# Introduction of Graph theory and Graph Coloring

## Introduction

Graph theory is one of the exhilarating and fast-growing areas having wide range of applications to many real-life problems due to its abundant theoretical notions. There exists a close inter relation between the branches of mathematics such as topology, group theory, matrix theory and probability with graph theory. It is one of the fastest growing areas in mathematics primarily scintillating due to its diverse applications. Adequate number of works have been carried out in various topics like decomposition, domination, factoring, orienting, colouring etc.,

In subjects like chemistry and physics, the graph theory is utilised to investigate molecules. By combining graph-theoretic properties and statistics with the atomic topology, the 3D structure of complex simulated atomic structures may be quantitatively evaluated in physics. In physics, link between the interrelating parts of a system, as well as, physical process on systems can be characterized the help of diagrammatic representation.

In chemistry, molecules are meant to be as atoms and bonds, which are represented as vertices and edges in graph. This method is particularly employed in Information processing in the form of molecular structures, which ranges from statistical analysis to database search. In biology and conservation efforts, theoretical notions of graphs are applied to some extent. In particular species, the areas are represented by a vertex, and the movement or path between the regions is represented by edges. This is necessary for investigating breeding patterns, illness and parasite spread, and the effects of migration on other species. Tree-based structures, whose expressive potential is based on the concept of compositionality and expressed in a hierarchical graph, have traditionally been used in syntax and compositional semantics. In general, graph theory has numerous applications and is gaining in popularity.

In Computer Science field, graphs have been used to depict communication networks, data organisation, computing devices, computation flow and so on. A directed graph, for example, can be used to depict the connection structure of a website, with web pages marked by vertices and links between pages denoted by directed edges.

Other areas in which graphs are used include modelling, activity networks. The most common and successful network applications in OR are project planning and scheduling for large, complicated projects. The Critical Path Method (CPM) and Project Evaluation Review Technique (PERT) are two well-known and useful ways to problem-solving.

In Sociology, mean is generally used in graph theory. To rationalise actor prestige or to investigate diffusion mechanisms, particularly using social network analysis software, for example. There are numerous types of graphs available under the umbrella of Social Network graphs.

In 1936, Lewin the famous psychologist proposed that a planar map could characterize the life space of an individual. It was pointed out that people and interpersonal relations are represented by points and by lines. D.N.A chains have been studied with the invention of interval graphs. During 1959, Benzer Deliberates the linearity of the chain for higher organisms where each gene is encoded as an interval under the hypothesis that mutations arises from modifications of associated segments. Variations in traits of micro organisms can be studied to identify whether their determining amino acid sets could intersect. This establishes a graph with traits as vertices and common alterations as edges.

**Origin of Graph Theory**

The main source of graph theory can be traced back to Euler's work on the Konigsberg bridges problem (1735), which subsequently led to the initiative of an Eulerian graph. The problem of Konigsberg Bridge was studied by Euler and found a structure to solve the problem.

Euler explained that the choice of route inside each land mass is improper. The necessary feature of a route is the sequences of bridges crossed. This is allowed to assemble the problem in abstract terms, eliminating all kind but the list of land masses and the bridges between them.

Each land mass replace with an abstract vertex or node, and each bridge replace with an abstract link, an edge, which only serves to prove which pair of vertices or land masses are associated by that bridge. Graph is a result of mathematical structure.

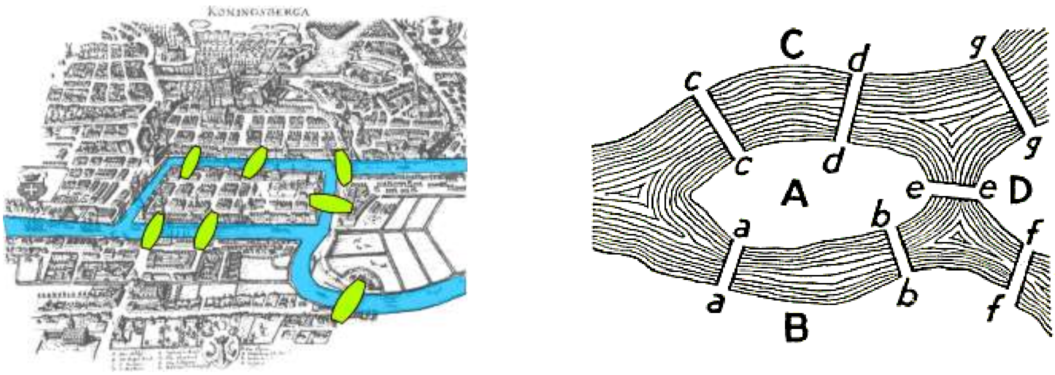


**Fig. 1:** Konigsberg Bridge Problem

Euler denoted the island Kneiphof in Konigsberg by the letter A and the remaining three land by B, C and D. The seven bridges that crossed the River Pregel were denoted by a, b, c, d, e, f and g (Figure 1). Euler explains what must occur if there was a route that crossed each of the seven bridges accurately once. These routes could be represented as a sequence of letters, each term of which is one of the letters **A, B, C** and **D**. A Specific term in this sequence would indicate that the route had reached that land region and the term immediately following it would indicate the land region to which the route had progressed after crossing a bridge. Due to seven bridges, the sequence should consist of eight terms.

