Chapter - IV

CHAPTER IV

INVERTIBLE MATRICES OVER SEMIRINGS

In this chapter S is always supposed to be a commutative antiring.

Theorem: 4.1

Let $A \in M_n(S)$. If A is invertible in $M_n(S)$, then AA^T and A^TA are invertible diagonal matrices.

Proof:

Suppose that A is invertible in $M_n(S)$. Then there exists a matrix B in $M_n(S)$ such that $AB = I_n$. By Result 1.38 (iii), we have

$(\Sigma_{k \in \underline{n}}\, a_{ik})\, (\Sigma_{l \in \underline{n}}\, b_{li}) = 1$ for all $i \in \underline{n}$, and so $\Sigma_{k \in \underline{n}}\, a_{ik} \in U(S)$ for all $i \in \underline{n}$.

Let $u_i = \Sigma_{k \in \underline{n}}\, a_{ik}$ for each $i \in \underline{n}$. Then $u_i^2 = (\Sigma_{k \in \underline{n}}\, a_{ik})^2 = \Sigma_{k \in \underline{n}}\, a^2_{ik} + \Sigma_{1 \le k \le l \le n}\, 2a_{ik}\, a_{il} = \Sigma_{k \in \underline{n}}\, a^2_{ik}$ (By Result 1.38(i)) and $u_i^2 \in U(S)$. (By Result 1.38 (i)), we have that for any i and j in $\underline{n}$,

$$(AA^T)_{ij} = \Sigma_{k \in \underline{n}}\, a_{ik}\, a_{jk} = \begin{cases} u_i^2 & i = j \\ 0 & i \neq j \end{cases}$$

By Result 1.37. $AA^T = \text{diag}\,(u_1^2, u_2^2, \ldots\ldots\ldots\, u_n^2)$ is an invertable matrix in $M_n(S)$.

Remark: 4.2

The diagonal matrices AA^T and A^TA in Theorem 4.1 need not to be equal in general. For example, consider the matrix $A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ over the antiring R^+. Then, it is clear that A is invertible in $M_2(R^+)$ and $AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$.

But $A^TA = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$.

Corollary: 4.3

If S satisfies U(S) = {1} and $A \in GL_n(S)$, then $AA^T = A^TA = I_n$.

Theorem: 4.4

Let $A \in M_n(S)$. If A is right invertible in $M_n(S)$, then $A^{[n]}$ is an invertible diagonal matrix in $M_n(S)$, where [n] denotes the least common multiple of the integers 1, 2,...n.

Proof:

Let $A \in M_n(S)$ be right invertible. Then by Result 1.36, A is invertible in $M_n(S)$, and so $A^{[n]}$ is invertible in $M_n(S)$. In the following we will prove that $A^{[n]}$ is a diagonal matrix.

Let w = [n]. If n = 2, then w = [2] and

$$A^w = A^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^2$$

$$= \begin{bmatrix} a_{11}{}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{21}a_{12} + a^2{}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & 0 \\ 0 & a_{21}a_{12} + a_{22}^2 \end{bmatrix} \quad \text{(By Result 1.38)}$$

Therefore A^2 is a diagonal matrix.

We now assume $n \geq 3$.

For any i, j$\in\underline{n}$. If $a_{ij}^{(w)} \neq 0$, then there exist $i_1, i_2, \ldots\ldots i_{w-1} \in \underline{n}$ such that $a_{ii}a_{ii_2}\ldots\ldots a_{i_{w-1}j} \neq 0$, where $w = [n]$. In the following we shall prove i = j.

We prove it in four steps.

(I) If $i_s = i_t$ for some s, t with $0 \leq s < t < w$ (taking $i = i_0$, $j = i_w$), then $i_{s+1} = i_{t+1}$. In fact, if $i_{s+1} \neq i_{t+1}$, then $a_{i_s i_{s+1}} a_{i_t} a_{i_{t+1}} = a_{i_s i_{s+1}} a_{i_s} a_{i_{t+1}} = 0$ (by Result 1.38),and so $a_{ii} a_{i,i_2}\ldots\ldots a_{i_{w-1}j} = a_{i_0 i_1}\ldots\ldots a_{i_s i_{s+1}}\ldots\ldots\ldots\ldots a_{i_t i_{t+1}}\ldots\ldots a_{i_{w-1} i_w} = 0$. This is a contradiction.

(II) If $i_s = i_t$ for some s, t with $0 \leq s \leq t \leq w$, then $i_{s-1} = i_{t-1}$. In fact, if $i_{s-1} \neq i_{t-1}$, then $a_{i_{s-1} i_s} a_{i_{t-1}} a_{i_t} = a_{i_{s-1} i_s} a_{i_{t-1} i_s} = 0$ (by Result 1.38), and so $a_{ii} a_{ii_2}\ldots\ldots a_{i_{w-1}j} = a_{i_0 i_1}\ldots\ldots a_{i_{s-1} i_s}\ldots\ldots\ldots\ldots a_{i_{t-1} i_t}\ldots\ldots a_{i_{w-1} i_w} = 0$. This is a contradiction.

(III) There exists a$d\in\underline{n}$ such that $i_0 = i_d$. In fact, since $i_0, i_1, \ldots\ldots i_n \in \underline{n}$ (note that $w>n$), there exist some u, v in {0, 1,……n} such that $i_u = i_v$ with u<v. If u = 0, then $i_0 = i_v$ and in this case, our statement is proved. If 0<u, then by (II); we have $i_{u-1} = i_{v-1}$, and if 0<u-1, then, again, we have $i_{u-2} = i_{v-2}$ (by (II)). Repeating this argument, we have $i_0 = i_{v-u}$. Taking d = v-u, we have $d\in\underline{n}$ and $i_0 = i_d$.

