CHAPTER – III

CHAPTER III

FUZZY SEMI IRRESOLUTE AND FUZZY STRONGLY IRRESOLUTE FUNCTIONS

Definition: 3.1

A fuzzy set α is said to be **quasi-coincident** with a fuzzy set β denoted by $\boldsymbol{\alpha}$ **q** $\boldsymbol{\beta}$ if and only if there exists a $x \in X$ such that $\alpha(x) > \beta'(x)$ or $\alpha(x) + \beta'(x) > 1$. If α does **not quasi-coincident** with β then we write $\boldsymbol{\alpha}$ $\boldsymbol{\not q}$ $\boldsymbol{\beta}$.

Definition: 3.2

A fuzzy set β is a **quasi-neighbourhood (q-nbd,** for short) of fuzzy set α iff there exists an open set λ such that $\alpha \, q \, \lambda \leq \beta$.

Definition: 3.3

A fuzzy set λ in X is said to be **semi-q-nbd** of x_α iff there exists a fuzzy semi-open set μ in X such that $x_\alpha q \, \mu \leq \lambda$.

Remark: 3.4

A **fuzzy semi-closure** scl (λ) of a fuzzy set λ in X is the union of all fuzzy singletons x_α such that every fuzzy semi-open semi-q-nbd of x_α is q-coincident with λ, equivalently scl λ is the intersection of all fuzzy semi-closed sets containing λ.

Definition: 3.5

The union of all fuzzy semi-open sets contained in a fuzzy set λ in X is called the **fuzzy semi-interior** of λ , to be denoted by sint λ.

Definition: 3.6

A fuzzy singleton x_α is said to be a **fuzzy semi-θ- cluster point** of a fuzzy set λ in X iff the fuzzy semi-closure of every fuzzy semi-open semi-q-nbd of x_α is q-coincident with λ.

Definition: 3.7

The union of all fuzzy semi-θ - cluster points of λ is called the **fuzzy semi-θ - closure** of λ and is denoted by $[\,\lambda\,]_{S-\theta}$ or $cl_{S-\theta}(\lambda)$.

Definition: 3.8

A fuzzy set λ is called **fuzzy semi-θ-closed** iff $\lambda = [\,\lambda\,]_{S-\theta}$ and complement of a fuzzy semi-θ -closed set is **fuzzy semi-θ - open.**

Note: 3.9

Every fuzzy semi-θ - closed set is fuzzy semi-closed.

Throughout the chapter by (X,τ) and (Y,τ_1) (or simply X and Y) we shall mean fuzzy topological spaces (fts, for short) in Chang's sense.

Definition: 3.10

A function f : X $\longrightarrow$ Y is said to be **fuzzy semi-irresolute (fuzzy strongly irresolute)** iff for any fuzzy singleton x_α in X and any fuzzy semi-open set μ in Y containing f(x_α) , there exists a fuzzy semi-open set λ in X containing x_α such that f(λ) $\leq$scl μ (respectively , f(scl λ) $\leq \mu$).

We first show that in the above definitions, containment by fuzzy semi-open set can be replaced by semi-q-nbds.

Theorem: 3.11

A function f : X $\longrightarrow$ Y is fuzzy semi-irresolute (irresolute, strongly irresolute) iff for each fuzzy singleton x_α in X and each semi-open semi-q- nbd μ of f(x_α) , there exists a fuzzy semi-open semi-q-nbd λ of x_α in X such that f(λ) $\leq$scl μ (respectively , f(λ) $\leq \mu$, f(scl λ) $\leq \mu$).

Proof:

We prove the case for fuzzy semi-irresolute function, the other cases are similar.

Let f be a fuzzy semi-irresolute and μ be any fuzzy semi-open semi-q- nbd of f(x_α) in Y.

Then μ(f(x)) $+\alpha$ >1. We choose a positive real number β such that

μ(f(x)) $> \beta > 1 - \alpha$.