Because there are five bridges leading into (or out of) land region **A** (the island Kneiphof), each occurrence of **A** has to indicate that either the route began at **A**, ended at **A** or had progressed to and then terminated from **A**. Thus, in the sequence **A** must appear three times. In related manner, each of **B, C** and **D** must appear twice in the sequence. But still, this gives that a sequence must contain nine terms, which highly impossible. So, there is no relevant route in Konigsberg that crosses each bridge accurately once.

**Operations on Graph Theory**

Operations on graphs produce new ones from older ones. There are several operations on Graph Theory that produce new graphs from old ones, which is mentioned in the following major categories.

- ❖ Unary Operations
- ❖ Binary Operations

**Unary Operations** [Pavol Hell et al., 1988]

Unary operations create a new graph from the old one. Elementary operations are sometimes called "editing operations" on graphs. They create a new graph from the original one by a simple or a local change, such as addition or deletion of a vertex or an edge, merging and splitting of vertices, edge contraction, etc., Advanced operations such as Line graph, Dual graph, Complement graph, Graph minor, Power of graph, etc.,also creates new graphs from the old ones.

**Binary Operations**

Binary operations create a new graph from two initial graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ :

- ❖ The disjoint union of graphs, sometimes referred as simply graph union, which is defined as: For two graphs with disjoint vertex sets $V_1$ and $V_2$ and disjoint edge sets E1 and E2 their disjoint union is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.
- ❖ The graph join of two graphs is their graph union with all the edges that connect the vertices of the first graph with the vertices of the second graph.

Graph products based on the Cartesian product of the vertex sets are:

**Cartesian product of graphs**

- ❖ Lexicographic product of graphs also called Graph composition Strong product of graphs.
- ❖ Tensor product of graphs, also called as direct product, categorical product.
- ❖ cardinal product or Kronecker product
- ❖ Zig-zag product of graphs

**Other graph operations called "products" are:**

- ❖ Rooted product of graphs.
- ❖ Corona product or simply corona of G1 and G2.

**Graph Colorings** [P. Francis, 2018]

Partitioning a set of objects into some defined classes based on certain rules is a fundamental process in Mathematics, and it appears in many actual situations. The theory of graph coloring deals precisely with this situation. Graph coloring is one of the most attractive field for graph theorists, not only for its applicability but also for its theoretical interest.

**History of Graph Coloring**

Coloring theory begins with challenge of coloring countries on a map in such a way that no two countries that share a common border receive the same colour. A planar graph is made by representing countries as points in the plane and linking each pair of points that share a border with a curve. All planar graphs can be colored with four different colors, according to the famous 4 Color Problem. In A.D. Morgan's letter to W.R. Hamilton, it appears to be voiced for the first time. No one realised it was the start of a new hypothesis at the time. Kempe provided the first "evidence" in 1879. It grew for more than a decade until Heawood discovered an inaccuracy in 1890. Heawood discovered that different map may be colored with only 5 colors.

The Four Color Problem has evolved into one of the most complicated Graph Theory issues. It influenced many other areas of graph theory through colorings. Coloring theory is a dispute resolution theory in general. In a graph, adjacent vertices must have different colors, indicating that they are in a stable dispute. If we have a "good" coloration, we respect any disagreements. We have a pair of nearby vertices colored with the same colour if we have a "poor" coloring. It appears to be a geographical map with two countries sharing a shared border coloured in the same color. Colors are used to denote the status of a vertex, and graphs are used to depict what is in conflict with what. Because we will only calculate "excellent" colorings, coloring theory is more specifically the theory of partitioning the sets containing internal unreconcilable conflicts. The most famous erroneous proof was illuminating.

On the other hand, it provided an excellent demonstration of how pictures may be employed in Graph Theory proofs. On the other thing, it was a clear demonstration of the limitations of drawings. Also, no one could have predicted the catastrophic history that lay ahead. Guthrie, Francis First proposed in 1852, Kempe verified it in 1879, and Heawood denied it in 1890. The Four Color Problem was one of the most well-known discrete mathematics problems in the twentieth century, until it was supplanted in 1977 by Appel and Haken's Four Color Theorem.

**Basic Principles for Calculating Chromatic Numbers**

A few basic principles come again in many chromatic-number calculations. They merge to provide a direct approach connecting two steps,

- Upper Bound Shows $\chi(G) \leq k$, most often by exhibiting a proper k-coloring of G.
- Lower Bound Shows $\chi(G) \geq k$, most especially, by finding a sub graph that requires k colors.

