Chapter 7

Separation Axioms using λ_g^δ-closed sets

7.1 Introduction

Separation axioms using λ_g^δ-closed sets namely λ_g^δ-T_i, $i = 0, 1, 2$ are introduced. The properties and characterizations of λ_g^δ-closed sets are studied under δ-T_i, $i = 0, 1, 2$ and λ_g^δ-T_i, $i = 0, 1, 2$ axioms. Further the relationships of λ_g^δ-T_i, δ-T_i and T_i, $i = 0, 1, 2$ axioms are also found. λ_g^δ-compactness is characterized using the finite intersection property and λ_g^δ-connectedness is characterized using λ_g^δ-Frontier.

7.2 Behaviour of λ_g^δ-closed sets under δ-T_i, $i = 0, 1, 2$ Axioms

Theorem 7.2.1. *If (X, τ) is δ-T_0, then λ_g^δ-closures of distinct points are distinct.*

Proof. Let $x \neq y \in X$. Since (X, τ) is δ-T_0, there exists a δ-open set U such that $x \in U$ and $y \notin U$. Therein $X \setminus U$ is a δ-closed set containing y but not x. Since every δ-closed set is λ_g^δ-closed as well, $X \setminus U$ is a λ_g^δ-closed set containing y but not x. $\lambda_g^\delta cl\{y\}$ is the intersection of all λ_g^δ-closed sets containing y and therefore $y \in \lambda_g^\delta cl\{y\}$ but $x \notin \lambda_g^\delta cl\{y\}$ as $x \notin X \setminus U$. Hence $\lambda_g^\delta cl\{x\} \neq \lambda_g^\delta cl\{y\}$. □

Remark 7.2.2. The converse of the above theorem need not be true in general as observed from the following Example.

Example 7.2.3. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b, c\}\}$. Then λ_g^δ-closure of distinct points are distinct. But (X, τ) is not δ-T_0 as for $a \neq b$, there exists no δ-open set U such that $a \in U$ but $b \notin U$.

Theorem 7.2.4. *If (X, τ) is δ-T_1 space, every singleton is λ_g^δ-closed.*

Proof. Let $x \neq y \in X$. Since (X, τ) is δ-T_1, there exist δ-open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$ and $y \in V$. Therefore $y \in V \subseteq X \setminus \{x\}$ where $X \setminus \{x\} = \cup\{y|x \neq y\} \subseteq \cup\{V \mid V$ is δ-open such that $y \in V$ and $x \notin V\} = A(say)$ and $A \subseteq X \setminus \{x\}$ as $x \notin A$. Hence $X \setminus \{x\} = A$, which is the arbitrary union of δ-open sets so that $X \setminus \{x\}$ is δ-open. Therefore $X \setminus \{x\}$ is λ_g^δ-open and hence {x} is λ_g^δ-closed. □

Remark 7.2.5. The converse of the above theorem need not be true in general as observed from the following example.

Example 7.2.6. Let X and τ be defined as in Example 7.2.3. Then every singleton is λ_g^δ-closed but (X, τ) is not δ-T_1 as for $a \neq b$, there exist no δ-open sets U, V such that $a \in U$, $b \notin U$ and $b \in V$, $a \notin V$.

Theorem 7.2.7. *If (X, τ) is δ-T_2 space, for each $x \in X$, $\{x\} = \cap\{\lambda_g^\delta cl(U)|U$ is λ_g^δ-open in X and $x \in U\}$.*

Proof. Let $x \neq y \in X$. Since (X, τ) is δ-T_2, there exist disjoint δ-open subsets U and V such that $x \in U$ and $y \in V$. Clearly $X \setminus V$ is δ-closed and hence λ_g^δ-closed. Since $U \cap V = \phi$ and $U \in X \setminus V$, $\lambda_g^\delta cl(U) \subseteq \lambda_g^\delta cl(X \setminus V) = X \setminus V$. Since $y \notin X \setminus V$, $y \notin \lambda_g^\delta cl(U)$. Therefore, $y \notin \cap\{\lambda_g^\delta cl(U)|U$ is δ-open in X and $x \in U\} \supseteq \cap\{\lambda_g^\delta cl(U)|U$ is λ_g^δ-open in X and $x \in U\}$. Therefore $y \notin \cap\{\lambda_g^\delta cl(U)|U$ is λ_g^δ-open in X and $x \in U\} = \{x\}$, as every δ-T_2 is δ-T_1 implying every singleton is δ-closed and hence λ_g^δ-closed. □

7.3 Separation Axioms

Some important separation axioms of λ_g^δ-closed sets are introduced along with some important characterizations besides which the relationship between them is remarked.