(IV) Since $1 \leq d \leq n$, we have $d|w$. Let $w = gd$, where g is a positive integer. Then by (I) and (III), we have $i = i_0 = i_d = i_{2d} = \ldots\ldots = i_{gd} = i_w = i_j$. Consequently, we have that $a_{ij}^{(w)} = 0$ for all i, j in $\underline{n}$ with $i \neq j$. Thus, A^w is a diagonal matrix.

Corollary: 4.5

If S satisfies U(S) = {1} and A is a right invertible matrix in $M_n(S)$, then $A^{[n]} = I_n$.

Definition: 4.6

Let $A \in M_n(S)$. The mapping $f_A : V_n(S) \to V_n(S)$ is defined by $f_A(x) = Ax$ for $x \in V_n(S)$.

Theorem: 4.7

Let $A \in M_n(S)$ and f_A is a surjective mapping, then A is right invertible in $M_n(S)$.

Proof:

Since f_A is a surjective mapping, ther exist column vectors $(x_{1i}, x_{2i}, \ldots\ldots x_{ni})^T \in V_n(S)$ $(i \in \underline{n})$ such that

$$A\begin{pmatrix} x_{11} \\ x_{21} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ A\begin{pmatrix} x_{12} \\ x_{22} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ldots\ldots\ldots, A\begin{pmatrix} x_{1n} \\ x_{2n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

Put $X = (x_{ij})$. Then $X \in M_n(S)$ and $AX = I_n$, i.e, A is right invertible.

Theorem: 4.8

Let $A \in M_n(S)$. If f_A is a injective mapping and for any $i, j \in \underline{n}$, $\Sigma_{s\in\underline{n}}\, a_{is}$, $\Sigma_{t\in\underline{n}}\, a_{tj} \in U(S)$, and $a_{ij}\,(\Sigma_{s\in\underline{n}}\, a_{is}) = a_{ij}\,(\Sigma_{t\in\underline{n}}\, a_{tj}) = a_{ij}^2$, then A is invertible in $M_n(S)$.

Proof:

For any $i, j \in \underline{n}$, take $b_{ij} = (\Sigma_{s\in\underline{n}}\, a_{is})^{-1}\, a_{ij}$. Then $\Sigma_{s\in\underline{n}}\, b_{is} = 1$ and $b_{ij}^2 = b_{ij}$. Let $d_i = \Sigma_{s\in\underline{n}} a_{is}$ for $1 \le i \le n$ and $D = \operatorname{diag}(d_1, d_2, \ldots\ldots, d_n)$ and $B = (b_{ij})$. Then D is invertible in $M_n(S)$ (by Result 1.37 (i)) and $B = D^{-1}A$, To show A is invertible in $M_n(S)$, it is sufficient to show that B is invertible in $M_n(S)$. We shall prove it in four steps.

(i) f_B is an injective mapping. In fact, suppose that $f_B(x) = f_B(y)$ for some $x, y \in V_n(S)$. Then $Bx = By$, i.e $D^{-1}Ax = D^{-1}Ay$, and so $Ax = Ay$, i.e, $f_A(x) = f_A(y)$. Since f_A is an injective mapping, we have $x = y$. Therefore, f_B is an injective mapping.

(ii) For any $j \in \underline{n}$, we have

$$\prod_{t \in \underline{n}} b_{tj} = 0 \qquad (*)$$

Assume that $\Pi_{t\in\underline{n}} b_{tj_0} \neq 0$ for some $j_0 \in \underline{n}$. Put $x = \Pi_{t\in\underline{n}} b_{tj_0} e_{j_0}$, $y = \Pi_{t\in\underline{n}} b_{tj_0} e$.

Then

$$\begin{aligned} Bx &= \left(b_{1j_0}^2 b_{2j_0} \cdots\cdots b_{nj_0}, b_{1j_0} b_{2j_0}^2 \cdots b_{nj_0}, \cdots, b_{1j_0} b_{2j_0} \cdots b_{nj_0}^2\right)^T \\ &= \left(\prod_{t\in\underline{n}} b_{tj_0}, \prod_{t\in\underline{n}} b_{tj_0} \cdots\cdots \prod_{t\in\underline{n}} b_{tj_0}\right)^T \qquad \text{(because } b_{ij}^2 = b_{ij} \text{ for all } i, j \in \underline{n}\text{)} \\ &= \prod_{t\in\underline{n}} b_{tj_0} e = y. \end{aligned}$$

and

$$\begin{aligned} By &= \prod_{t\in\underline{n}} b_{tj_0} Be \\ &= \prod_{t\in\underline{n}} b_{tj_0} \left(\sum_{s\in\underline{n}} b_{1s}, \sum_{s\in\underline{n}} b_{2s}, \cdots\cdots\cdots\cdots \sum_{s\in\underline{n}} b_{ns}\right)^T \\ &= \prod_{t\in\underline{n}} b_{tj_0} (1, 1, \ldots\ldots\ldots\ldots 1)^T \qquad \text{(since } \sum_{s\in\underline{n}} b_{is} = 1 \text{ for all } i \in \underline{n}\text{)} \\ &= \prod_{t\in\underline{n}} b_{tj_0} e = y. \end{aligned}$$

Thus $f_B(x) = Bx = By = f_B(y)$. But $x \neq y$. This contradicts the fact that f_B is an injective mapping.