The following result is an easy upper bound for $\chi(G)$, it is complemented by the easy lower bound of clique number $\omega(G)$.

**Proposition:**

Let G be a simple graph. Then $\chi(G) \leq \delta_{\max}(G) + 1$.

**Proof:**

The sequential coloring algorithm never uses more than $\delta_{\max}(G) + 1$ colors, no matter how the vertices are ordered, since a vertex cannot have more than $\delta_{\max}(G)$ neighbours.

**Proposition**:

Let G be a graph that has k mutually neighbouring vertices. Then $\chi(G) \geq k$

**Proof:**

Using less than k colors on graph G would result in a pair from the equally adjacent set of k vertices being assigned the similar colour.

**Properties of Chromatic Number** [Gary Chartrand, et al., 2009]

- $\chi(G) = 1$, iff G is completely disconnected.
- $\chi(G) \geq 3$, if G has an odd cycle, equivalently if G is not bipartite.
- $\chi(G) \leq \Delta(G) + 1$.

- $\chi(G) \leq \Delta(G)$, unless G is a complete graph or an odd cycle (Brookes Theorem).
- $\chi(G) \leq 4$ , for every planar graph (Four colour theorem).
- $\chi(G) \geq \omega(G)$ clique number.
- If G is a graph of order n and α is the independence number, then

$$n\ \alpha(G) \leq \chi(G) \leq n - \alpha(G) + 1.$$

- If every vertex of a graph G lies on at most k odd cycles for some non negative integer k, then

$$\chi(G) \leq 1\ 1+\sqrt{}\ 8k+9\ 2\ m$$

**Decision Problem** [Roopesh, N et al., 2010]

A decision problem is a language L of strings greater than an alphabet.

**P and NP**

P is the class of decision problems for which we can discover a answer in polynomial time. NP is the class of decision problems for which we can confirm solutions in polynomial time. NP represents for Non deterministic Polynomial time.

**NP-Hard**

If for every language $L'$ in NP, $L'$ is polynomial time reducible to L then a decision problem L is NP-hard. (when f(x) belongs to L , a language $L'$ polynomial time reducible to a language L if there be a polynomial time function f(x) from string to end string such that x is in $L'$).

**NP-Complete**

A decision problem L is NP-complete if it is in NP and it is NP-hard.

**Graph coloring is hard**

Computational complexity theory classifies problems according to their usage of resources (time, storage, etc.) which discusses some issues in complexity theory [Sipser, M 2005], for a comprehensive treatment. The turing engine is a well known formal model of a universal purpose computer. It is a made-up machine on which algorithms are run. The complexity class P is the set of decision problems (with a yes-no answer) which can be solved on a problems in which “yes” answers can be verified on a turing machine in polynomial time. Clearly P is contained in

NP. Whether P and NP are the same, is one of seven Millennium Prize Problems published by the Clay Mathematics Institute.

G is a graph indeed properly colorable with 3 colors, one such proper coloring is verifiable in polynomial time. This 3-Colorable is in NP. In fact it is in NP-Complete [Michael, R.G et al., 1979] the subset of NP problems such that if there is a problem in it which is also in P, then P is NP. The NP-Complete problems are the most composite problems in NP.

A problem M is considered NP-Hard if there is a NP-Complete problem N such that an instance of N can be converted into an instance of M in polynomial time.The NP-Hard problems are as complex as NP problems atleast. Even approximating the chromatic number to within $n^{1-\epsilon}$ where n is the number of vertices in G and $\epsilon$ is any positive number, is NP-Hard [Subhash, K 2001].

**Types of Coloring** [Gary Chartrand, et al., 2009]

Graph coloring is pretty old still a incredibly active field of research. The essential definitions together with some special conditions will form some types of colorings. Some variations of coloring had been found and studied by many researchers by different ways.

**Vertex coloring** [Bondy, J.A et al., 1976]

A **vertex coloring** of a graph is an assignment of colors to the vertices so that adjacent vertices have distinct colors. A k-coloring of a graph uses at most k colors. A 5-coloring of a graph is illustrated in Fig.2 (a), where positive integer designates colors. The vertex coloring of graph in Fig. 2(a) is not optimal since there is another vertex coloring of the graph with three colors as illustrated in Fig. 2(b). The smallest integer k such that a graph G has a k-coloring is called the chromatic number of G and is denoted by $\chi(G)$. If $\chi(G) = k$, the graph G is said to be k-chromatic.

Since the graph *G* in Fig. 2(a) has a triangle, one can observe that *G* cannot have a vertex coloring with less than three colors, and hence $\chi(G) = 3$ for the graph in Fig. 2(a). It is clear that $\chi(K_n) = n$. On the other hand, if *G* is a null graph then $\chi(G) = 1$. Every bipartite graph is 2-chromatic since to color the vertices in an independent set, we need only one color. Since a tree is a bipartite graph, every tree is 2-chromatic.

