x, y^2, z2) : x \in X\}$, define $\tau/\widetilde{X}_j$, $\tau^*/\widetilde{X}_j : I^{\widetilde{X}_j} \to I$ by

$$\tau/\widetilde{X}_j(\mu) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{2}{3} & \text{if } \mu = \underline{0.1}, \\ \frac{1}{4} & \text{if } \mu = \underline{0.7}, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau^*/\widetilde{X}_j(\mu) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \frac{1}{2} & \text{if } \mu = \mu_1, \\ \frac{1}{3} & \text{if } \mu = \underline{0.1}, \\ \frac{3}{4} & \text{if } \mu = \underline{0.7}, \\ 1 & \text{otherwise}, \end{cases}$$

where $\mu_1(x^1, y^2, z^2) = 0.5$, $\mu_1(x^2, y^2, z^2) = 0.2$ and $\mu_1(x^3, y^2, z^2) = 0.3$.

Then the projection map $\pi_j/\widetilde{X}_j : \widetilde{X}_j \to X$ is bijective IF continuous, but $\pi_j/\widetilde{X}_j$ is not IF open, because

$$\frac{2}{3} = \tau/\widetilde{X}_j(\underline{0.1}) \not\leq \tau_1\left(\pi_1/\widetilde{X}_j(\underline{0.1})\right) = 0.$$

Hence, $\widetilde{X}_j$ and X are not homeomorphic.

Theorem: 3.2.9

Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) \,/\, i\in\Gamma\}$ of IFTSs. Then, the following properties hold:

1) $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \leq \prod_{i\in\Gamma} C_{\tau,\tau^*}(\lambda_i, r, s)$, $\forall \lambda_i \in I^{X_i}$, $r\in I_0$, $s\in I_1$,

2) if $C_{\tau_i,\tau_i^*}(\lambda_i, r, s) = \lambda_i$, $\forall \lambda_i \in I^{X_i}$, $r\in I_0$, $s\in I_1$, then $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) = \prod_{i\in\Gamma}\lambda_i$.

Proof:

Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) \,/\, i\in\Gamma\}$ of IFTSs.

1) **To prove:** $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \le \prod_{i\in\Gamma} C_{\tau,\tau^*}(\lambda_i, r, s)$ $\forall \lambda_i \in I^{X_i}$, $r\in I_0$, $s\in I_1$.

Suppose $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \not\le \prod_{i\in\Gamma} C_{\tau,\tau^*}(\lambda_i, r, s)$.

Then there exists $x\in X$ and $t\in(0, 1)$ such that

$$C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s)(x) \ge t > \prod_{i\in\Gamma} C_{\tau_i,\tau_i^*}(\lambda_i, r, s)(x) \quad \text{(D)}$$

Since $\prod_{i\in\Gamma} C_{\tau_i,\tau_i^*}(\lambda_i, r, s) < t$, there exists $j\in\Gamma$

such that $\prod_{i\in\Gamma} C_{\tau_i,\tau_i^*}(\lambda_i, r, s) \le \pi_j^{-1}(C_{\tau_i,\tau_i^*}(\lambda_i, r, s)) < t$.

Put $\pi_j(x) = x^j$.

$\Rightarrow \quad C_{\tau_j,\tau_j^*}(\lambda_j, r, s)(x^j) < t.$

From the definition of C_{τ_j,τ_j^*}, there exists $\mu_j \in I^{X_j}$ with $\lambda_j \le \mu_j$ and

$\tau_j(\tilde{1}-\mu_j) \ge r$, $\tau_j^*(\tilde{1}-\mu_j) \le s$ such that $C_{\tau_j,\tau_j^*}(\lambda_j, r, s)(x^j) \le \mu_j(x^j) < t$.

On the otherway, we have

$$\lambda_j \le \mu_j \Rightarrow \pi_j^{-1}(\lambda_j) \le \pi_j^{-1}(\mu_j)$$

$$\Rightarrow \prod_{i\in\Gamma}\lambda_i = \bigwedge_{i\in\Gamma}\pi_j^{-1}(\lambda_i) \le \pi_j^{-1}(\mu_j)$$

$$\Rightarrow C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \le \pi_j^{-1}(\mu_j).$$

Since $\tau(\tilde{1}-\pi_j^{-1}(\mu_j)) = \tau(\pi_j^{-1}(\tilde{1}-\mu_j)) \ge \tau_j(\tilde{1}-\mu_j) \ge r$

and $\tau^*(\tilde{1}-\pi_j^{-1}(\mu_j)) = \tau^*(\pi_j^{-1}(\tilde{1}-\mu_j)) \le \tau_j^*(\tilde{1}-\mu_j) \le s$.

Hence $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s)(x) \le \pi_j^{-1}(\mu_j)(x) = \mu_j(x^j) < t$.

It is a contradiction to (D).

Hence $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \leq \prod_{i\in\Gamma} C_{\tau_i,\tau_i^*}(\lambda_i, r, s)$.

2) **To prove:** if $C_{\tau_i,\tau_i^*}(\lambda_i, r, s) = \lambda_i,\ \forall\lambda_i \in I^{X_i}, r\in I_0, s\in I_1$.

then $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) = \prod_{i\in\Gamma}\lambda_i$.