Then μ is fuzzy semi-open set containing $f(x_\beta)$. By hypothesis, there exists a fuzzy semi-open set γ containing x_β such that $f(\gamma) \leq$ scl μ .

Now $\gamma(x) \geq \beta$ implies $\gamma(x) > 1-\alpha$ and thus γ is a fuzzy semi-open semi-q-nbd of x_α .

Conversely, let x_α be any fuzzy singleton in X and μ be any fuzzy semi-open set in Y containing $f(x_\alpha)$. Then $x_\alpha \in f^{-1}(\mu) = \gamma$ (say).

Let m be a positive integer such that $(1/m) \leq \gamma(x)$. For any positive integer n $\geq$ m, we put $\alpha_n = 1 + (1/n) - \gamma(x)$. Then $0 \leq \alpha_n \leq 1$ for all $n \geq m$.

Now, μ is a fuzzy semi-open semi-q-nbd of $Y\alpha_n$ for all $n \geq m$, where

y = f(x).

In fact, $\mu(y) + \alpha_n = \mu(y) + 1 + (1/n) - f^{-1}(\mu)(x)$

$= 1 + (1/n) > 1$.

Then by hypothesis, there exists a fuzzy semi-open set λ_n in X such that $X\alpha_n q \lambda_n$ and $f(\lambda_n) \leq$ scl μ , for all $n \geq m$.

We put $\lambda = \bigcup_{n \geq m} (\lambda_n)$. Then λ is a fuzzy semi-open set in X such that

$f(\lambda) = \bigcup_{n \geq m} f(\lambda_n) \leq$ scl μ.

Now, it suffices to show that $x_\alpha \in \lambda$. Indeed we have,

$\lambda_n(x) + \alpha_n > 1$, for all $n \geq m$.

So that $\lambda_n(x) > 1-\alpha_n = \gamma(x) - (1/n)$, for all $n \geq m$.

Hence $\lambda(x) > \gamma(x) - (1/n)$, for all $n \geq m$.

which implies $\lambda(x) \geq \gamma(x) \geq \alpha$ (since $x_\alpha \in \gamma$) and thus $x_\alpha \in \lambda$.

Lemma: 3.12

For a fuzzy semi-open set λ in an fts X, scl $\lambda = [\lambda]_{S-\theta}$.

Proof:

It is easy to see that scl $\lambda \leq [\lambda]_{S-\theta}$. So it suffices to show that

$[\lambda]_{S-\theta} \leq$ scl λ .

In fact, x_α is any fuzzy singleton in X such that $x_\alpha \notin$ scl λ . Hence there exists a fuzzy semi-open semi-q-nbd μ of x_α such that $\mu \not{q} \lambda$.

This means $\mu \leq \lambda'$ so that scl $\mu \leq scl\, \lambda' = \lambda'$.

Then scl $\mu \not{q} \lambda$ and SO $x_\alpha \notin [\lambda]_{S-\theta}$.

Corollary: 3.13

If λ is a fuzzy semi-clopen in X, then λ is fuzzy semi-θ- closed as well as fuzzy semi-θ- open.

Theorem: 3.14

For a function f : X⟶Y the following are equivalent :

a) f is fuzzy semi-irresolute,

b) scl $(f^{-1}(\beta)) \leq f^{-1}([\beta]_{S-\theta})$, for any fuzzy set β in Y,

c) f(scl λ) $\leq [f(\lambda)]_{S-\theta}$, for each fuzzy set λ in X,

d) $f^{-1}(\gamma)$ is fuzzy semi-closed, for every semi-θ-closed set γ in Y,

e) $f^{-1}(\mu)$ is fuzzy semi-open, for every semi-θ-open set μ in Y.

Proof:

(a)⟹(b):

Let β be any fuzzy set in Y, and $x_\alpha \notin f^{-1}([\beta]_{S-\theta})$.