The definition of *k*-coloring implies that the *k*-coloring of a graph $G=(V,E)$ partitions the vertex set *V* into *k* independent set $V_1,V_2,...,V_k$ such that $G=V_1\cup V_2\cup ...V_k$. The independent sets $V_1,V_{2,}...V_k$ are called the color classes. The vertex coloring problem, i.e., coloring vertices of a graph *G* with $\chi(G)$ colors, is a NP-hard problem.

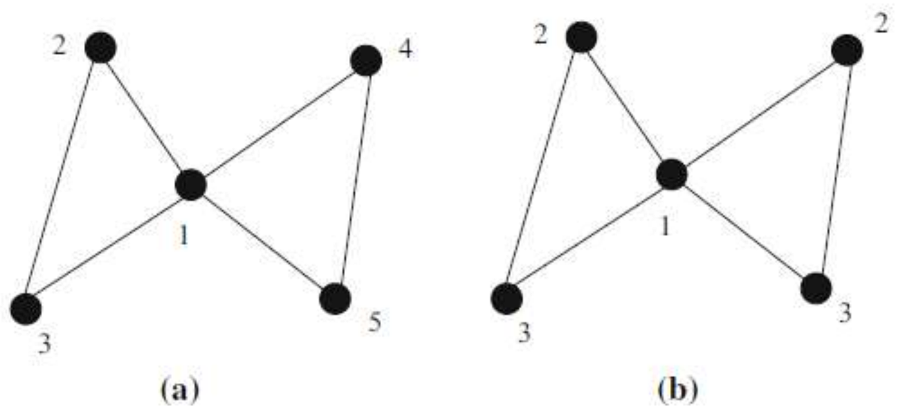


**Fig. 2: (a)** A 5-coloring and **(b)** a 3-coloring of a graph

**Lemma**

Every simple graph of the maximum degree $\Delta$ has a $(\Delta+1)$ coloring.

**Proof :**

We confirm the claim using induction on the number of vertices of G. Every simple graph of n $\leq$ 2 vertices has maximum degree 1 and the graph is 2-colorable. We now assume that n >2 and the claim is true for every graph of less than n vertices. Let G be a simple graph of n vertices, and let v be a vertex in G. Let $G'$ be the graph obtained from G by deleting v. Then $G'$ has n − 1 vertices and has maximum degree at most $\Delta$. By induction hypothesis, $G'$ is $(\Delta+1)$colorable, where the neighbours of v in $G'$ used at most $\Delta$ colors. Then a $(\Delta+1)$coloring of G can be obtained from the $(\Delta+1)$ coloring of $G'$ by coloring v with a different color from the vertices adjacent to v.

**Edge Coloring** [Tiago Januario et al., 2016]

An edge coloring of a graph is a handing over of colors to the edges so that the edges incident to a vertex has different colors. If *G* has an edge coloring with *k* colors then a graph *G* is *k*-edge colorable. The chromatic index of a graph *G* is *k* if

*G* is *k*-edge colorable but not *(k* − 1*)* edge colorable. The chromatic index of a graph is indicated by $\chi(G)$. The definition of *k*-edge coloring provides that the *k*-edge coloring of a graph $G=(V,E)$ partitions the edge set *E* into *k* matching's $E_1, E_2, \ldots, E_k$ such that $E = E_1 \cup E_2 \cup \ldots E_k$. The sets $E_1, E_2, \ldots, E_k$ are called the color classes.

**Interval edge/vertex coloring**

Interval edge/vertex coloring is a coloring of a graph G with colors 1, 2, ..., k such that the edges/vertices which are incident/adjacent to any vertex of G are distinct and consecutive.

**Equitable edge/vertex coloring**

If colors are assigned to the edges/vertices of a graph such that any two adjacent edges/vertices should not have the same color and the numbers of edges/vertices in any two color classes differ by at most one, this type of coloring is known as equitable edge/vertex coloring.

**Interval equitable edge/vertex coloring**

An interval equitable edge coloring is an assignment of colors (positive integers) to the edges of the graph if it satisfies the following conditions:

(i) No two adjacent edges have the same color

(ii) The set of colors defined on the edges incident to any vertex of the graph forms an interval

(iii) The cardinality of the edges in any two color classes differ by at most one.

**Total Coloring** [Somasundaram, K et al., 2019]

A total coloring of G is a mapping $f: V(G) \cup E(G) \rightarrow C$, where C is a set of colors, satisfying the following three conditions

(a) $f(u) \neq f(v)$ For any two adjacent vertices $u, v \in V(G)$.

(b) $f(e) \neq f(e')$ For any two adjacent edges $e, e' \in E(G)$ and

(c) $f(v) \neq f(e)$ For any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The total chromatic number of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring.