It is clear that

$$\prod_{i\in\Gamma}\lambda_i \leq C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \leq \prod_{i\in\Gamma} C_{\tau_i,\tau_i^*}(\lambda_i, r, s) = \prod_{i\in\Gamma}\lambda_i.$$

Hence $C_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) = \prod_{i\in\Gamma}\lambda_i$.

SECTION: 3.3

SOME PROPERTIES OF (r, s)-T_0 AND (r, s)-T_1 SPACES

Definition: 3.3.1

An IFTS (X, τ, τ^*) is

1) **(r, s)-quasi-T_0 space** if for each $x_t, x_m\in P_t(x)$ and $t<m$, there exists $\lambda\in Q(x_m, r, s)$ such that $x_t\bar{q}\lambda$.

2) **(r, s)-sub-T_0 space** if for each $x \neq y\in X$, there exists $t\in I_0$ such that there exists $\lambda\in Q(x_t, r, s)$ such that $y_t\bar{q}\lambda$, or there exists $\mu\in Q(y_t, r, s)$ such that $x_t\bar{q}\mu$.

3) **(r, s)-T_0 space** if for each $x_t, y_m\in P_t(X)$, there exists $\lambda\in Q(x_t, r, s)$ such that $y_m\bar{q}\lambda$, or there exists $\mu\in Q(y_m, r, s)$ such that $x_t\bar{q}\mu$.

4) **(r, s)-T_1 space** if for each x_t, $y_m \in P_t(X)$, such that $x_t \not\leq y_m$, there exists $\lambda \in Q(x_t, r, s)$ such that $y_m \bar{q} \lambda$.

Theorem: 3.3.2

Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent :

1) (X, τ, τ^*) is (r, s)-T_0 space;

2) for each x_t, $y_m \in P_t(X)$, $Q(x_t, r, s) \neq Q(y_m, r, s)$;

3) for each x_t, $y_m \in P_t(X)$, then $x_t \notin C_{\tau,\tau^*}(y_m, r, s)$ or $y_m \notin C_{\tau,\tau^*}(x_t, r, s)$.

Proof:

(1) $\Rightarrow$ (2): It is trivial.

(2) $\Rightarrow$ (3): Let $\lambda \in Q(x_t, r, s)$ and $\lambda \notin Q(y_m, r, s)$.

Since $\lambda \notin Q(y_m, r, s)$, we have

$$y_m \leq \tilde{1} - \lambda,\ \tau(\lambda) \geq r,\ \tau^*(\lambda) \leq s.$$

By definition 3.1.3, we have $C_{\tau,\tau^*}(y_m, r, s) \leq \tilde{1} - \lambda$.

Since $x_t q \lambda$ and $\lambda \leq \tilde{1} - C_{\tau,\tau^*}(y_m, r, s)$, then $x_t q[\tilde{1} - C_{\tau,\tau^*}(y_m\ r, s)]$.

Hence $x_t \notin C_{\tau,\tau^*}(y_m, r, s)$.

(3) $\Rightarrow$ (1):

Let x_t, $y_m \in P_t(X)$ and $x_t \notin C_{\tau,\tau^*}(y_m, r, s)$.

Then $t > C_{\tau,\tau^*}(y_m, r, s)(x)$ implies $x_t q[\tilde{1} - C_{\tau,\tau^*}(y_m, r, s)]$.

Since $C_{\tau,\tau^*}(y_m, r, s) = \wedge\{\mu | \mu \geq y_m, \tau(\tilde{1}-\mu) \geq r, \tau^*(\tilde{1}-\mu) \leq s\}$.

Since $\tau(\vee(\tilde{1}-\mu)) \geq \wedge\, \tau(\tilde{1}-\mu)$ and $\tau^*(\vee(\tilde{1}-\mu)) \leq \vee\, \tau^*(\tilde{1}-\mu)$,

we have $\tau(\tilde{1}-C_{\tau,\tau^*}(y_m, r, s)) \geq r$ and $\tau^*(\tilde{1}-C_{\tau,\tau^*}(y_m, r, s)) \leq s$.

Hence $[\tilde{1}-C_{\tau,\tau^*}(y_m, r, s)] \in Q(x_t, r, s)$.

Since $y_m \in C_{\tau,\tau^*}(y_m, r, s)$ and $y_m\ \bar{q}\ [\tilde{1}-C_{\tau,\tau^*}(y_m, r, s)]$.

Hence (X, τ, τ^*) is (r, s)-T_0 space.

Corollary: 3.3.3

Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:

1) (X, τ, τ^*) is (r, s)-quasi-T_0 space;

2) for each $x_t, x_m \in P_t(X)$, $Q(x_t, r, s) \neq Q(x_m, r, s)$;

3) for each $x_t, x_m \in P_t(X)$, then $x_t \notin C_{\tau,\tau^*}(x_m, r, s)$ or $x_m \notin C_{\tau,\tau^*}(x_t, r, s)$.

Corollary: 3.3.4

Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:

1) (X, τ, τ^*) is (r, s)-sub-T_0 space;

2) for each $x \neq y \in X$, there exists $t \in I_0$ such that $Q(x_t, r, s) \neq Q(y_t, r, s)$;

3) for each $x \neq y \in X$, there exists $t \in I_0$ such that $x_t \notin C_{\tau,\tau^*}(y_t, r, s)$ or

$y_t \notin C_{\tau,\tau^*}(x_t, r, s)$.