Then f $(x_\alpha) \notin ([\beta]_{S-\theta})$. Thus there exists a fuzzy semi-open semi-q-nbd μ

of f (x_α) such that scl $\mu \, \not q \, \beta$.

By (a), there exists a fuzzy semi-open semi-q-nbd λ of x_α such that

$f(\lambda) \leq$ scl μ .

Hence $f(\lambda) \not q \, \beta$ so that $f(\lambda) \leq \beta'$ which implies

$\lambda \leq f^{-1}(\beta') = (f^{-1}(\beta))'$. Therefore $\lambda \not q \, f^{-1}(\beta)$ and thus $x_\alpha \notin$ scl $f^{-1}(\beta)$.

Hence scl $f^{-1}(\beta) \leq f^{-1}([\beta]_{S-\theta})$.

(b)⇒(c):

For any fuzzy set λ *in X*, we have scl $\lambda \leq scl\, f^{-1}$(f ($\lambda$)).

By (b), scl (f^{-1}(f (λ))) $\leq f^{-1}$([f (λ)]$_{S-\theta}$), and hence

f(scl λ) $\leq f f^{-1}$([f(λ)]$_{S-\theta}$) $\leq$ [f(λ)]$_{S-\theta}$.

(c)⇒(b):

For any fuzzy set β in Y, we have [f($f^{-1}(\beta)$)]$_{S-\theta}$ $\leq [\beta]_{S-\theta}$.

By (c), we obtain f(scl$f^{-1}(\beta)$) $\leq [f f^{-1}(\beta)]_{S-\theta}$.

And hence scl $f^{-1}(\beta) \leq f^{-1}([\beta]]_{S-\theta})$.

(b)⇒(d):

Let γ be any fuzzy semi-θ closed set in Y.

By (b), we have scl ($f^{-1}(\gamma)$) $\leq f^{-1}([\gamma]_{S-\theta}) = f^{-1}(\gamma)$

and hence $f^{-1}(\gamma)$ is fuzzy semi-closed in X.

(d)⇒(e):

Let μ be any fuzzy semi-θ -open set in Y.

Then μ' is fuzzy semi-θ- closed in Y.

By (d), $f^{-1}(\mu') = (f^{-1}(\mu))'$ is fuzzy semi-closed in X.

Thus $f^{-1}(\mu)$ is fuzzy semi-open in X.

(e)⇒(a):

Let x_α be a fuzzy singleton in X and μ a fuzzy semi-open semi-q-nbd of

f (x_α).

Then scl μ is fuzzy semi-clopen and hence by corollary 3.13, it is fuzzy

semi-θ-open in Y.

Let $\lambda = f^{-1}(\text{scl } \mu)$

By (e), λ is a fuzzy semi-open semi-q-nbd of x_α and

$f(\lambda) = f(f^{-1}(\text{scl } \mu)) \leq \text{scl } \mu$.

Hence f is fuzzy semi-irresolute.

Lemma: 3.15

If λ is any fuzzy set and β a fuzzy semi-open set in a fts X such that $\lambda \not{q} \beta$, then

scl $\lambda \not{q} \beta$.

Proof:

If there exists an x∈X such that scl $\lambda(x) + \beta(x) > 1$, then putting scl $\lambda(x) = \alpha$, we see that β is a fuzzy semi-open semi-q-nbd of x_α with $\alpha \not{q} \beta$, whereas $x_\alpha \epsilon$ scl λ.

Thus we have arrive at a contradiction.

Theorem: 3.16

If f: X→Y is a function, then the following are equivalent:

a) f is fuzzy semi-irresolute.

b) $f^{-1}(\mu) \leq$sint(f^{-1}(scl μ)), for every fuzzy semi-open set μ in Y.

c) $scl\, f^{-1}(\mu) \leq f^{-1}(scl\, \mu)$ for every fuzzy semi-open set μ in Y.

Proof:

We supply only a sketch of the proof.