Definition 7.3.1. A topological space (X, τ) is said to be $\boldsymbol{\lambda_g^\delta}$**-**$\boldsymbol{T_0}$ space if for each pair of distinct points $x, y \in X$, there exists a λ_g^δ-open set containing one point but not the other.

Example 7.3.2. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then (X, τ) is a λ_g^δ-T_0 space.

Theorem 7.3.3. *A topological space (X, τ) is λ_g^δ-T_0 iff λ_g^δ-closures of distinct points are distinct.*

Proof. *Necessity :* Let $x, y \in X$ such that $x \neq y$. Since (X, τ) is λ_g^δ-T_0, there exists a λ_g^δ-open set G such that $x \in G$ and $y \notin G$. Consequently, $X \setminus G$ is a λ_g^δ-closed set such that $x \notin X \setminus G$ and $y \in X \setminus G$. Since $\lambda_g^\delta cl\{y\}$ is the intersection of all λ_g^δ-closed sets containing y, $y \in \lambda_g^\delta cl\{y\}$ but $x \notin \lambda_g^\delta cl\{y\}$ as $x \in X \setminus G$. Therefore $\lambda_g^\delta cl\{x\} \neq \lambda_g^\delta cl\{y\}$.
Sufficiency : Let $\lambda_g^\delta cl\{x\} \neq \lambda_g^\delta cl\{y\}$ for $x \neq y$. Then there exists at least one point $z \in X$ such that $z \in \lambda_g^\delta cl\{x\}$ but $z \notin \lambda_g^\delta cl\{y\}$. We claim $z = x$ because if $x \in \lambda_g^\delta cl\{y\}$ then $\{x\} \subseteq \lambda_g^\delta cl\{y\} \Rightarrow \lambda_g^\delta cl\{x\} \subseteq \lambda_g^\delta cl\{y\} \Rightarrow z \in \lambda_g^\delta cl\{y\}$, a contradiction. Hence $x \notin \lambda_g^\delta cl\{y\} \Rightarrow x \in X \setminus \lambda_g^\delta cl\{y\}$, which is a λ_g^δ-open set containing x but not y. This implies (X, τ) is λ_g^δ-T_0. □

Definition 7.3.4. A topological space (X, τ) is said to be $\boldsymbol{\lambda_g^\delta}$**-**$\boldsymbol{T_1}$ space if for each pair of distinct points $x, y \in X$, there exist λ_g^δ-open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

Example 7.3.5. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then (X, τ) is a λ_g^δ-T_1 space.

Theorem 7.3.6. *If singletons are λ_g^δ-closed then (X, τ) is λ_g^δ-T_1 but not conversely.*

Proof. Suppose $\{x\}$ is λ_g^δ-closed, for all $x \in X$. Let $x, y \in X$ such that $x \neq y$ which implies $y \in X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a λ_g^δ-open set containing y but not x. Changing the role of x and y, $x \in X \setminus \{y\}$ and hence $X \setminus \{y\}$ is a λ_g^δ-open set containing x but not y. Therefore (X, τ) is λ_g^δ-T_1. □

Example 7.3.7. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ then (X, τ) is λ_g^δ-T_1 but not all singletons in (X, τ) are λ_g^δ-closed.

Definition 7.3.8. A topological space is said to be $\boldsymbol{\lambda_g^\delta}$-$\boldsymbol{T_2}$ space if for any pair of distinct points x and y in X, there exist disjoint λ_g^δ-open sets U and V such that $x \in U$ and $y \in V$.

Example 7.3.9. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then (X, τ) is a λ_g^δ-T_2 space.

Theorem 7.3.10. *In a topological space (X, τ), the following are equivalent.*

(i) (X, τ) is λ_g^δ-T_2.