(iii) For any $i, j, p \in \underline{n}$ with $i \neq j$, we have

$$b_{ip} b_{jp} = 0 \qquad (*_1)$$

Assume that $b_{i_0p_0}b_{j_0p_0} \neq 0$ for some i_0, j_0, $p_0 \in \underline{n}$ with $i_0 \neq j_0$.

Let M be a maximal subset of the set $\underline{n} \times \underline{n}$ such that (i_0, p_0), $(j_0, p_0) \in M$ and $\Delta = \Pi_{(s,t)\in M}\, b_{st} \neq 0$, where $\underline{n} \times \underline{n} = \{(i, j) : i \in \underline{n}, j \in \underline{n}\}$. Then $M \neq \phi$ since (i_0, p_0) and $(j_0, p_0) \in M$, and $M \neq \underline{n} \times \underline{n}$ since $\Pi_{t\in\underline{n}} b_{tj} = 0$ for any $j \in \underline{n}$ (By $*$).

$$b_{ij}\,\Delta = \begin{cases} \Delta & \text{if } (i, j) \in M \\ 0 & \text{if } (i, j) \notin M \end{cases} \qquad (*_2)$$

In fact, If $(i, j) \in M$, then

$$b_{ij}\,\Delta = \left(\prod_{(s,t)\in M} b_{st}\right) b_{ij} = \left(\prod_{\substack{(s,t)\in M \\ (s,t)\neq(i,j)}} b_{st}\right) b_{ij}^2$$

$$= \left(\prod_{\substack{(s,t)\in M \\ (s,t)\neq(i,j)}} b_{st}\right) b_{ij} \qquad \text{(since } b_{ij}^2 = b_{ij}\text{)}$$

$$= \prod_{(s,t)\in M} b_{st} = \Delta$$

if $(i, j) \notin M$, then by the definitions of M and Δ, we have $b_{ij}\Delta = 0$.

Thus, $(*_2)$ holds.

In the following, we will prove that for any $i \in \underline{n}$, there exists a $j \in \underline{n}$ such that $(i, j) \in M$. In fact, since $\Sigma_{s\in\underline{n}}\, b_{is} = 1$ for each $i \in \underline{n}$, we have $\Delta = \Delta(\Sigma_{s\in\underline{n}}\, b_{is}) = \Sigma_{s\in\underline{n}}(b_{is}\Delta) \neq 0$, and so $b_{ij}\,\Delta \neq 0$ for some $j \in \underline{n}$. By $(*_2)$, we have $b_{ij}\,\Delta = \Delta$ and $(i, j) \in M$.

For each $i \in \underline{n}$, let $M(i) = \{s \in \underline{n}, (i, s) \in M\}$. It is clear that $M(i) \neq \phi$. Let $k_i = w(M(i))$. Then $1 \leq k_i \leq n$ and $\Delta = \Delta.1 = \Delta\,(\Sigma_{s\in\underline{n}}(b_{is}) = \Sigma_{s\in\underline{n}}\Delta b_{is} = \Sigma_{(i,\, s)\in M}\,\Delta$ $(*2)$ $= \Sigma_{s\in M(i)}\Delta = k_i\,\Delta$. That is, $\Delta = k_i\,\Delta$ for each $i \in \underline{n}$.

In the following we shall prove that $k_1 = k_2 = \ldots\ldots\ldots\ldots = k_n = 1$.

To do this, we first prove that for any given $j \in \underline{n}$, there exists an $i \in \underline{n}$ such that $b_{ij}\Delta = 0$. In fact, if there exists some $j \in \underline{n}$ such that $b_{ij}\Delta \neq 0$ for all $i \in \underline{n}$, then $b_{ij}\Delta = \Delta$ for all $i \in \underline{n}$ $(*_2)$, and so $\Pi_{t\in\underline{n}}\, b_{tj}\Delta = \Delta$. But $\Pi_{t\in\underline{n}}\, b_{tj} = 0$ $(*)$, we have $\Delta = 0$. This contradicts the fact that $\Delta \neq 0$.

Suppose that there exists an $i \in \underline{n}$ such $k_i \geq 2$. Taking $k = \max\{k_1, k_2, \ldots\ldots, k_n\}$, we have that $k \geq 2$ and $\Delta = k\Delta$ (since $\Delta = k_i\Delta$ for every k_i). Let $H = \{i \in \underline{n}, k_i \geq 2\}$. Then $w(H) \geq 1$ since $i \in H$.

We now choose an s_0 in $\underline{n}$ such that $b_{ts_0}\Delta = 0$ for all $t \in \underline{n} \setminus H$ (note that if $H = \underline{n}$ then s_0 may be chosen arbitrarily in $\underline{n}$), and let

$$x = \Delta\left(\sum_{s\,\in\,s_0} s \in \underline{n}\right) + (k-1)\,\Delta e s_0,\ y = \Delta\left(\sum_{s \neq s_0} s \in \underline{n}\ e_s\right),\ u = Bx \text{ and } v = By.$$

For any $i \in \underline{n}$, if $b_{is_0}\Delta = 0$, then

$$\begin{aligned} u_i &= (Bx)i = \sum_{s\in\underline{n}} b_{is}x_s \\ &= \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\right)\Delta + b_{is_0}(k-1)\Delta \\ &= \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\right)\Delta \text{ (since } b_{is_0}\Delta = 0) \\ &= (By)_i = v_i \end{aligned}$$