Example: 3.3.5

Let X = {x, y} be a set. We define an IGO (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{1} \text{ or } \underline{0}, \\ \frac{1}{2} & \text{if } \lambda = x_{0.7}, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{1} \text{ or } \underline{0}, \\ \frac{1}{2} & \text{if } \lambda = x_{0.7}, \\ 1 & \text{otherwise,} \end{cases}$$

For each $x \neq y \in X$, there exists $0.4 \in I_0$

such that $x_{0.7} \in Q(x_{0.4}, 1/2, 1/2)$ and $y_{0.4}\, \overline{q}\, x_{0.7}$.

Hence, (X, τ, τ^*) is (1/2, 1/2)-sub-T_0 space.

On the other hand, since $Q(y_{0.5}, 1/2, 1/2) = Q(y_{0.6}, 1/2, 1/2) = \{\underline{1}\}$,

by **Corollary 3.3.3(2)**, (X, τ, τ^*) is not (1/2, 1/2)-quasi-T_0-space.

Theorem: 3.3.6

Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:

1) (X, τ, τ^*) is (r, s)-T_1 space;

2) for each $x_t \in P_t(X)$, $x_t = C_{\tau,\tau^*}(x_t, r, s)$;

3) for each $\lambda \in I^X$, $\lambda = \wedge\{\mu / \lambda \leq \mu, \tau(\mu) \geq r, \tau^*(\mu) \leq s\}$.

Proof:

(1) $\Rightarrow$ (2):

Assume (X, τ, τ^*) is (r, s)-T_1 space.

Claim: for each $x_t \in P_t(X)$, $x_t = C_{\tau,\tau^*}(x_t, r, s)$;

It is enough to prove that $C_{\tau,\tau^*}(x_t, r, s) \leq x_t$.

Let $y_m \in C_{\tau,\tau^*}(x_t, r, s)$.

Suppose that $y_m \not\leq x_t$

Since (X, τ, τ^*) is (r, s)-T_1 space, there exists $\lambda \in Q(y_m, r, s)$ such that $x_t \bar{q} \lambda$.

$\Rightarrow$ $x_t \leq \tilde{1} - \lambda$ with $\tau(\lambda) \geq r$ and $\tau^*(\lambda) \leq s$.

Hence $C_{\tau,\tau^*}(x_t, r, s) \leq \tilde{1} - \lambda$.

Since $y_m \in C_{\tau,\tau^*}(x_t, r, s) \leq \tilde{1} - \lambda$, we have $\lambda \notin Q(y_m, r, s)$,

which is a contradiction.

Hence $y_m \leq x_t$.

Since $y_m \in C_{\tau,\tau^*}(x_t, r, s)$.

$\Rightarrow$ $y_m \leq x_t$, then $C_{\tau,\tau^*}(x_t, r, s) \leq x_t$.

(2) $\Rightarrow$ (3):

Assume $x_t = C_{\tau,\tau^*}(x_t, r, s)$, for each $x_t \in P_t(X)$.

Claim:

$\lambda = \wedge\{\mu \,/\, \lambda \leq \mu,\ \tau(\mu) \geq r,\ \tau^*(\mu) \leq s\}$ for each $\lambda \in I^X$.

Let $\rho = \wedge\{\mu \,/\, \lambda \leq \mu,\ \tau(\mu) \geq r,\ \tau^*(\mu) \leq s\}$.

It is enough to show that $\rho \leq \lambda$.

Suppose there exists $x \in X$ and $t \in (0, 1)$

$\ni$ $\rho(x) > 1 - t \geq \lambda(x)$.

Then $\lambda \leq \tilde{1} - x_t$.

Since $x_t = C_{\tau,\tau^*}(x_t, r, s)$,

$\tau(\tilde{1} - x_t) = \tau(\tilde{1} - C_{\tau,\tau^*}(x_t, r, s)) \geq r$, $\tau^*(\tilde{1} - C_{\tau,\tau^*}(x_t, r, s)) \leq s$.

Hence $\rho \leq \tilde{1} - x_t$, which is a contradiction.

Therefore, $\rho \leq \lambda$.

(3) $\Rightarrow$ (1):

Assume $\lambda = \wedge\{\mu / \lambda \leq \mu, \tau(\mu) \geq r, \tau^*(\mu) \leq s\}$, for each $\lambda \in I^X$.

Claim:

(X, τ, τ^*) is (r, s)-T_1 space.

For each $x_t, y_m \in P_t(X)$ such that

$x_t \not\leq y_m$, $\tilde{1} - x_t \not\geq \tilde{1} - y_m$.

From (3), since $\tilde{1} - y_m = \wedge\{\mu / \tilde{1} - y_m \leq \mu, \tau(\mu) \geq r, \tau^*(\mu) \leq s\}$, there exists $\mu = \tilde{1} - y_m \in I^X$ such that

$\tau(\tilde{1} - y_m) \geq r$, $\tau^*(\tilde{1} - y_m) \leq s$.

Since $\tilde{1} - x_t \not\geq \tilde{1} - y_m$, we have $x_t\, q\, [\tilde{1} - y_m]$.

Thus $\tilde{1} - y_m \in Q(x_t, r, s)$ such that $y_m\, \bar{q}\, [\tilde{1} - y_m]$.

Hence (X, τ, τ^*) is (r, s)-T_1 space.