It is clear that $\Delta(G)+1 \leq \chi''(G)$. In the 1965's [Behzad, M and Vizing, V.G] independently conjectured that for every graph G $\chi''(G) \leq \Delta(G)+2$ . This conjecture is also called "Total Coloring Conjecture" (TCC). The TCC asserts that the graphs fall into two types. Type I graphs have $\chi''(G) = \Delta(G)+1$, where as G is called Type II if $\chi''(G) = \Delta(G)+2$.

The total graph T(G) of a graph G is a graph such that the vertex set of T(G) corresponds to the vertices and edges of G and two vertices are adjacent in T(G) iff their corresponding elements are either adjacent or incident in G.

A great graph is a graph in which the chromatic amount of every induced subgraph equals the size of the main clique of that sub graph.

**Fall coloring** [Dunbar, et al., 2000]

A fall coloring of a graph G is a good coloring such that every vertex of G has neighbours in all the other color classes.

**Fall achromatic coloring** [Roopesh, N et al., 2010]

A fall achromatic coloring is an exacting case of b-coloring in which every vertex is colourful. A fall achromatic number, the greatest cardinality of a fall coloring of G which we denote by $\psi_f(G)$ .

***K*-coloring** [Gary Chartrand, et al., 2009]

An oriented *k*-coloring of an oriented graph G is a partition of V (G) into k color classes such that:

(i) No two adjacent vertices belong to the same color class and

(ii) All the arcs linking two color classes have the same direction

**Star coloring** [Gary Chartrand, et al., 2009]

A star coloring of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

**Exact coloring** [Francis, P et al., 2018]

An exact coloring of a graph G is a proper vertex coloring in which every pair of colors appears on exactly one pair of adjacent vertices. That is, if a coloring is both harmonious and complete, then it is an exact coloring.

**Co-coloring** [Alkhateeb, M 2012]

Co-coloring of a graph G is an project of colors to the vertices such that each color class forms an independent set in G or in the complement of G. The co chromatic number $z(G)$ of G is the least number of colors needed in any co-colorings of G. The graphs with co chromatic number two are exactly the bipartite graphs.

**Dominator coloring** [Gera, R.M 2007]

A dominator coloring is a coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other class. The dominator chromatic number of a graph G, is denoted by $\chi_d(G)$ .

**Grundy coloring** [Alkhateeb, M 2012]

A grundy coloring of a graph is a proper vertex coloring $\phi : V \to N$ such that every vertex v has a neighbour of color i for all $1 \le i \le \phi(v)$.

**Harmonious coloring** [Bondy, J.A et al., 1976]

A harmonious coloring is a good vertex coloring in which every pair of colors appears on at most one pair of adjacent vertices. The harmonious chromatic number $\chi_H(G)$ of a graph G is the minimum number of colors needed for any harmonious coloring of G.

**Face Coloring (Map Coloring)** [Clark, J et al., 1991]

A face coloring of a plane graph is a coloring of its features such that no two adjacent faces get the similar color. A *k*-face coloring of a plane graph is a face coloring of the graph using *k* colors. If a plane graph admits a *k*-face coloring, then it is *k*-face colorable.

**Result:**

- All map can be colored in four or less colors.
- G is k-vertex colorable if and only if the dual graph of G is k-face colorable.

**Acyclic Coloring** [Francis, P et al., 2018]

An acyclic coloring of a graph *G* is a vertex coloring of *G* such that no cycle of *G* is bi-chromatic. That is, the vertices on a cycle in *G* cannot be colored with exactly two colors in an acyclic coloring of *G*. An acyclic *k*-coloring of *G* is an acyclic coloring of *G* using at most *k* colors. The smallest number of colors needed to acyclically color the vertices of a graph is called its acyclic chromatic number.

**Dynamic Colorings** [Jonathan Gross et al., 2004]

A graph G is defined to be a vertex-critical graph for dynamic colorings if $\chi_d(G-v) < \chi_d(G)$ for any vertex v of G. Any graph $G = K_{2k-1} - E(C_{2k-1})$ for $k = \chi_d(G) \geq 4$ is a vertex-critical graph for dynamic colorings, since the proper coloring specified for G and G − v is in each case also a dynamic coloring.

**Equitable Coloring** [Furmanczyk, H et al., 2018]

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots V_K$ such that each $V_i$ is an independent set and the condition $\left\| V_i \right| - \left| V_j \right\| \leq 1$ holds for every $i \neq j$, then G is said to be equitably *k*-colorable $(V_1, V_2, \ldots V_K)$ is called an equitable independence-partition. The smallest integer k for which G is equitable *k*− colorable is known as the equitable chromatic of G and denoted by $\chi(G)$.

**The b-Coloring Graph [**Irving et al., 1999]

The motivation for the b-chromatic number, as in the case of achromatic number, comes from algorithmic graph theory. Irving and Manlove introduced the concept of b-chromatic number by considering proper colorings that are minimal with respect to a partial order defined on the set of all partitions of $V(G)$.