If $b_{is_0}\Delta \neq 0$, then $i \in H$ (since $b_{ts_0}\Delta = 0$ for all $t \in \underline{n} \setminus H$) and $(i, s_0) \in M$ (or $s_0 \in M(i)$), and so $b_{is_0}\Delta = \Delta$ $(*_2)$. Then

$$U_i = (Bx)_i = \sum_{s\in\underline{n}} b_{is}x_s$$

$$= \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\Delta\right) + (k-1)\Delta = \left(\sum_{\substack{(i,s)\in M\\ s\neq s_0}} b_{is}\Delta\right) + (k-1)\Delta$$

$$= \left(\sum_{\substack{(i,s)\in M\\ s\neq s_0}} \Delta\right) + (k-1)\Delta = \left(\sum_{\substack{s\in M(i)\\ s\neq s_0}} \Delta\right) + (k-1)\Delta$$

$$= (k_i - 1)\Delta + (k-1)\Delta \quad (\text{since } S_0\in M(i)).$$

Since $i\in H$, we have $k_i \geq 2$. If $k_i = 2$ then $u_i = \Delta+(k-1)\Delta = (1+(k-1))\Delta = k\Delta = \Delta$ (since $\Delta = k\Delta$) $= (k_i-1)\Delta$ and if $k_i \geq 3$ then

$$\begin{aligned} u_i &= (k_i-1)\Delta + (k-1)\Delta \\ &= (k_i-2)\Delta + \Delta + (k-1)\Delta \quad (\text{since } k_i \geq 3) \\ &= (k_i-2)\Delta + k\Delta = (k_i-2)\Delta+\Delta \quad (\text{since } \Delta = k\Delta) \\ &= (k_i-1)\Delta \end{aligned}$$

Since

$v_i = (By)_i = \sum_{s\in\underline{n}} b_{is}y_s = \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\right)\Delta = \sum_{\substack{(i,s)\in M\\ s\neq s_0}} b_{is}\Delta = \sum_{\substack{(i,s)\in M\\ s\neq s_0}} \Delta = (k_i-1)\Delta$,, we have $u_i = v_i$.

Consequently, we have $u = v$, i.e., $f_B(x) = f_B(y)$. But $x \neq y$. This contradicts the fact that f_B is an injective mapping. Therefore $k_1 = k_2 = \ldots\ldots = k_n = 1$.

Since $k_1 = k_2 = \ldots\ldots = k_n = 1$, we have that for each $i\in\underline{n}$, there exists a unique $j\in\underline{n}$ such that $(i, j)\in M$. Since (i_0, p_0), $(j_0, p_0)\in M$, there exists a $q_0\in\underline{n}$ such that $(i, q_0)\notin M$ for each $i\in\underline{n}$, that is, $b_{iq_0}\Delta = 0$ for each $i\in\underline{n}$. Then $a_{iq_0}\Delta = d_ib_{iq_0}\Delta = 0$ for all $i\in\underline{n}$, and so

$$\left(\sum_{t\in\underline{n}} a_{tq_0}\right)\Delta = \sum_{t\in\underline{n}} a_{tq_0}\Delta = 0.$$

But $\sum_{t\in\underline{n}} a_{tq_0} \in U(S)$, we have $\Delta = 0$. This contradicts the fact that $\Delta \neq 0$. Thus $b_{ip}b_{jp} = 0$ for any i, j, $p\in\underline{n}$ with $i\neq j$.

(iv) Now

$$BB^T = \left(\sum_{s\in\underline{n}} b_{is}b_{js}\right)_{n\times n}$$

$$= \operatorname{diag}\left(\sum_{s\in\underline{n}} b^2_{1s}, \ldots\ldots\ldots, \sum_{s\in\underline{n}} b^2_{ns}\right) \qquad \text{(By } *_1\text{)}$$

$$= \operatorname{diag}\left(\sum_{s\in\underline{n}} b_{1s}, \ldots\ldots\ldots, \sum_{s\in\underline{n}} b_{ns}\right) \qquad \text{(because } b^2_{1s} = b_{ij} \text{ for all } i, j\in\underline{n}\text{)}$$

$$= \operatorname{diag}(1, 1, \ldots\ldots\ldots, 1) \qquad \text{(because } \sum_{s\in\underline{n}} b_{is} = 1 \text{ for all } i\in\underline{n}\text{)} = I_n.$$

Then B is invertible in $M_n(S)$.

Theorem: 4.9

If $A \in M_n(S)$ is right (left) invertible and $a_{11}, a_{22}, \ldots\ldots\ldots, a_{nn} \in U(S)$, then A is a diagonal matrix.

Proof:

We assume that A is right invertible in $M_n(S)$, i.e., $AB = I_n$ for some $B\in M_n(S)$. Then for any $i, j\in\underline{n}$

$$a_{ii}\, b_{ij} + \sum_{\substack{k\in\underline{n}\\ k\neq i}} a_{ik}b_{kj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Consequently $a_{ii}\, b_{ij} = 0$ for $i\neq j$, and so $b_{ij} = 0$ for $i\neq j$ (since $a_{ii}\in U(S)$). Therefore B is a diagonal matrix. Since $AB = I_n$, B is an invertible diagonal matrix, and so $A = B^{-1}$ is an invertible diagonal matrix by Result 1.37(ii). If A is left invertible, then by result 1.36, A is right invertible, according to what we have proved, A is a diagonal matrix.