Theorem: 3.3.7

Let (X, τ, τ^*) be a stratified IFTS. Then (X, τ, τ^*) is (r, s)-quasi-T_0 space for all $r \in I_0$, $s \in I_1$.

Proof: Let (X, τ, τ^*) be a stratified IFTS.

Claim:

(X, τ, τ^*) is (r, s)-quasi-T_0 space for all $r \in I_0$, $s \in I_1$.

Let x_t, $x_m \in P_t(X)$ such that $t < m$.

Then there exists $\alpha \in I_0$ such that $t \leq 1-\alpha < m$.

Since (X, τ, τ^*) is stratified IFTS, we have $\tau(\underline{\alpha}) = 1$ and $\tau^*(\underline{\alpha}) = 0$.

Hence $\underline{\alpha} \in Q(x_m, r, s)$ such that $x_t \bar{q}\ \underline{\alpha}$.

Hence (X, τ, τ^*) is (r, s)-quasi-T_0 space for all $r \in I_0$, $s \in I_1$.

Theorem: 3.3.8

1) Every (r, s)-T_0 space is both (r, s)-quasi-T_0 and (r, s)-sub T_0.

2) Every (r, s)-T_1 space is (r, s)-T_0 space.

Proof:

1) Assume (X, τ, τ^*) is (r, s)-T_0 space.

Claim: (X, τ, τ^*) is (r, s)-quasi-T_0 and (r, s)-sub T_0 space.

For each x_t, $x_m \in P_t(X)$ such that $t < m$.

Claim: 1

(X, τ, τ^*) is (r, s)-quasi-T_0 space.

Since (r, s)-T_0 space, there exists $\lambda \in Q(x_t, r, s)$ such that $x_m \bar{q}\ \lambda$.

Since $t < m$, we have $x_t \bar{q}\ \lambda$.

Hence X is (r, s)-quasi-T_0 space.

Hence Claim 1.

Claim: 2

(X, τ, τ^*) is (r, s)-sub T_0 space.

For each x_t, $y_t \in P_t(X)$ such that $x_t \neq y_t$.

Since (X, τ, τ^*) is (r, s)-T_0 space,

$\Im\ \lambda \in Q(x_t\ r, s) \ni y_t \bar{q} \lambda$.

Since $x_t \neq y_t$, we may also have,

$\mu \in Q(y_t\ r, s) \ni x_t \bar{q}\ \mu$.

$\therefore$ For each $x \neq y \in X$, there exists $t \in I_0 \ni$

$Q(x_t, r, s) \neq Q(y_t, r, s)$.

$\Rightarrow$ (X, τ, τ^*) is a (r, s)-sub T_0 space.

Hence Claim 2.

Hence X is both (r, s)-quasi-T_0 and (r, s)-sub-T_0 space.

2) Assume (X, τ, τ^*) is (r, s)-T_1 space.

To prove:

(X, τ, τ^*) is (r, s)-T_0 space.

For each x_t, $y_m \in P_t(X)$ if $x_t \not\leq y_m$

by (r, s)-T_1 space, $\Im\ \lambda \in Q(x_t, r, s) \ni y_m \bar{q}\ \lambda$.

Also, if $y_m \not\leq x_t$,

by (r, s)-T_1 space, $\Im\ \mu \in Q(y_m, r, s) \ni x_t \bar{q}\ \mu$.

Hence X is (r, s)-T_0 space.

Example: 3.3.9

Let X = {x, y} be a set.

We define an IGO (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \frac{1}{3} & \text{if } \lambda = \mu_{pq}, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \frac{1}{3} & \text{if } \lambda = \mu_{pq}, \\ 0 & \text{otherwise,} \end{cases}$$

where for each $0 < p \leq 0.4$, $\mu_{pq}(x) = p$ and $\mu_{pq}(y) = q$, $0 \leq q < p$.

Since (X, τ, τ^*) is a stratified IFTS, by Theorem 3.3.7, (X, τ, τ^*) is (r, s)-quasi-T_0 space for each $r \in I_0$ and $s \in I_1$.

If r > 1/3, s < 2/3, and $t \in I_0$, then for each $x_t, y_t \in P_t(X)$, we have $Q(x_t, r, s) = Q(y_t, r, s) = \{\underline{\alpha} / 1 - t < \alpha \leq 1\}$.

By **Theorem 3.3.2 (2)** and **Corollary 3.3.4 (2)**, (X, τ, τ^*) is neither (r, s)-sub-T_0 nor (r, s)-T_0 for r > 1/3 and s < 2/3.

If $0 < r \leq 1/3$, $2/3 \leq s \leq 1$ and $x \neq y \in X$, there exists $0.7 \in I_0$ such that there exists $\mu_{(2/5)0} \in Q(x_{0.7}, r, s)$ with $y_{0.7}\ \overline{q}\ \mu_{(2/5)0}$.

Hence, (X, τ, τ^*) is (r, s)-sub-T_0 for $0 < r \leq 1/3$ and $2/3 \leq s < 1$.

For $x_{0.3}, y_{0.3} \in P_t(X)$, $0 < r \leq 1/3$ and $2/3 \leq s < 1$,

we have $Q(x_{0.3}, r, s) = Q(y_{0.3}, r, s) = |\ \underline{\alpha}\ |\ 0.7 < \alpha\ |$.