**Definition:**

A b-coloring of a graph G is a proper coloring of the vertices of G such that each color class contains a vertex that has a neighbour in all other color classes. Any such vertex is called as a colourful vertex. The b-chromatic number ϕ (G) is the largest integer *k* such that G admits a b-coloring with *k* colors. For any graph G, $m(G)$ is the largest integer such that a graph G has at least $m(G)$ vertices with degree at least $m(G)-1$. The b-coloring number of several graphs using the parameter $m(G)$.

**Theorem:**

b-chromatic number is NP-complete. [Michael, R.G et al., 1979]

**Proof:**

In 2002, Kratochvil et al showed that for a d-regular graph G with at least d vertices, $\varphi(G) = d + 1$. In 2011, M. Kouider et al., obtained some lower bounds for the b-chromatic number of connected bipartite graphs. They also discuss some algorithmic consequences of such lower bounds on some subfamilies of connected bipartite graphs.

For a given graph G, it may be easily remarked that

$$\chi(G) \le \varphi(G) \le \Delta(G) + 1.$$

An important problem is to characterize those graphs G such that $\varphi(G) = \Delta(G) + 1$.

In fact, it was shown that deciding whether a graph admits a b-coloring with a given number of colors is a NP-complete problem, even for connected bipartite graphs [Kouider, M 2011]. One general upper bound for the b-chromatic number of a graph, has been proved to be very useful [Irving and Manlove 1999].

**Lemma:** [Kamalian, R.R 1989]

Let G be a non-trivial connected graph. Then $\varphi(G) = 2$ if and only if G is bipartite and has a full vertex in each part of the bipartition.

**Theorem:** [Karthick, T 2014]

Let G be a $K_{1,s}$ free graph where $s \ge 3$, then. $\varphi(G) \le (s-1)(\chi(G)-1)+1$.

**Theorem:** [Khachatrian, H et al., 2015]

Let G be a connected graph of stability number $\alpha$. Then $\varphi(G) \le n + 1 - \alpha$

**Example:**

The b-coloring of the grid graph $G_2 = P_2 \blacksquare P_4$. The following figure is an example of b-coloring of a graph $G_2$ with two colors, namely 1 and 2.

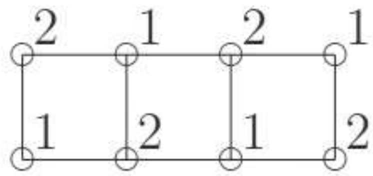


**Fig. 3:** b-coloring of $G_2$ with 2 colors

In the above coloring, all the vertices are colourful. The following figures illustrate the b-coloring of the same graph $G_2$ with three colors and four colors respectively.

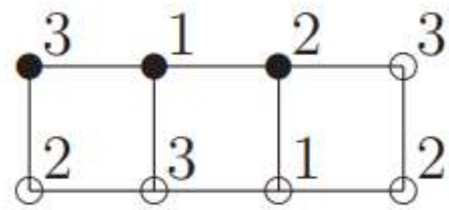


**Fig. 4:** b-coloring of $G_2$ with 3 colors

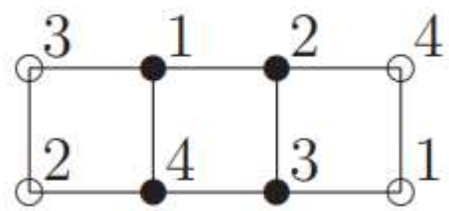


**Fig. 5:** b-coloring of $G_2$ with 4 colors

Note that for the graph $G_2, \chi(G_2) = 2$ . Since $\Delta(G_2) = 3$ , is easy to see that $\chi(G_2) \leq 4$. Hence the b-chromatic number $\chi(G_2) = 4$.

**Remark:**

The maximum degree of $G_1$ is $\Delta(G_1) = 4$ but it is not b-colorable with $\Delta(G_1) + 1 = 5$ colors. But $\Delta(G_2) = 3$ and it is b-colorable with $\Delta(G_2) + 1 = 4$ colors.

**APPLICATIONS OF GRAPH COLORINGS**

**Application 1: University Course Scheduling** [Bondy, J.A et al., 1976 and Aycan, E et al., 2006]

Suppose that the vertices of a simple graph G represent the courses at a university. Two vertices are close to this model when, for both classes, at least 1 student has pre-registered. Clearly, it would be undesirable for two such courses to be planned at the same time. Then the vertex-chromatic number $\chi$(G) gives the minimum number of class scheduling times in which to list the classes so that no student has a conflict between two courses.

**Application 2: Fast-Register Allocation for Computer Programming**

In some computers, there are a limited number of special registers that permit faster execution of arithmetic operations than ordinary memory locations. The program variables that are used most often can be declared to have the register storage class. Unfortunately, if the programmer declares more register variables than the number of hardware registers available, then the program execution may waste more time swapping variables between ordinary memory and the quick memory variables than is saved by using the fast registers. The programmer can check the designation of the register so that variables simultaneously active in various registers are allocated. The graph model has one vertex for each variable, with two vertices activated the corresponding variables simultaneously. Then the chromatic number equals the number of registers needed to avoid the exchange phenomena. There are many problems that can be characterized by a graph and whose solution involves finding the chromatic number of this graph.