Corollary: 4.10

If $A \in M_n(S)$ is right (left) invertible with $a_{11} = a_{22} = \ldots\ldots = a_{nn} = 1$, then $A = I_n$.

Definition: 4.11

Let $x, y \in V_n(S)$, the scalar product of x and y is defined as $x^T y \in S$.

Theorem: 4.12

Let $A \in M_n(S)$. Then the following statements are equivalent:

(i) A is right invertible.

(ii) A is left invertible.

(iii) A is invertible.

(iv) AA^T is an invertible diagonal matrix.

(v) $A^T A$ is an invertible diagonal matrix.

(vi) AA^T and $A^T A$ are invertible diagonal matrices.

(vii) A^d is an invertible diagonal matrix for some positive integer d.

(viii) f_A is a surjective mapping.

(ix) f_A is a bijecive mapping.

(x) There exist $u_1, u_2, \ldots\ldots u_n \in U(S)$ such that $(f_A(x))^T f_A(y) = \Sigma_{i \in \underline{n}} u_i x_i y_i$ for any $x = (x_1, x_2, \ldots\ldots x_n)^T$ and $y = (y_1, y_2 \ldots\ldots y_n)^T \in V_n(S)$.

(xi) f_A is a injective mapping and for any $i, j \in \underline{n}$, $\Sigma_{s \in \underline{n}}\, a_{is}, \Sigma_{s \in \underline{n}}\, a_{tj} \in U(S)$ and $a_{ij} (\Sigma_{s \in \underline{n}}\, a_{is}) = a_{ij} (\Sigma_{t \in \underline{n}}\, a_{tj}) = a_{ij}^2$.

(xii) f_A is an injective mapping and $A^{k+d} = A^k \bullet D$ for some invertible diagonal matrix D and positive integers k and d.

Proof:

By the Result 1.36 and Theorems 4.1 & 4.2, we have that statements (i)-(vii) are equivalent. Also, the implications (iii) $\Rightarrow$ (ix) and (ix) $\Rightarrow$ (viii) are obvious.

(viii) $\Rightarrow$ (i). It is Theorem 4.7.

(vii) $\Rightarrow$ (xii). It is obvious.

(xii) $\Rightarrow$ (vii). Since f_A is an injective mapping, f_{A^k} is also an injective mappings. Since $A^k \bullet A^d = A^{k+d} = A^k \bullet D$ for some invertible diagonal matrix D and positive integers k and d, we have $A^k \bullet$ (D_{*j}) for all $j \in \underline{n}$, i.e., $f_{A^k}((A^d)_{*j}) = f_{A^k}(D_{*j})$. Then $(A^d)_{*j} = D_{*j}$ for all $j \in \underline{n}$, (since f_A is an injective mapping and so $A^d = D$.

(xi) $\Rightarrow$ (iii). It is Theorem 4.8

(iii) $\Rightarrow$ (xi). By Result 1.39 (i) & (iii)

(v) $\Rightarrow$ (x). Let $A^TA = D$, where D is an invertible diagonal matrix. Then for any $x = (x_1, x_2, \ldots\ldots, x_n)^T$ and $y = (y_1, y_2, \ldots\ldots, y_n)^T \in V_n(S)$, we have $(f_A(x)^T f_A(y) = (Ax)^T(Ay) = x^T (A^TA) y = x^TDY$. Let $D = \text{diag}(u_1, u_2, \ldots\ldots., u_n)$. Then $u_1, u_2, \ldots\ldots.., u_n \in U(S)$ and $(f_A(x))^T = f_A(y) = \Sigma_{i\in\underline{n}}\, u_i\, x_i\, y_i$.

(x) $\Rightarrow$ (v). Suppose that (x) holds. Then for any $x = (x_1, x_2, \ldots\ldots, x_n)^T$ and $y = (y_1, y_2, \ldots\ldots, y_n)^T \in V_n(S)$, we have $x^T\, A^T\, Ay = (Ax)^T\,(Ay) = (f_A(x)^T\, f_A(y) = \Sigma_{i\in\underline{n}}\, u_i\, x_i\, y_i$. Let $D = \text{diag}\,(u_1, u_2, \ldots\ldots.., u_n)$.

Then D is an invertible diagonal matrix in $M_n(S)$ and $x^T\,(A^TA)\,y = x^TDY$. Therefore, $e_i^T(A^TA)\,e_j = e_i^TDe_j$ for any i and $j \in \underline{n}$, i.e., $\sum_{s\in\underline{n}} a_{si}\, a_{sj} = \begin{cases} u_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Hence $A^TA = D$ is an invertible diagonal matrix in $M_n(S)$.

Corollary: 4.13

If S is an incline and A $\in M_n(S)$. Then the following statements are equivalent:

(i) A is right invertible.

(ii) A is left invertible.

(iii) A is invertible.

(iv) $AA^T = I_n$.

(v) $A^TA = I_n$.

(vi) $AA^T = A^TA = I_n$.

(vii) $A^k = In$ for some positive integer k.

(viii) f_A is a surjective mapping.

(ix) f_A is a bijecive mapping.

(x) f_A preserves the scalar products, that is, for any x, $y \in V_n(S)$, $(f_A(x))^T f_A(y) = x^Ty$.

(xi) $A \in M_n(I(S))$ and f_A is an injective mapping, where $M_n(I(S))$ denotes the set of all $n \times n$ matrices over the set $I(S)$.

(xii) f_A is an injective mapping and $A^{k+d} = A^k$ for some integers k and d.