Hence, it is not (r, s)-T_0 for $0 < r \leq 1/3$ and $2/3 \leq s < 1$. For $0 < r \leq 1/3$ and $2/3 \leq s < 1$, (X, τ, τ^*) is both (r, s)-quasi-T_0 and (s, s)-sub-T_0, but not (r, s)-T_0.

Example: 3.3.10

Let X = {x, y} be a set.

We define an IGO (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \dfrac{1}{2} & \text{if } \lambda = \mu_{pq}, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \dfrac{1}{2} & \text{if } \lambda = \mu_{pq}, \\ 1 & \text{otherwise,} \end{cases}$$

where for each $0 < p < 1$, $\mu_{pq}(x) = p$ and $\mu_{pq}(y) = q$, $0 \leq q < p$.

Let z_t, $z_m \in P_t(X)$ with $t \neq m$ for z = x or y.

We have

$$Q\left(z_t, \frac{1}{2}, \frac{1}{2}\right) \neq Q\left(z_m, \frac{1}{2}, \frac{1}{2}\right)$$

For x_t, $y_m \in P_t(X)$, for $p > 1-t$, we have $\mu_{p0} \in Q(x_t, 1/2, 1/2)$ with $y_m \overline{q} \mu_{p0}$. Hence, (X, τ, τ^*) is (1/2, ½)-T_0 space.

On the other hand, let $y_{0.5} \not\leq x_{0.5}$.

For each $\mu_{pq} \in Q(x_t, 1/2, 1/2)$, since q + 0.5 > 1 and p > q,

we have $x_{0.5}\, q\, \mu_{pq}$, that is $Q(y_{0.5}, 1/2, 1/2) \subset Q(x_{0.5}, 1/2, 1/2)$.

Thus (X, τ, τ^*) is not (r, s)-T_1 space.

Theorem: 3.3.11

Every subspace of (r, s)-quasi-T_0 (resp. (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1) space is (r, s)-quasi-T_0 (resp. (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) space.

Proof: Let (X, τ, τ^*) be an IFTS.

Let A be a subspace of X.

Claim: 1

Every subspace of (r, s)-quasi-T_0 space is (r, s)-quasi-T_0 space.

Let (X, τ, τ^*) be (r, s)-quasi-T_0 space.

Let $a_t, a_m \in P_t(A)$ such that $t < m$.

Then $a_t, a_m \in P_t(X)$ such that $t < m$.

Since X is (r, s)-quasi-T_0 space,

$\Im\ \lambda \in Q(a_m, r, s) \ni a_t \bar{q}\ \lambda$.

Since $\tau_A(i^{-1}(\lambda)) \geq \tau(\lambda) \geq r$ and $\tau^*_A(i^{-1}(\lambda)) \leq \tau^*(\lambda) \leq s$.

We have $i^{-1}(\lambda) \in Q_{\tau/A, \tau^*/A}(a_m, r, s) \ni a_t \bar{q}^{i^{-1}}(\lambda)$.

Hence claim 1.

Claim: 2

Every subspace of (r, s)-sub-T_0 space is (r, s)-sub-T_0 space.

Let (X, τ, τ^*) be (r, s)-sub-T_0 space.

Let $a_t, b_t \in P_t(A)$ such that $a_t \neq b_t$.

Then $a_t, b_t \in P_t(X)$ such that $a_t \neq b_t$.

Since X is (r, s)-sub-T_0 space,

$\Im t \in I_0$, for each $a \neq b \in X$ such that

$\Im\ \lambda \in Q(a_t, r, s) \ni b_t \bar{q}\ \lambda$

(or) $\Im\ \mu \in Q(b_t, r, s) \ni a_t \bar{q}\ \mu$.

Since $\tau_A(i^{-1}(\lambda)) \geq \tau(\lambda) \geq r$ and $\tau^*_A(i^{-1}(\lambda)) \leq \tau^*(\lambda) \leq s$

(or) $\tau_A(i^{-1}(\mu)) \geq \tau(\mu) \geq r$ and $\tau^*_A(i^{-1}(\mu)) \leq \tau^*(\mu) \leq s$

We have $i^{-1}(\lambda) \in Q_{\tau/A, \tau^*/A}(a_t, r, s) \ni b_t \overline{q}^{i^{-1}}(\lambda)$.

(or) $i^{-1}(\mu) \in Q_{\tau/A, \tau^*/A}(b_t, r, s) \ni a_t \overline{q}^{i^{-1}}(\mu)$.

Hence claim 2.

Claim: 3

Every subspace of (r, s)-T_0 space is (r, s)-T_0 space.

Let (X, τ, τ^*) be (r, s)-T_0 space.

Let $a_t, b_m \in P_t(A)$

Then $a_t, b_m \in P_t(X)$.

Since (X, τ, τ^*) is (r, s)-T_0 space,

$\exists$ $\lambda \in Q(a_t, r, s) \ni b_m \overline{q}\, \lambda$

(or) $\exists$ $\mu \in Q(b_m, r, s) \ni a_t \overline{q}\, \mu$.

Since $\tau_A(i^{-1}(\lambda)) \geq \tau(\lambda) \geq r$ and $\tau^*_A(i^{-1}(\lambda)) \leq \tau^*(\lambda) \leq s$

(or) $\tau_A(i^{-1}(\mu)) \geq \tau(\mu) \geq r$ and $\tau^*_A(i^{-1}(\mu)) \leq \tau^*(\mu) \leq s$

We have $i^{-1}(\lambda) \in Q_{\tau/A, \tau^*/A}(a_t, r, s) \ni b_m \overline{q}^{i^{-1}}(\lambda)$.