(ii) For each $x \neq y$ in X, there exists a λ_g^δ-open set U such that $x \in U$ and $y \notin \lambda_g^\delta cl\{U\}$.

(iii) For each $x \in X$, $\{x\} = \cap\{\lambda_g^\delta cl(U) |$ U is λ_g^δ-open set in X and $x \in U\}$.

Proof. (i) ⇒ (ii) Let $x, y \in X$ such that $x \neq y$ then there exist disjoint λ_g^δ-open sets U and V such that $x \in U$ and $y \in V$. Clearly $X \setminus V$ is λ_g^δ-closed. Since $U \cap V = \phi$, $U \subseteq X \setminus V \Rightarrow \lambda_g^\delta cl(U) \subseteq \lambda_g^\delta cl(X \setminus V) = X \setminus V$. Now $y \notin X \setminus V \Rightarrow y \notin \lambda_g^\delta cl(U)$.
(ii) ⇒ (iii) For each $x \neq y$ in X, there exists a λ_g^δ-open set U such that $x \in U$ and $y \notin \lambda_g^\delta cl(U)$. Suppose there exists $z \notin \cap\{\lambda_g^\delta cl(U)|U$ is λ_g^δ-open in X and $x \in U\}$ such that $z \neq x$. Then by hypothesis, there exists a λ_g^δ-open set U_z containing x and $z \notin \lambda_g^\delta cl(U_z)$ which is a contradiction. Hence $\{x\} = \cap\{\lambda_g^\delta cl(U)|U$ is λ_g^δ-open set in X and $x \in U\}$.
(iii) ⇒ (i) Let $x, y \in X$ such that $x \neq y$. By hypothesis, there exists a λ_g^δ-open set U such that $x \in U$ and $y \notin \lambda_g^\delta cl(U) = V$(say). Therefore $y \in X \setminus V$, where $y \in X \setminus V$ is λ_g^δ-open. Thus there exist two disjoint λ_g^δ-open sets U and $X \setminus V$ such that $x \in U$ and $y \in X \setminus V$. Hence (X, τ) is λ_g^δ-T_2. □

Proposition 7.3.11. For a topological space (X, τ), the following conditions are true.

(i) Every δ-T_2 space is a λ_g^δ-T_2 space but not conversely.

(ii) Every δ-T_1 space is a λ_g^δ-T_1 space but not conversely.

(iii) Every δ-T_0 space is a λ_g^δ-T_0 space but not conversely.

Proof. (i) Let (X, τ) be a δ-T_2 space. Then for $x \neq y$, there exist disjoint δ-open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. By Proposition 2.2.3 (i), every δ-open set is λ_g^δ-open. Therefore there exist disjoint λ_g^δ-open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

(ii) and (iii) are similar to (i).

□

Remark 7.3.12. The following Example demonstrates that the converse statements of the above theorem are not true.

Example 7.3.13. (i) Let X, τ be defined as in Example 7.3.9. Then (X, τ) is a λ_g^δ-T_2 space but not a δ-T_2 space.

(ii) Let X, τ be defined as in Example 7.3.5. Then (X, τ) is a λ_g^δ-T_1 space but not a δ-T_1 space.

(iii) Let X, τ be defined as in Example 7.3.2. Then (X, τ) is a λ_g^δ-T_0 space but not a δ-T_0 space.

Proposition 7.3.14. For a topological space (X, τ), the following conditions are true.

(i) Every λ_g^δ-T_2 space is a λ_g^δ-T_1 space.

(ii) Every λ_g^δ-T_1 space is a λ_g^δ-T_0 space but not conversely.

(iii) Every λ_g^δ-T_2 space is a λ_g^δ-T_0 space.

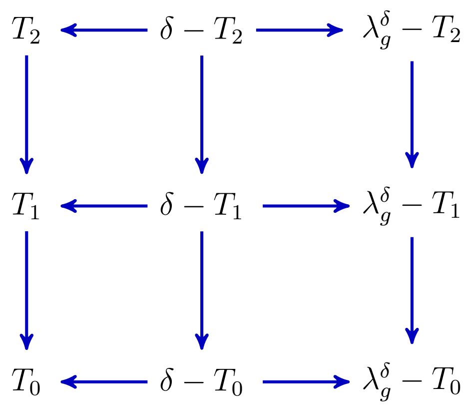
Proof. Follows from the definitions. □

Remark 7.3.15. The converse of (ii) in the above theorem is not true as seen from the following example.