(a)⇒(b):

For any $x_\alpha \in f^{-1}(\mu)$ there exists a fuzzy semi-open set λ containing x_α such that f $(\lambda) \leq scl\, \mu$ so that $x_\alpha \in \lambda \leq$ sint(f^{-1}(scl μ)).

(b)⇒(c):

If $x_\alpha \in f^{-1}$(scl μ) , then there exists a fuzzy semi-open semi-q-nbd G of f(x_α) such that G $\not{q}$ μ and hence scl G $\not{q}$ μ (by lemma 3.15) then x_α q f^{-1}(G) ≤ sint(f^{-1}(scl G)) $\not{q}$ $f^{-1}(\mu)$ (Using (b)). And hence $x_\alpha \notin$scl($f^{-1}(\mu)$).

(c)⇒(a):

For any fuzzy singleton x_α in X and any fuzzy semi-open set μ containing f(x_α), we have scl μ is fuzzy-clopen so that x_αqf^{-1}(scl(scl $\mu)'$)).

Using (c), we have $x_\alpha \in [\text{scl} f^{-1}(scl\ \mu)']' = \lambda$ (say) such that

$f(\lambda) \leq f[(f^{-1}(\text{scl}\ \mu)')'] = ff^{-1}(\text{scl}\ \mu) \leq \text{scl}\ \mu$.

Theorem: 3.17

For any function f: X→Y, the following statements are equivalent:

a) f is fuzzy semi-irresolute.

b) For each fuzzy singleton x_α in X and each fuzzy semi-open set λ containing x_α such that f (scl λ) $\leq$ scl μ.

c) $f^{-1}(\gamma)$ is fuzzy semi-clopen for every fuzzy semi-clopen set γ in Y.

Theorem: 3.18

For a function f: X→Y, the following are equivalent:

a) f is fuzzy strongly irresolute,

b) $[f^{-1}(\beta)]_{S-\theta} \leq f^{-1}(\text{scl}\ \beta)$, for every fuzzy set β in Y,

c) $f([\lambda]_{S-\theta}) \leq \text{scl}[f(\lambda)]$, for every fuzzy set λ in X,

d) $f^{-1}(\beta)$ is fuzzy semi-θ-closed for every fuzzy semi-closed set β in Y,

e) $f^{-1}(\beta)$ is fuzzy semi-θ-open for every fuzzy semi-open set β in Y,

Proof:

(a)⇒(b):

Let β be a fuzzy set in Y. Now $x_\alpha \notin f^{-1}(\text{scl}\ \beta)$ implies $f(x_\alpha) \notin \text{scl}\ \beta$.

So that there exists a fuzzy semi-open semi-q-nbd G of f(x_α)such that

G $\not q$ β.

By (a), there exists a fuzzy semi-open semi-q-nbd μ of x_α such that

f (scl μ) ≤ G. Then f(scl μ) $\not q$ β so that scl μ $\not q$ $f^{-1}(\beta)$.

Hence $x_\alpha \notin [f^{-1}(\beta)]_{s-\theta}$ and we conclude that $[f^{-1}(\beta)]_{s-\theta} \leq f^{-1}$(scl β).

(b)⇒(c):

For any fuzzy set λ in X, by (b),

We have $[\lambda]_{S-\theta} \leq [f^{-1}$(f($\lambda$))$]_{S-\theta} \leq f^{-1}$(scl(f($\lambda$))) and thus

f($[\lambda])_{S-\theta}$) ≤ ff^{-1}(scl f(λ)) ≤ scl f(λ).

(c)⇒(d):

For any semi-closed set β in Y.

we have by (b), f($[f^{-1}(\beta)]_{S-\theta}$) ≤ scl f($f^{-1}(\beta)$) ≤ scl β=β and hence

$[f^{-1}(\beta)]_{S-\theta} \leq f^{-1}(\beta)$, proving that $f^{-1}(\beta)$ is fuzzy semi-θ-closed in Y.