(or) $i^{-1}(\mu) \in Q_{\tau/A, \tau^*/A}(b_m, r, s) \ni a_t \overline{q}^{i^{-1}}(\mu)$.

Hence claim 3.

Claim: 4

Every subspace of (r, s)-T_1 space is (r, s)-T_1 space.

Let (X, τ, τ^*) be (r, s)-T_1 space.

Let $a_t, b_m \in P_t(A)$ such that $a_t \not\leq b_m$.

Then $a_t, b_m \in P_t(X)$ such that $a_t \not\leq b_m$.

Since (X, τ, τ^*) is (r, s)-T_1 space,

$\exists\ \lambda \in Q(a_t, r, s) \ni b_m \bar{q}\ \lambda$.

Since $\tau_A(i^{-1}(\lambda)) \geq \tau(\lambda) \geq r$ and $\tau^*_A(i^{-1}(\lambda)) \leq \tau^*(\lambda) \leq s$,

We have $i^{-1}(\lambda) \in Q_{\tau/A, \tau^*/A}(a_t, r, s) \ni b_m \bar{q}^{i^{-1}}(\lambda)$.

Hence claim 4.

Hence the proof.

Theorem: 3.3.12

Every IF homeomorphic space of (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1) space is (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) space.

Theorem: 3.3.13

Let $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ be a family of (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1) space. Let (τ, τ^*) be the product IGO on $X = \prod_{i \in \Gamma} X_i$.

Then (X, τ, τ^*) is (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1) space.

Proof:

Let (τ,τ^*) be the product IGO on $X = \prod_{i\in\Gamma} X_i$.

Claim: 1

If $\{(X_i, \tau_i, \tau_i^*) / i\in\Gamma\}$ is a family of (r, s)-quasi-T_0 space then (X, τ, τ^*) is (r, s)-quasi-T_0 space.

Let (X_i, τ_i, τ_i^*) be a (r, s)-quasi-T_0 space.

Let $x_t, x_m \in P_t(X)$ such that $t < m$.

Then there exists $i\in\Gamma$ such that $(\pi_i(x))_t < (\pi_i(x))_m$.

Since X_i is (r, s)-quasi-T_0 space, there exists $\lambda \in I^{X_i}$ ∍

$\lambda \in Q_{\tau_i,\tau_i^*}((\pi_i(x))_m, r, s)$, $(\pi_i(x))_t \, \bar{q} \, \lambda$.

Since $\pi_i(x_m) = (\pi_i(x))_m \, q \, \lambda$ iff $x_m \, \bar{q}^{\pi_i^{-1}} (\lambda)$, we have

$\pi_i^{-1}(\lambda) \in Q(x_m, r, s)$, $x_t \, \bar{q}^{\pi_i^{-1}} (\lambda)$.

Hence X is (r, s)-quasi-T_0-space.

Hence claim 1.

Claim: 2

If $\{(X_i, \tau_i, \tau_i^*) / i\in\Gamma\}$ is a family of (r, s)-sub-T_0 space then (X, τ, τ^*) is (r, s)-sub-T_0 space.

Let (X_i, τ_i, τ_i^*) be a (r, s)-sub-T_0 space.

Let $x_t, y_t \in P_t(X)$ such that $x_t \neq y_t$.

Then there exist $i\in\Gamma$ such that $(\pi_i(x))_t \neq (\pi_i(y))_t$.

Since X_i is (r, s)-sub-T_0 space, $\Im$ $\lambda \in I_0$ for each $x \neq y \in X$ and

$\Im$ $\lambda \in I^{X_i}$ (or) $\mu \in I^{X_i}$ $\ni$

$\lambda \in Q_{\tau_i, \tau_i^*}((\pi_i(x))_t, r, s)$, $(\pi_i(y))_t \, \bar{q} \, \lambda$.

(or) $\mu \in Q_{\tau_i, \tau_i^*}((\pi_i(y))_t, r, s)$, $(\pi_i(x))_t \, \bar{q} \, \mu$.

Since $\pi_i(x_t) = (\pi_i(x))_t \, q \, \lambda$ iff $x_t \, q^{\pi_i^{-1}} (\lambda)$,

(or) $\pi_i(y_t) = (\pi_i(y))_t \, q \, \mu$ iff $y_t \, q^{\pi_i^{-1}} (\mu)$, we have

$\pi_i^{-1}(\lambda) \in Q(x_t, r, s)$, $y_t \, \bar{q}^{\pi_i^{-1}} (\lambda)$

(or) $\pi_i^{-1}(\mu) \in Q(y_t, r, s)$, $x_t \, \bar{q}^{\pi_i^{-1}} (\mu)$.

Hence X is (r, s)-sub T_0-space.

Hence claim 2.

Claim: 3

If $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ is a family of (r, s)-T_0 space

then (X, τ, τ^*) is (r, s)-T_0 space.

Let (X_i, τ_i, τ_i^*) be a (r, s)-T_0 space.

Let $x_t, y_m \in P_t(X)$. Then there exists $i \in \Gamma$.