Example 7.3.16. (ii) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then (X, τ) is λ_g^δ-T_0 but not λ_g^δ-T_1 as for $c \neq d$, there exist no λ_g^δ-open sets U, V such that $c \in U$, $d \notin U$ and $d \in V$, $c \notin V$.

Remark 7.3.17. The following figure portrays the dependence relationship between the newly introduced separation axioms and some of the other already existing separation axioms. δ-$T_i \to \lambda_g^\delta$-T_i, for $i = 0, 1, 2$ is obvious from the fact that every δ-closed set is λ_g^δ-closed.

Figure 7.1: **Relation between the Separation Axioms**



7.4 λ_g^δ-compactness and λ_g^δ-connectedness

Definition 7.4.1. A collection $\mathcal{A}$ of a topological space (X, τ) is said to cover X (or) to be a covering of X if the union of elements of $\mathcal{A}$ is equal to X. It is said to be a ***λ_g^δ-open covering of*** X if its elements are λ_g^δ-open subsets of (X, τ).

Definition 7.4.2. A non-empty collection $\{A_i | i \in I\}$ of λ_g^δ-open sets in a topological space (X, τ) is called a ***λ_g^δ-open cover*** of a subset B of (X, τ) if $B \subseteq \cup\{A_i | i \in I\}$.

Definition 7.4.3. A topological space (X, τ) is called ***λ_g^δ-compact*** if every λ_g^δ-open cover of X has a finite subcover.

Definition 7.4.4. A subset B of a topological space (X, τ) is called ***λ_g^δ-compact***

relative to X if for every collection $\{A_i | i \in I\}$ of λ_g^δ-open subsets of (X, τ) such that $B \subseteq \cup\{A_i | i \in I\}$ there exists a finite subset I_0 of I such that $B \subseteq \cup\{A_i | i \in I_0\}$.

Theorem 7.4.5. *A λ_g^δ-closed subset A of a λ_g^δ-compact space X is λ_g^δ-compact relative to X.*

Proof. Let A be a λ_g^δ-closed subset of a λ_g^δ-compact space X. Let Ω be an λ_g^δ-open cover of A in X. Then $\Omega \cup \{X \setminus A\}$ is a λ_g^δ-open cover of (X, τ). Since (X, τ) is λ_g^δ-compact, Ω has a finite subcover (say) $\{P_1, P_2, P_3, ..., P_n\} = \Omega_1$.
case(i) : If $X \setminus A \notin \Omega_1$ then $A \in \Omega_1$. Thus Ω_1 is a finite subcover of A.
case(ii) : If $X \setminus A \in \Omega_1$ then $\Omega_1 \cup \{X \setminus A\}$ is a finite subcover of A.
Therefore A is λ_g^δ-compact relative to X. □

Theorem 7.4.6. *Let $f : X \to Y$ be a surjective, λ_g^δ-continuous function. If X is λ_g^δ-compact then Y is compact.*

Proof. Let $\{V_i | i \in I\}$ be an open cover of Y. Since f is λ_g^δ-continuous, $\{f^{-1}(V_i) | i \in I\}$ is a λ_g^δ-open cover of X. Since X is λ_g^δ-compact, there exists a finite subcover $\{f^{-1}(V_1), f^{-1}(V_2), ..., f^{-1}(V_n)\}$ of $\{f^{-1}(V_i) | i \in I\}$. Since f is surjective, $\{V_1, V_2, ..., V_n\}$ is a finite open cover of Y. Hence Y is compact. □

Theorem 7.4.7. *Let $f : X \to Y$ be a surjective, quasi λ_g^δ-continuous function. If X is compact then Y is λ_g^δ-compact.*

Proof. Let $\{V_i | i \in I\}$ be a λ_g^δ-open cover of Y. Since f is quasi λ_g^δ-continuous, $\{f^{-1}(V_i) | i \in I\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{f^{-1}(V_1), f^{-1}(V_2), ..., f^{-1}(V_n)\}$ of $\{f^{-1}(V_i) | i \in I\}$. Since f is surjective, $\{V_1, V_2, ..., V_n\}$ is a finite subcover of Y and hence Y is λ_g^δ-compact. □

Corollary 7.4.8. Let $f : X \to Y$ be a surjective, perfectly λ_g^δ-continuous function. If X is compact then Y is λ_g^δ-compact.