**Application 3:** [Arthur Benjamin et al., 2015]

An eight faculty members are denoted by $f_1, f_2, f_3, \ldots f_8$ have been invited to discuss their topics of interest. It is decided that it would be more efficient to divide these faculty members into seven committees of three each, namely,

$$S_1 = \{f_1, f_2, f_3\}, S_2 = \{f_2, f_3, f_4\}, S_3 = \{f_4, f_5, f_6\}, S_4 = \{f_5, f_6, f_7\},$$

$$S_5 = \{f_1, f_7, f_8\}, S_6 = \{f_1, f_5, f_7\}, S_7 = \{f_2, f_5, f_8\},$$

each committee must meet as per the following time periods 9 - 10 am, 10 - 11 am, 11am - 12 noon, 2 - 3 pm, 3 - 4 pm, 4 - 5 pm, Then what is the earliest time that all committee meetings can be completed?

**Solution.**

No two committees can meet during the same time period if some faculty member belongs to both committees. Define a graph G by $V(G) = \{S_1, S_2, \ldots, S_7\}$

where two vertices $S_i$ and $S_j$ are adjacent if $S_i \cap S_j \neq \phi$; (and so $S_i$ and $S_j$ must meet at different times).

The graph G is shown in Figure 6 (a). The answer to this question is $\chi(G)$. Here, $\omega(G) = 4$ and so $\chi(G) \geq 4$. The 5-coloring in Figure 6(b) shows that $\chi(G) \leq 5$. The sub graph induced by $\{S_3, S_4, S_6, S_7\}$ is $K_4$ and so any vertex coloring of G must assign distinct colors to these four vertices, say $S_3 - 1, S_4 - 2, S_6 - 3, S_7 - 4$. If only four colors are used to color G, then we must have $S_5$ colored 1 and $S_2$ colored 2. However then, $S_1$ cannot be colored by any of 1, 2, 3 or 4. Since no 4-coloring of G is possible, $\chi(G) = 5$.

Possible meeting times for these committees are therefore

9 - 10 am: $S_1$     10 - 11 am: $S_6$     11am - 12 noon: $S_7$

2 - 3 pm: $S_2, S_4$     3 - 4 pm: $S_3, S_5$

and so all committee meetings are over by 4 pm.

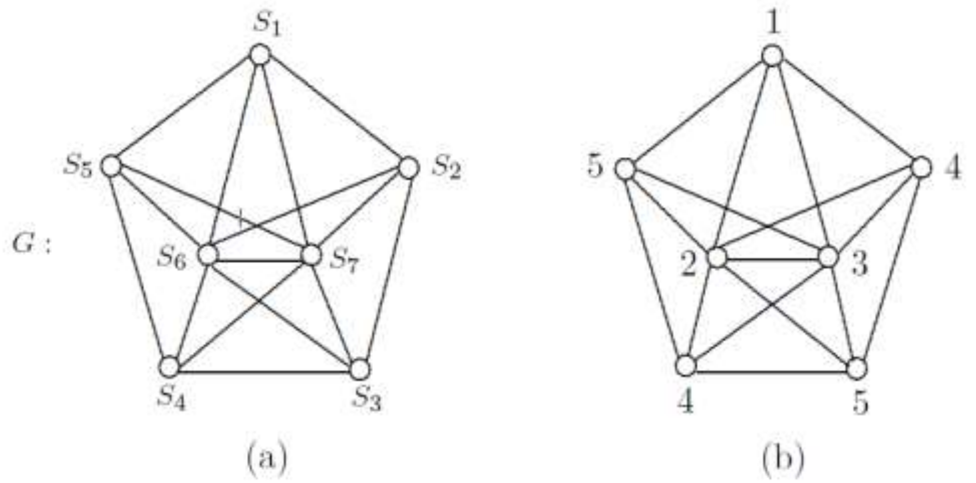


**Fig. 6:** The graph G is a 5-coloring of G

**Application 4:** [Arthur Benjamin et al., 2015] Figure 7 shows six traffic lanes $L_1, L_2, ..., L_6$ at the junction between two streets. A traffic signal is located at the intersection. During each phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the crossing into certain permitted lanes. What is the minimum number of phases needed for the traffic light so that all cars may proceed safely through the junction?