Since X_i is (r, s)-T_0 space,

$\Im$ $\lambda \in I^{X_i}$ (or) $\mu \in I^{X_i}$ such that

$\lambda \in Q_{\tau_i, \tau_i^*}((\pi_i(x))_t, r, s)$, $(\pi_i(y))_m \, \bar{q} \, \lambda$.

(or) $\mu \in Q_{\tau_i, \tau_i^*}(\pi_i(y))_m, r, s)$, $(\pi_i(x))_t \, \bar{q} \, \mu$.

Since $\pi_i(x_t) = (\pi_i(x))_t\ q\ \lambda$ iff $x_t\ \overline{q}^{\pi_i^{-1}}(\lambda)$,

(or) $\pi_i(y_m) = (\pi_i(y))_m\ q\ \mu$ iff $y_m\ \overline{q}^{\pi_i^{-1}}(\mu)$, we have

$\pi_i^{-1}(\lambda) \in Q(x_t, r, s)$, $y_m\ \overline{q}^{\pi_i^{-1}}(\lambda)$,

(or) $\pi_i^{-1}(\mu) \in Q(y_m, r, s)$, $x_t\ \overline{q}^{\pi_i^{-1}}(\mu)$.

Hence X is (r, s)-T_0-space.

Hence claim 3.

Claim: 4

If $\{(X_i, \tau_i, \tau_i^*) / i\in\Gamma\}$ is a family of (r, s)-T_1 space then (X, τ, τ^*) is (r, s)-T_1 space.

Let (X_i, τ_i, τ_i^*) be a (r, s)-T_1 space.

Let $x_t, y_m \in P_t(X)$ such that $x_t \not\leq y_m$.

Then there exists $i\in\Gamma$ such that $((\pi_i(x))_t \not\leq ((\pi_i(y))_m$.

Since (X_i, τ_i, τ_i^*) is (r, s)-T_1 space, $\exists\ \lambda\in I^{X_i}$ such that

$\lambda\in Q_{\tau_i, \tau_i^*}((\pi_i(x))_t, r, s)$, $(\pi_i(y))_m\ \overline{q}\ \lambda$.

Since $\pi_i(x_t) = (\pi_i(x))_t\ q\ \lambda$ iff $x_t\ \overline{q}^{\pi_i^{-1}}(\lambda)$, we have

$\pi_i^{-1}(\lambda) \in Q(x_t, r, s)$, $y_m\ \overline{q}^{\pi_i^{-1}}(\lambda)$.

Hence (X, τ, τ^*) is (r, s)-T_1 space.

Hence the proof.

Theorem: 3.3.14

Let $\{(X_i, \tau_i, \tau_i^*) / i\in\Gamma\}$ be a family of IFTSs. Let (τ,τ^*) be the product IGO on $X = \prod_{i\in\Gamma} X_i$. If (X, τ, τ^*) is (r, s)-sub-T_0 space, then (X_i, τ_i, τ_i^*) is $(r-\varepsilon, s+\varepsilon)$-sub-$T_0$ space for each $\varepsilon > 0$ and for each $i\in\Gamma$.

Proof:

Let $x^j, y^j\in X_j$ such that $x^j \neq y^j$.

Then there exists $x^i\in X_i$ for all $i\in\Gamma-\{j\}$ such that $x \neq y\in X$ and

$$\Pi_i(x) = \begin{cases} x^i & \text{if } i \in \Gamma - \{j\}, \\ x_j & \text{if } i = j \end{cases},$$

$$\Pi_i(y) = \begin{cases} x^i & \text{if } i \in \Gamma - \{j\}, \\ y_j & \text{if } i = j \end{cases},$$

Since (X, τ, τ^*) is (r, s)-sub-T_0 space,

there exists $t\in(0, 1)$ such that $\rho\in Q(x_t, r, s)$, $y_t \bar{q} \rho$.

Let (β, β^*) be a base for (τ, τ^*).

Since $\tau(\rho) \geq r$ and $\tau^*(\rho) \leq s$, by theorem 3.1.9, for $\varepsilon > 0$, there exists a family $\{\rho_k / \rho = \vee_{k\in\Delta} \rho_k\}$ such that

$$\tau(\rho) \geq \wedge_{k\in\Delta} \beta(\rho_k) > r - \varepsilon, \ \tau^*(\rho) \leq \vee_{k\in\Delta} \beta^*(\rho_k) < s + \varepsilon.$$

Since x_t q $[\rho = \vee_{k\in\Delta} \rho_k]$, there exists $k\in\Gamma$ such that x_t q ρ_k, $\beta(\rho_k) > r - \varepsilon$ and $\beta^*(\rho_k) < s + \varepsilon$.

Then, there exists a family $\{\lambda_i \,/\, \rho_k = \underset{i\in F}{\wedge} \pi_i^{-1}(\lambda_i)\}$ in which F is a finite subset of Γ such that

$$\beta(\rho_k) \geq \underset{i\in F}{\wedge} \tau_i(\lambda_i) > r-\varepsilon,\ \ \beta^*(\rho_k) \leq \underset{i\in F}{\vee} \tau_i^*(\lambda_i) < s+\varepsilon \qquad \text{(E)}$$

By without loss of generality, we assume $j\in F$.

Take $F_1 = F\cup\{j\}$ such that $\lambda_j = \tilde{1}$; $\tau_j(\tilde{1}) = 1$ and $\tau_j^*(\tilde{1}) = 0$.