Proof. Since every perfectly λ_g^δ-continuous function is a quasi λ_g^δ-continuous function, the result follows. □

Theorem 7.4.9. *If $f : X \to Y$ is λ_g^δ-irresolute and $B \subseteq X$ is λ_g^δ-compact relative to X then $f(B)$ is λ_g^δ-compact relative to Y.*

Proof. Let $\{A_i | i \in I\}$ be an λ_g^δ-open cover of $f(B)$ i.e., $f(B) \subseteq \cup\{A_i | i \in I\} \Rightarrow B \subseteq \cup\{f^{-1}(A_i) | i \in I\}$. Since B is λ_g^δ-compact relative to X, $\cup\{f^{-1}(A_i) | i \in I\}$ has a finite subcover $\cup\{f^{-1}(A_i) | i \in I_0\}$(say), where I_0 is a finite subset of I such that $B \subseteq \cup\{f^{-1}(A_i) | i \in I_0\} \Rightarrow f(B) \subseteq \cup\{A_i | i \in I_0\}$, where $\{A_i | i \in I_0\}$ is a finite subcover of $\{A_i | i \in I\}$. Therefore $f(B)$ is λ_g^δ-compact relative to Y. □

Theorem 7.4.10. *A topological space X is λ_g^δ-compact iff every family of λ_g^δ-closed subsets of X with the finite intersection property has a non-empty intersection.*

Proof. Suppose a collection $\mathcal{G}$ of subsets of X is taken, let $\mathcal{H} = \{P \setminus G \mid G \in \mathcal{G}\}$ be the collection of the complements of $G \in \mathcal{G}$. Then the following results are true:

(i) $\mathcal{G}$ is a collection of λ_g^δ-open sets iff $\mathcal{H}$ is a collection of λ_g^δ-closed sets.

(ii) The collection $\mathcal{G}$ covers X iff the intersection of all elements of $\mathcal{H}$ is non-empty.

(iii) The finite sub-collection $\{G_1, G_2, ..., G_n\}$ of $\mathcal{G}$ covers X iff the intersection of the corresponding elements $H_i = X \setminus G_i$ of $\mathcal{H}$ is empty.

Statement (i) is obvious whereas (ii) and (iii) follow from DeMorgan's law: $X \setminus \bigcup\limits_{\alpha \in J} A_\alpha = \bigcap\limits_{\alpha \in J} (X \setminus A_\alpha)$. Now we prove the theorem by contra positive approach which is equivalent to the following:

Let $\mathcal{G}$ be any collection of λ_g^δ-open sets in X. If no finite sub-collection of $\mathcal{G}$ covers X, then $\mathcal{G}$ does not cover X. Now applying (i) to (iii), we observe that this statement is equivalent to the following:

Given any collection $\mathcal{H}$ of λ_g^δ-closed sets, if every finite intersection of elements of $\mathcal{H}$ is non-empty then the intersection of all elements of $\mathcal{H}$ is non-empty. Hence the condition in the theorem is obtained. □

Definition 7.4.11. A topological space (X, τ) is called ***λ_g^δ-connected*** if X cannot be expressed as a union of two disjoint, non-empty, λ_g^δ-open sets.

Theorem 7.4.12. *For a topological space X, the following are equivalent.*

(i) X is λ_g^δ-connected;

(ii) X and ϕ are the only subsets of X which are both λ_g^δ-open and λ_g^δ-closed;

(iii) Each λ_g^δ-continuous function of X into a discrete space Y with atleast two points is a constant function.