Proof:

Since S is an incline, we have U(S) = {1} and so any invertible diagonal matrix in $M_n(S)$ is the identity matrix I_n. Therefore, by Theorem 4.12, the statements (i) - (x) and (xii) are equivalent.

(iii) $\Rightarrow$ (xi). If (iii) holds, then by Theorem 4.12 and the condition U(s) = {1}, we have that for any $i \in \underline{n}$, $\sum_{s \in \underline{n}} a_{is} = \sum_{t \in \underline{n}} a_{tj} = 1$ and that for any i, $j \in \underline{n}$ $a^2_{ij} = a_{ij}$, and so $A \in M_n(I(S))$. It is clear that f_A is an injective mapping. Thus the statement (XI) holds.

(xi) $\Rightarrow$ (xii). Since $\mathrm{I}(S)$ is a distributive lattice (by the result which and states that, "If S is an incline and $\mathrm{I}(S)$ is the set of all idempotent elements in S, then $\mathrm{I}(S)$ is a distributive lattice and $A \in M_n(\mathrm{I}(S))$, it follows from "If S is a distributive lattice and $A \in M_n(S)$ then there exist positive integers k and d such that $A^{k+d} = A^k$ for some positive integers k and d. Since f_A is an injective mapping, (XII) holds.

Theorem: 4.14

If S satisfies U(S) = {1} and $A \in M_n(S)$, then the following statements are equivalent.

(i) A is invertible

(ii) A is an orthogonal combination of some permutation matrices of order n.

Proof:

(i) $\Rightarrow$ (ii). If A is invertible in $M_n(S)$, then by Theorem 4.12, we have $A^T A = AA^T = \mathrm{I}_n$ (since U(S) = {1}). Therefore $a_{ij}\, a_{ik} = a_{ji}\, a_{ki} = 0$ for all i, j, k in $\underline{n}$ with $j \neq k$ and $\sum_{s\in\underline{n}} a_{is}^2 = \sum_{t\in\underline{n}} a_{tj}^2 = 1$ for all $i, j \in \underline{n}$, and so $\left(\sum_{s\in\underline{n}} a_{is}\right)^2 = \sum_{s\in\underline{n}} a_{is}^2 + \sum_{s_1\neq s_2} a_{is_1} a_{is_2} = \sum_{s\in\underline{n}} a_{is}^2 = 1$ and $\left(\sum_{t\in\underline{n}} a_{tj}\right)^2 = \sum_{t\in\underline{n}} a_{tj}^2 + \sum_{t_1\neq t_2} a_{t_1} a_{t_2 j} = \sum_{t\in\underline{n}} a_{tj}^2 = 1$. Thus, $\sum_{s\in\underline{n}} a_{is} = 1$ and $\sum_{t\in\underline{n}} a_{tj} = 1$, that is, each row and each column of A is an orthogonal decomposition of 1 in S. Also, for any $i, j \in \underline{n}$, $a_{ij} = a_{ij}\left(\sum_{s\in\underline{n}} a_{is}\right) = a_{ij}^2 + \sum_{s\neq j} a_{is} a_{ij} = a_{ij}^2$, that is, each entry of A is idempotent.

Since $\sum_{s\in\underline{n}} a_{is} = 1$ for all $i \in \underline{n}$, we have $1 = \Pi_{i\in\underline{n}} \left(\sum_{s\in\underline{n}} a_{is}\right) = \sum_{1\leq s_1, s_2, \ldots\ldots, s_n \leq m} a_{1s_1} a_{2s_2}, \ldots\ldots, a_{ns_n}$. If $s_i = s_j$ for some $i, j \in \underline{n}$ with $i \neq j$, then $a_{is_i} \cdot a_{js_j} = 0$ and so $a_{1s_1} a_{2s_2}, \ldots\ldots, a_{ns_n} = 0$.

Therefore $1 = \sum_{\sigma\in s_n} a_{1\sigma(1)} a_{2\sigma(2)} \ldots\ldots a_{n\sigma(n)}$.

Let $a_\sigma = a_{1\sigma(1)}a_{2\sigma(2)}$ .... $a_{n\sigma(n)}$ for $\sigma \in s_n$ and $G = \{\sigma : \sigma \in s_n, a_\sigma \neq 0\}$. Then, it is clear that $\sum_{\sigma \in G} a_\sigma = 1$ and $a_\sigma a_\tau$ for any $\sigma, \tau \in G$ with $\sigma \neq \tau$, and so the set $\{a_\sigma : \sigma \in G\}$ is an orthogonal decomposition of 1 in S. Also, for any $\sigma \in G$.

$$a_\sigma\, a_{ij} = \begin{cases} a_\sigma & \text{if } j = \sigma(i) \\ 0 & \text{if } j \neq \sigma(i) \end{cases} \quad i, j \in \underline{n}.$$

Therefore, there exists a unique (0, 1) matrix $P_\sigma \in M_n(S)$ such that $a_\sigma A = a_\sigma p_\sigma$ holds. It follows that

$$a_\sigma^2 P_\sigma P_\sigma^T = (a_\sigma P_\sigma)(a_\sigma P_\sigma)^T = (a_\sigma A)(a_\sigma A)^T = a_\sigma^2 AA^T = a_\sigma^2 I_n,$$ and so $P_\sigma P_\sigma^T = I_n$. Then P_σ is a permutation matrix for each $\sigma \in G$. Clearly,

$$\sum_{\sigma \in G} a_\sigma P_\sigma = \sum_{\sigma \in G} a_\sigma A = \left(\sum_{\sigma \in G} a_\sigma \right) A = A.$$

i.e., A is an orthogonal combination of some permutation matrices.