Since $x_t\, q\, \rho_k$ and $y_t\, \bar{q}\, \rho_k$,

$$t > [\underset{i\in F-\{j\}}{\vee} (\tilde{1}-\lambda_i)(\pi_i(x))] \vee (\tilde{1}-\lambda_j)(x^j) \qquad \text{(F)}$$

$$t \leq [\underset{i\in F-\{j\}}{\vee} (\tilde{1}-\lambda_i)(\pi_i(x))] \vee (\tilde{1}-\lambda_j)(y^j) \qquad \text{(G)}$$

If $(\underset{i\in F-\{j\}}{\vee} (\tilde{1}-\lambda_i)(\pi_i(x))) \geq t$, it is a contradiction to (F) and (G).

Thus $\underset{i\in F-\{j\}}{\vee} (\tilde{1}-\lambda_i)(\Pi_i(x)) < t$.

$\Rightarrow \quad t > (\tilde{1}-\lambda_j)(x^j),\ \ t \leq (\tilde{1}-\lambda_j)(y^j)$.

By (E), we have

$\tau_j(\lambda_j) > r-\varepsilon$ and $\tau_j^*(\lambda_j) < s+\varepsilon$.

Hence $\lambda_j \in Q_{\tau_j,\tau_j^*}((x^j)_t, r-\varepsilon, s+\varepsilon)$, $(y^j)_t\, \bar{q}\, \lambda_j$.

Therefore, (X_j, τ_j, τ_j^*) is $(r-\varepsilon, s+\varepsilon)$-sub-$T_0$ space.

Example: 3.3.15

Let X = {x} and Y = {y} be sets.

Define IGO (τ_1, τ_1^*) on X as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 1 & \text{otherwise,} \end{cases}$$

and IGO (τ_2, τ_2^*) on Y as follows

$$\tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0} \text{ for or } \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = y_{0.2}, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0} \text{ for or } \underline{1}, \\ \frac{1}{2} & \text{if } \lambda = y_{0.2}, \\ 1 & \text{otherwise,} \end{cases}$$

Let $X \times Y = \{(x, y)\}$ be a product set and $(\tau_1 \otimes \tau_2, \tau_1^* \otimes \tau_2^*)$ the product IGO $X \times Y$.

Since $(x, y)_{0.2} = \pi_1^{-1}(\underline{0.2}) = \pi_2^{-1}(y_{0.2})$, by Theorem 3.1.9, we have

$(\tau_1 \otimes \tau_2)(\underline{0.2}) = \tau_1(\underline{0.2}) \vee \tau_2(y_{0.2}) = 1$, $(\tau_1^* \otimes \tau_2^*)(\underline{0.2}) = \tau_2(\underline{0.2}) \wedge \tau_2^*(y_{0.2}) = 0$.

We can obtain the product IGO, $(\tau_1 \otimes \tau_2, \tau_1^* \otimes \tau_2^*)$ as follows:

$$\tau_1 \otimes \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_1^* \otimes \tau_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 1 & \text{otherwise,} \end{cases}$$

Then, $(X \times Y, \tau_1 \otimes \tau_2, \tau_1^* \otimes \tau_2^*)$ are (r, s)-T_0 and (r, s)-quasi-T_0 for all $r \in I_0$, $s \in I_1$.

But (Y, τ_2, τ_2^*) is not (r, s)-quasi-T_0 for all $r \in I_0$, $s \in I_1$.

Hence, it is neither (r, s)-T_0 nor (r, s)-T_1 for all $r \in I_0$, $s \in I_1$.

Theorem: 3.3.16

Let $\{(X_i, \tau_i, \tau_i^*) / i \in \Gamma\}$ be a family of IFTSs and (τ, τ^*) be the product IGO on $X = \prod_{i \in \Gamma} X_i$. If (X, τ, τ^*) is (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) and (X_j, τ_j, τ_j^*) is stratified for $j \in \Gamma$, then (X_j, τ_j, τ_j^*) is (r, s)-quasi-T_0 (resp., (r, s)-sub T_0, (r, s)-T_0 and (r, s)-T_1) space.

Proof:

Let (X, τ, τ^*) be (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) space.

Let (X_j, τ_j, τ_j^*) be a stratified for $j \in \Gamma$.

To prove:

(X_j, τ_j, τ_j^*) is (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) space.

Let (X, τ, τ^*) and $\tilde{X}_j = X_j \times \{\prod y^i / i \neq j\}$ of X parallel to X_j.

Since $\left(\tilde{X}_j, \tau/\tilde{X}_j, \tau^*/\tilde{X}_j\right)$ is a subspace of (X, τ, τ^*), by Theorem 3.3.11 $\left(\tilde{X}_j, \tau/\tilde{X}_j, \tau^*/\tilde{X}_j\right)$ is (r, s)-quasi-T_0 (resp, (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) space.

Since (X_j, τ_j, τ_j^*) is stratified, by theorem 3.2.7,

$$\pi_j / \tilde{X}_j : \left(\tilde{X}_j, \tau/\tilde{X}_j, \tau^*/\tilde{X}_j\right) \rightarrow (X_j, \tau_j, \tau_j^*) \text{ is IF homeomorphism.}$$

From theorem 3.3.12 (X_j, τ_j, τ_j^*) is (r, s)-quasi-T_0 (resp, (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1) space.

Hence the proof.