Proof. (i) $\Rightarrow$ (ii) Let U be a λ_g^δ-open and λ_g^δ-closed subset of X. Then $X \setminus U$ is both λ_g^δ-open and λ_g^δ-closed in X. Since X is the disjoint union of λ_g^δ-open sets U and $X \setminus U$, one of these must be empty. That is, $U = \phi$ or $U = X$.
(ii) $\Rightarrow$ (i) Let X and ϕ be the only subsets of X which are both λ_g^δ-open and λ_g^δ-closed. Suppose $X = A \cup B$, where A and B are two non-empty, disjoint, λ_g^δ-open subsets of X. Since $X \setminus B = A$, A is λ_g^δ-closed. By assumption, $A = \phi$ or $A = X$, which is a contradiction. Hence X is λ_g^δ-connected.
(ii) $\Rightarrow$ (iii) Let $f : X \to Y$ be a λ_g^δ-continuous function and Y be a discrete space with at least two points. Then for each $y \in Y$, $\{y\}$ is both open and closed. Since f is λ_g^δ-continuous, $f^{-1}\{y\}$ is λ_g^δ-open as well as λ_g^δ-closed in X and $X = \cup\{f^{-1}\{y\}|y \in Y\}$. By hypothesis $f^{-1}\{y\} = \phi$ or X for each $y \in Y$. If $f^{-1}\{y\} = \phi$, for all $y \in Y$ then f will not be a function. If $f^{-1}\{y\} = X$, for a single point $y \in Y$ then there cannot exist another point $y_1 \in Y$ such that $f^{-1}\{y_1\} = X$. Hence there exists only one $y \in Y$ such that $f^{-1}\{y\} = X$ and $f^{-1}\{y_1\} = \phi$, where $y_1 \in Y$ and $y_1 \neq y$. This proves that f is a constant function.
(iii) $\Rightarrow$ (ii) Let U be λ_g^δ-open as well as λ_g^δ-closed in X. Suppose $U \neq \phi$. Let $f : X \to Y$ be a λ_g^δ-continuous function defined by $f(U) = \{y\}$ and $f(X \setminus U) = \{w\}$, for $y, w \in Y$ such that $y \neq w$. By assumption, f is a constant function. Thus $y = w$ and therefore $U = X$. □

Theorem 7.4.13. *Let $f : X \to Y$ be a surjective, λ_g^δ-continuous function. If X is λ_g^δ-connected then Y is connected.*

Proof. Suppose Y is not connected. Then $Y = A \cup B$, where A and B are two disjoint, non-empty, λ_g^δ-open subsets of Y. Since f is surjective and λ_g^δ-continuous, $X =$

$f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint, non-empty and λ_g^δ-open sets in X. But this is a contradiction to the fact that X is λ_g^δ-connected. Hence Y is connected. $\square$

Theorem 7.4.14. *Let $f : X \to Y$ be a surjective, λ_g^δ-irresolute function. If X is λ_g^δ-connected then Y is λ_g^δ-connected.*

Proof. Similar to previous Theorem. $\square$

Theorem 7.4.15. *Every topological space which is both ${}_{\lambda_g^\delta}T_\delta$ and connected is λ_g^δ-connected.*

Proof. The proof follows from the definitions. $\square$

Definition 7.4.16. A subset A of a topological space (X, τ) is called

(i) λ_g^δ***-regular closed*** if $A = \lambda_g^\delta cl(\lambda_g^\delta int(A))$.

(ii) λ_g^δ***-regular open*** if $A = \lambda_g^\delta int(\lambda_g^\delta cl(A))$.

(iii) λ_g^δ***-regular*** if it is both λ_g^δ-regular closed and λ_g^δ-regular open.