(ii) ⇒ (i). If A is such combination, say

$$A = \sum_{s \in \underline{m}} a_s P_s,$$

then

$$\begin{aligned} AA^T &= \left(\sum_{s \in \underline{m}} a_s P_s \right) \left(\sum_{t \in \underline{m}} a_t P_t \right)^T = \sum_{s \in \underline{m}} \sum_{t \in \underline{m}} a_s a_t P_s P_t^T = \sum_{s \in \underline{m}} a_s^2 I_n \\ &= \sum_{s \in \underline{m}} \sum_{t \in \underline{m}} a_s a_t I_n \quad (\text{since } a_s a_t = 0 \text{ with } s \neq t) \\ &= \left(\sum_{s \in \underline{m}} a_s \right)^2 I_n = I_n. \end{aligned}$$

Thus A is invertible in $M_n(S)$.

CRAMER'S RULE OVER COMMUTATIVE ANTIRINGS:

In this part, Cramer's rule for a matrix equation over a commutative antiring S is presented.

Lemma: 4.15

Let $A \in M_{m\times n}(S)$ $(m \leq n)$, and $U \in GL_m(S)$ and $V \in GL_n(S)$. Then per (UAV) = per U per A per V.

Proof:

We first prove that per (UA) = per U per A for any $A \in M_{m\times n}(S)$ and U in $GL_m(S)$. Let W = UA. Then for any $i \in \underline{m}$ and $j \in \underline{n}$, $w_{ij} = \sum_{k=1}^{m} u_k a_{kj}$ .Thus

$$\text{per (UA)} = \text{per } W = \sum_{\sigma \in s_{m,n}} \prod_{i \in \underline{m}} w_{i\sigma(i)}$$

$$= \sum_{\sigma \in s_{m,n}} \prod_{i \in \underline{m}} \left(\sum_{k \in \underline{m}} U_{ik} a_{k\sigma(i)} \right)$$

$$= \sum_{\sigma \in s_{m,n}} \sum_{1 \leq k_1, k_2, \ldots\ldots k_m \leq m} u_{1k_1} a_{k_1\sigma(1)} u_{2k_2} a_{k_2\sigma(2)} \ldots\ldots\ldots u_{mk_m} a_{k_m\sigma(m)} .$$

$$= \sum_{\sigma \in s_{m,n}} \sum_{1 \leq k_1, k_2, \ldots\ldots k_m \leq m} \left(u_{1k_1} u_{2k_2} \ldots\ldots\ldots u_{mk_m}\right) \left(a_{k_1\sigma(1)} a_{k_2\sigma(2)} \ldots\ldots\ldots a_{k_m\sigma(m)}\right).$$

If $k_s = k_t$ for some s, $t \in \underline{m}$ with $s \neq t$, then $u_{sk_s} u_{tk_t} = u_{sk_s} u_{tk_s} = 0$ (by Result 1.40 (i))

and so

$$\text{per (UA)} = \sum_{\sigma \in s_{m,n}} \sum_{\substack{1 \leq k_1, k_2, \ldots\ldots k_m \leq m \\ k_s \neq k_t (s \neq t)}} \left(u_{1k_1} u_{2k_2} \ldots\ldots\ldots u_{mk_m}\right) \left(a_{k_1\sigma(1)} a_{k_2\sigma(2)} \ldots\ldots\ldots a_{k_m\sigma(m)}\right)$$

$$= \sum_{\sigma \in s_{m,n}} \sum_{\rho \in s_m} \left(u_{1\rho(1)} u_{2\rho(2)} \ldots\ldots\ldots u_{m\rho(m)}\right) \left(a_{\rho(1)\sigma(1)} a_{\rho(2)\sigma(2)} \ldots\ldots\ldots a_{\rho(m)\sigma(m)}\right)$$

$$= \sum_{\rho \in s_m} u_{1\rho(1)} u_{2\rho(2)} \ldots\ldots\ldots u_{m\rho(m)} \left(\sum_{\sigma \in s_{m,n}} a_{\rho(1)\sigma(1)} a_{\rho(2)\sigma(2)} \ldots\ldots\ldots a_{\rho(m)\sigma(m)} \right)$$

$$= \sum_{\rho \in s_m} u_{1\rho(1)} \ldots\ldots\ldots u_{m\rho(m)} \left(\sum_{\sigma \in s_{m,n}} a_{\rho(1)\sigma(\rho^{-1}(\rho(1)))} \ldots\ldots\ldots a_{\rho(m)\sigma(\rho^{-1}(\rho(m)))} \right)$$

$$= \sum_{\rho \in S_m} u_{1\rho(1)} \cdots\cdots\cdots u_{m\rho(m)} \left(\sum_{\sigma \in S_{m,n}} a_{1\sigma(\rho^{-1}(1))} \cdots\cdots\cdots a_{m\sigma(\rho^{-1}(m))} \right)$$

(because ρ is a bijection of the set $\underline{m}$)

$$= \sum_{\rho \in S_m} u_{1\rho(1)} \cdots\cdots\cdots u_{m\rho(m)} \left(\sum_{\sigma\rho^{-1} \in S_{m,n}} a_{1(\sigma\rho^{-1})(1)} \cdots\cdots\cdots a_{m(\sigma\rho^{-1})(m)} \right)$$

$$= \sum_{\rho \in S_m} u_{1\rho(1)} \cdots\cdots\cdots u_{m\rho(m)} \left(\sum_{\sigma \in S_{m,n}} a_{1\sigma(1)} \cdots\cdots\cdots a_{m\sigma(m)} \right)$$

= per U per A.