(d)⇒(e):

Obvious.

(e)⇒(a):

Let x_α be a fuzzy singleton in X and μ be a fuzzy semi-open semi-q-nbd of f (x_α).

Then $f^{-1}(\mu) = \delta$ is a fuzzy semi-θ-open set in X. Let y = f(x). Now, $y_\alpha q\mu$ implies $y_\alpha \notin \mu'$ and so $x_\alpha \notin f^{-1}(\mu')$=$(f^{-1}(\mu))'$.

As $(f^{-1}(\mu))'$ is fuzzy semi-θ-closed, there exists a fuzzy semi-open set δ in X such that $x_\alpha\, \not{q}\, \delta$ and scl δ q $(f^{-1}(\mu))'$.

Then scl $\delta \leq f^{-1}(\mu)$ so that f (scl δ)$\leq \mu$.

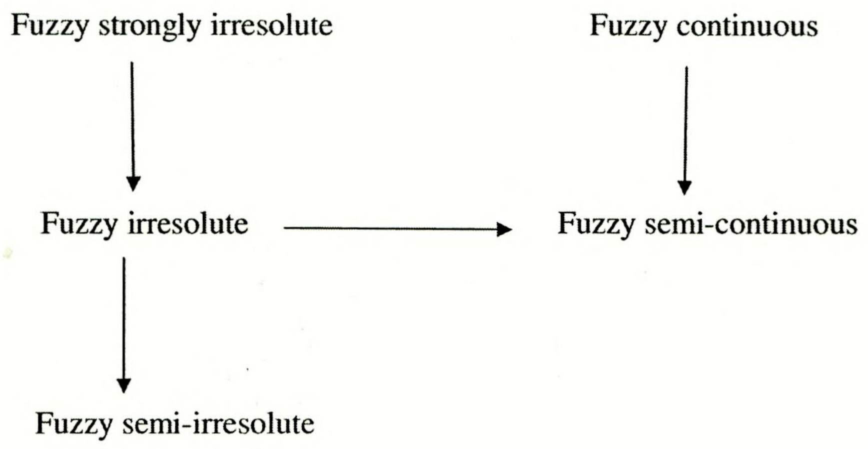
Hence f is fuzzy strongly irresolute.

Interrelations:

This section ascertains the mutual correlations among fuzzy irresolute, fuzzy strongly irresolute, fuzzy semi-irresolute, fuzzy continuous and fuzzy semi-continuous functions.

Note: 3.19

It is known that every fuzzy irresolute function is fuzzy semi-continuous and every fuzzy continuous function is fuzzy semi-continuous. Thus we have the following diagram.



It is shown by means of counter examples that no implications other than those in the above diagram exists, in general.

Example: 3.20

Fuzzy semi-irresolute but not fuzzy irresolute.

Let X be a non-empty set, Fix an element $a \in X$.

Let $\tau=\{0_X,1_X,\lambda\}$ and $\tau_1=\{0_X,1_X,\beta\}$, where $\lambda(a)=2/3$, $\beta(a)=3/5$ and

$\lambda(x)=\beta(x)=0$, for all $x \in X-\{a\}$.

Let $f:(X,\tau) \longrightarrow (X,\tau_1)$ be the identity mapping. Now β is fuzzy open

in (X, τ_1) but $f^{-1}(\beta) = \beta$ is not fuzzy semi-open in (X,τ).

Consequently, f is not fuzzy semi-continuous and hence not fuzzy

irresolute. But it is clear that for any fuzzy semi-open set μ in (X,τ_1),

scl $\mu = 1_x$.

Hence f is obviously fuzzy semi-irresolute.

Example: 3.21

Fuzzy strongly irresolute but not fuzzy continuous.

Let X be non empty set and a∈X. Consider the topologies

$\tau = \{1_X, 0_X, \lambda\}$ and $\tau_1 = \{1_X, 0_X, \lambda, \beta\}$ on X, where $\lambda(a) = 1/3$, $\beta(a) = ½$ and

$\lambda(x) = \beta(x) = 0$, for all $x \in X - \{a\}$.