Theorem 7.4.17. *A subset A of a topological space (X, τ) is λ_g^δ-regular iff $\lambda_g^\delta Fr(A) = \phi$.*

Proof. *Necessity :* Let A be λ_g^δ-regular then (i) $A = \lambda_g^\delta cl(\lambda_g^\delta int(A))$ and (ii) $A = \lambda_g^\delta int(\lambda_g^\delta cl(A))$. Now, (i) $\Rightarrow \lambda_g^\delta cl(A) = \lambda_g^\delta cl(\lambda_g^\delta cl(\lambda_g^\delta int(A))) = \lambda_g^\delta cl(\lambda_g^\delta int(A)) = A$ and (ii) $\Rightarrow \lambda_g^\delta int(A) = \lambda_g^\delta int(\lambda_g^\delta int(\lambda_g^\delta cl(A))) = \lambda_g^\delta int(\lambda_g^\delta cl(A)) = A$. Thus $\lambda_g^\delta Fr(A) = \lambda_g^\delta cl(A) \setminus \lambda_g^\delta int(A) = \phi$.
Sufficiency : Let $\lambda_g^\delta Fr(A) = \phi$. This implies $\lambda_g^\delta cl(A) = \lambda_g^\delta int(A)$ which means $\lambda_g^\delta int(A) = A = \lambda_g^\delta cl(A)$. Thus we have $\lambda_g^\delta cl(\lambda_g^\delta int(A)) = \lambda_g^\delta cl(A) = A$ and $\lambda_g^\delta int(\lambda_g^\delta cl(A)) = \lambda_g^\delta int(A) = A$. Hence A is λ_g^δ-regular. $\square$

Theorem 7.4.18. *For a topological space (X, τ), the following are equivalent:*

(i) X is λ_g^δ-connected.

(ii) X and ϕ are the only λ_g^δ-regular subsets of X.

(iii) Each λ_g^δ-continuous function of X into a discrete space Y with atleast two points is a constant function.

(iv) Every non-empty proper subset has a non-empty λ_g^δ-Frontier.

Proof. (i) $\Rightarrow$ (ii) Let U be a λ_g^δ-regular subset of X. Then $X \setminus U$ is both λ_g^δ-open and λ_g^δ-closed in X. Since X is the disjoint union of λ_g^δ-open sets U and $X \setminus U$, X is not λ_g^δ-connected which is a contradiction to (i) and hence one of these must be empty. That is, $U = \phi$ or $U = X$.

(ii) $\Rightarrow$ (i) Suppose $X = A \cup B$, where A and B are non-empty, λ_g^δ-open sets. Then $A = X \setminus B$ is λ_g^δ-closed. Then A is a non-empty, proper subset that is λ_g^δ-regular. This is a contradiction to (ii). Hence X is λ_g^δ-connected.

(ii) $\Rightarrow$ (iii) Let $f : X \to Y$ be a λ_g^δ-continuous function and Y be a discrete space with atleast two points. Then for each $q \in Y$, $\{q\}$ is both open and closed. Since f is λ_g^δ-continuous, $f^{-1}\{q\}$ is λ_g^δ-open as well as λ_g^δ-closed in X and $X = \cup\{f^{-1}\{q\} \mid q \in Y\}$. By hypothesis $f^{-1}\{q\} = \phi$ or X for each $q \in Y$. If $f^{-1}\{q\} = \phi$, for all $q \in Y$ then f will not be a function. If $f^{-1}\{q\} = X$, for a single point $q \in Y$ then there cannot exist another point $q_1 \in Y$ such that $f^{-1}\{q_1\} = X$. Hence there exists only one $q \in Y$ such that $f^{-1}\{q\} = X$ and $f^{-1}\{q_1\} = \phi$, where $q_1 \in Y$ and $q_1 \neq q$. This proves that f is a constant function.

(iii) $\Rightarrow$ (ii) Let U be a λ_g^δ-regular subset in X. We wish to prove that the only λ_g^δ-regular subsets are ϕ and X. Suppose $U \neq \phi$ then we claim $U = X$. Let $q_1, q_2 \in Y$. Define $f : X \to Y$ by

$$f(p) = \begin{cases} q_1, & p \in U \\ q_2, & \text{otherwise.} \end{cases}$$

Then for any open set V in Y,

$$f^{-1}(V) = \begin{cases} U, & \text{if } V \text{ contains } q_1 \text{ only} \\ X \setminus U, & \text{if } V \text{ contains } q_2 \text{ only} \\ X, & \text{if } V \text{ contains } q_1, q_2 \\ \phi, & \text{otherwise.} \end{cases}$$