**Solution.**

A graph G is constructed with vertex set $V(G)=\{L_1, L_2,..., L_6\}$ , where $L_i$ is adjacent to $L_j\,(i \neq j)$ if cars in lanes $L_i$ and $L_j$ cannot proceed safely through the intersection at the same time. (See Figure.8 (a).) The minimum number of phases needed for the traffic light so that all cars may proceed, in time, through the intersection is $\chi(G)$. Since $\{L_1, L_2, L_4\}$ induces a triangle, $\chi(G)\geq 3$. Since there is a 3-coloring of G (see Figure 8 (b)), it follows that $\chi(G)=3$. For example, since $L_1$; $L_5$ and $L_6$ belong to the same color class, cars in those three lanes may proceed safely through the intersection at the same time.

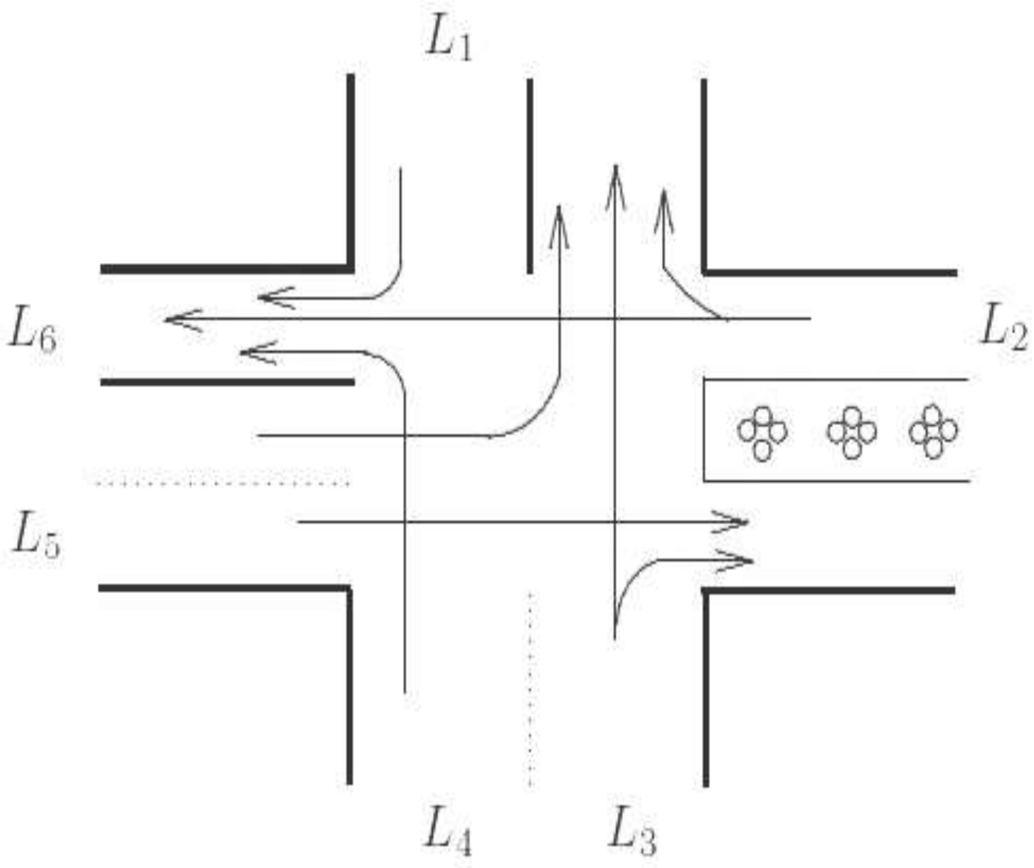


**Fig .7:** The Traffic lanes at street intersections in application.4

Therefore three possible phases for all cars to proceed safely through the intersection

Phase #1: $L_1, L_6$ , Phase #2: $L_{2,}\ L_5$ Phase #3: $L_3, L_4$ .

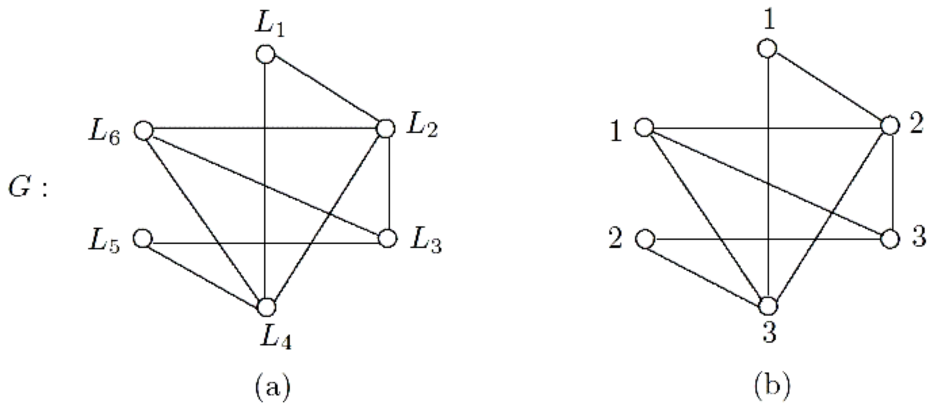


**Fig. 8:** The graph G is a 3-coloring of G