Similarly, we can prove that per (AV) = per A per V for any $A \in M_{m\times n}(S)$ and $V \in GL_n(S)$. Therefore, per (UAV) = per U per (AV) = per U per A per V.

Corollary: 4.16

If $A \in GL_n(S)$, then per $A \in U(S)$.

Proof:

If $A \in GL_n(S)$, then there exists a $B \in GL_n(S)$ such that $AB = I_n$. By Lemma 4.15, we have 1 = per I_n = per (AB) = per A per B, and so par $A \in U(S)$.

Corollary: 4.17

If S satisfies U(S) = {1}, then

(i) for any $A \in GL_n(S)$, we have per A = 1;

(ii) for any $A \in M_{m\times n}(S)$ $(m \le n)$, and $U \in GL_m(S)$ and $V \in GL_n(S)$,

We have per (UAV) = per A.

Lemma: 4.18

Let $A \in GL_n(S)$. Then $A^{-1} = (\text{per } A)^{-1}$ adj A.

Proof:

Let $A \in GL_n(S)$. Then $a_{ik}a_{il} = 0$ for an i, k, $l \in \underline{n}$ with $k \neq l$ (by Result 1.42 (i)). Let B = A adj A. Then for any i, $j \in \underline{n}$, we have $b_{ij} = \sum_{k \in \underline{n}} a_{ik}$ per (j / k).

If i = j, then $b_{ii} = \sum_{k \in \underline{n}} a_{ik}$ per A (i | k) = per A ; if $i \neq j$,

(by Result 1.39)

$$\text{then } b_{ij} = \sum_{k \in \underline{n}} a_{ik} \left(\sum_{\substack{l \in \underline{n} \\ l \neq k}} a_{il} \text{ per } A\,(ij / kl) \right) \qquad \text{(by Result 1.39)}$$

$$= \sum_{k \in \underline{n}} \sum_{\substack{l \in \underline{n} \\ l \neq k}} a_{il} \text{ per } A\,(ij / kl) = 0 \qquad \text{(by Result 1.42(i))}$$

Therefore, B = A adj A = (per A) I_n. Since per $A \in U(S)$ (by Corollary 4.16),

$A^{-1} = (\text{per } A)^{-1}$ adj A.

Corollary: 4.19

If S satisfies U(S) = {1} and $A \in GL_n(S)$, then adj A = A^T.

Proof:

By Corollary 4.13 and Corollary 4.19 (i) and Lemma 4.18, we have $A^T = A^{-1}$ = adj A.

The following Theorem is Cramer's rule for a matrix equation over a commutative antiring.

Theorem: 4.20

Let $A \in M_n(S)$ and $b = (b_1, b_2, \ldots, b_n)^T \in V_n(S)$. If A is invertible in $M_n(S)$, then the matrix equation $Ax = b$ has a unique solution $x = (d^{-1}d_1, d^{-1}d_2, \ldots\ldots, d^{-1}d_n)^T \in V_n(S)$, where $d = \text{per } A$ and

$$d_j = \text{per}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & & & & \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}, \quad j = 1, 2, \ldots\ldots n$$

Proof:

It is clear that the equation $Ax = b$ has a solution $x = A^{-1}b$. Let $y \in V_n(S)$ be any solution of this equation. Then $Ay = b$, and so $y = I_n\, y = (A^{-1}A)y = A^{-1}(Ay) = A^{-1}b$, which means that the equation $Ax = b$ has a unique solution.

Let now $(\text{adj } A)\, b = (d_1, d_2, \ldots\ldots, d_n)^T$. Then for any j in $\underline{n}$, we have $d_j = \sum_{i \in \underline{n}} \text{per } A\,(i \mid j)\, b_i$. On the other hand, we have

$$\text{per}\begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & & & \cdots & & & \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}$$

$$= \sum_{i \in \underline{n}} b_i \text{ Per } A\,(i \mid j) \qquad \text{(by Result 1.39)}$$

Therefore

$$x = A^{-1}b = (\text{per } A)^{-1}\, (\text{adj } A)b \qquad \text{(by Lemma 4.18)}$$

$$= d^{-1}(\text{adj } A)\, b = (d^{-1}d_1, d^{-1}d_2, \ldots\ldots\ldots, d^{-1}d_n)^T.$$

Corollary: 4.21

If S satisfies $U(S) = \{1\}$ and $A \in GL_n(S)$ and $(b_1, b_2, \ldots\ldots, b_n)^T \in V_n(S)$, then the matrix equation $Ax = b$ has a unique solution $x = (d_1, d_2, \ldots\ldots\ldots, d_n)^T \in V_n(S)$, we have

$$d_j = \text{per}\begin{bmatrix} a_{11} & \ldots\ldots & a_{1,j-1} & b_1 & a_{1,j+1} & \ldots\ldots & a_{1n} \\ a_{21} & \ldots\ldots & a_{2,j-1} & b_2 & a_{2,j+1} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & \ldots\ldots & a_{n,j-1} & b_n & a_{n,j+1} & \ldots\ldots & a_{nn} \end{bmatrix}, \; j = 1, 2, \ldots\ldots\ldots\ldots, n.$$