Let $f:(X,\tau) \longrightarrow (X,\tau_1)$ denote the identify map. Now, the fuzzy

semi-open sets of (X,τ) are $0_x, 1_x$ and any fuzzy set γ such that

$1/3 \leq \gamma(a) \leq 2/3$ and $0 \leq \gamma(x) \leq 1$ for all $x \in X - \{a\}$.

Also, clearly these are precisely the fuzzy semi-closed sets in (X,τ). Again a fuzzy semi-open set in (X, τ_1) is either 1_x or 0_x or a fuzzy set δ such that

$1/3 \leq \delta(a) \leq ½$ and $0 \leq \delta(x) \leq 1$ for all $x \in X - \{a\}$.

Then f is clearly fuzzy strongly irresolute, but it is not fuzzy continuous.

Example: 3.22

Fuzzy continuous but not fuzzy semi-irresolute.

We consider the identity map $f:(X,\tau) \longrightarrow (X,\tau_1)$, where X, τ and τ_1 are same as in example 3.21 above.

Clearly f is continuous (since $T \subset T_1$).

Now consider the fuzzy singleton $a_{7/12}$ and the fuzzy semi-open set λ

in (X,τ) given by $\lambda(a) = 7/12$ and $\lambda(x) = 1$, for all $x \in X - \{a\}$.

Then the only fuzzy semi-open set in (X,τ) containing $a_{7/12}$ is 1_X and

$F(1_X) \nleq \text{scl}\ \lambda = \lambda$. Thus f is not fuzzy semi-irresolute.

Example: 3.23

Fuzzy irresolute but not fuzzy strongly irresolute.

On a non-empty set X we consider the fuzzy topology $\tau = \{1_X, 0_X, \lambda\}$, where $\lambda(x) = 2/3$ for all $x \in X$. Then the identity map $f:(X,\tau) \longrightarrow (X,\tau)$ is obviously fuzzy continuous and irresolute.

For any point a of X, λ is a fuzzy semi-open semi-q-nbd of $a_{1/2}$. But for any

fuzzy semi-open set β in X, $\text{scl}\ \beta = 1_X$ and hence $f(\text{scl}\ \beta) > \lambda$.

Thus f is not fuzzy strongly irresolute.

Example: 3.24

Fuzzy semi - continuous but not fuzzy irresolute.

This case is already discussed in example 2.9.

We now investigate the suitable conditions under which the implications concerning fuzzy irresolute, semi-irresolute and strongly irresolute functions can be reversed.

Definition: 3.25

A fts X is said to be **fuzzy s- regular** iff for each fuzzy singleton x_α in X and each fuzzy semi-open semi-q-nbd $\lambda\ of\ x_\alpha$, there exists a fuzzy semi-open semi-q-nbd μ of x_α such that scl $\mu \leq \lambda$.

Theorem: 3.26

Let f: X→Y be a function

a. If f is fuzzy semi-irresolute and Y is fuzzy s-regular, Then is fuzzy irresolute.
b. If f is fuzzy irresolute and X is fuzzy s-regular, Then f is fuzzy strongly irresolute.

Proof:

a. Let f: X→Y be fuzzy semi irresolute and Y fuzzy s-regular.

Let x_α be any fuzzy singleton in X and μ any fuzzy semi-open semi-q-nbd of f(x_α). Since Y is fuzzy s-regular, there exists a fuzzy semi-open semi-q-nbd λ of f(x_α) such that scl $\lambda \leq \mu$.

By fuzzy semi irresolute property of f, there is a fuzzy semi-open-q-nbd γ of x_α such that f(γ) $\leq scl\ \lambda \leq \mu$.

Then by theorem 3.11, f is fuzzy irresolute.

Proof of (b) is similar to (a).