In all the cases, $f^{-1}(V)$ is λ_g^δ-open in X. Also, f is a non-constant, λ_g^δ-continuous function. This is a contradiction. Hence the only λ_g^δ-clopen subsets of X are ϕ and X.
(ii) $\Rightarrow$ (iv) Let A be a non-empty, proper subset of X. Suppose $\lambda_g^\delta Fr(A) = \phi$. Then A is both λ_g^δ-open and λ_g^δ-closed which is a contradiction to (ii).
(iv) $\Rightarrow$ (ii) Suppose that A is a non-empty, proper subset of X which is both λ_g^δ-closed and λ_g^δ-open. This implies A is λ_g^δ-regular and hence by Theorem 7.4.17, $\lambda_g^\delta Fr(A) = \phi$, which is a contradiction. □

Theorem 7.4.19. *Let $f : X \to Y$ be a λ_g^δ-open, λ_g^δ-closed(resp. δ-open, δ-closed) injection. If Y is λ_g^δ-connected then X is also λ_g^δ-connected.*

Proof. Let A be a λ_g^δ-regular set in X. Since A is λ_g^δ-open and λ_g^δ-closed, $f(A)$ is λ_g^δ-regular in Y. Since Y is λ_g^δ-connected, $f(A) = \phi$ or Y. Since f is an injection, $A = \phi$ or X. Hence X is λ_g^δ-connected. □

Theorem 7.4.20. *If $f : X \to Y$ is a totally λ_g^δ-continuous function from a λ_g^δ-connected space X to Y then Y has the indiscrete topology.*

Proof. Let V be open in Y. Since f is a totally λ_g^δ-continuous function, $f^{-1}(V)$ is λ_g^δ-regular in X. Since X is λ_g^δ-connected, $f^{-1}(V) = \phi$ or X. Since f is an injection, $V = \phi$ or Y. Hence Y has the indiscrete topology. □

Theorem 7.4.21. *If $f : X \to Y$ is a strongly λ_g^δ-continuous bijective function and Y is a topological space with atleast two points then X is not λ_g^δ-connected.*

Proof. Let $y \in Y$. Then $f^{-1}(\{y\})$ is a non-empty proper subset of X which is λ_g^δ-regular, as f is strongly λ_g^δ-continuous. Therefore X is not λ_g^δ-connected from Theorem 7.4.12. □

Theorem 7.4.22. *If a topological space X is almost weakly Hausdorff and connected then it is λ_g^δ-connected.*

Proof. Suppose X is not λ_g^δ-connected. Then $X = A \cup B$, where A and B are non-empty, disjoint, λ_g^δ-open sets of X. Since X is almost weakly Hausdorff, A and B are open in X[?]. This contradicts the connectedness of X. Hence X is λ_g^δ-connected. □

7.5 λ_g^δ-Lindelof

Definition 7.5.1. A topological space (X, τ) is λ_g^δ***-Lindelof*** if every λ_g^δ-open cover of X contains a countable subcover.

Theorem 7.5.2. *Every λ_g^δ-compact space is λ_g^δ-Lindelof.*

Proof. Follows from the fact that every finite collection is countable. □

Theorem 7.5.3. *A surjective, λ_g^δ-irresolute image of a λ_g^δ-Lindelof space is λ_g^δ-Lindelof.*

Proof. Let $f : X \to Y$ be a λ_g^δ-irresolute, surjection and X be a λ_g^δ-Lindelof space. Let $\{V_i \mid i \in I\}$ be an λ_g^δ-open cover of Y. Then $\{f^{-1}(V_i) \mid i \in I\}$ is a λ_g^δ-open cover of X. Since X is λ_g^δ-Lindelof, it has a countable subcover namely $\{f^{-1}(V_1), f^{-1}(V_2), ..., f^{-1}(V_n), ...\}$. Since f is surjective, $\{V_1, V_2, ..., V_n, ...\}$ is a countable subcover of Y. Hence Y is λ_g^δ-Lindelof. □

Theorem 7.5.4. *A surjective, λ_g^δ-continuous image of a λ_g^δ-Lindelof is Lindelof.*

Proof. Let $f : X \to Y$ be a surjective, λ_g^δ-continuous function from a λ_g^δ-Lindelof space X to Y. Let $\{V_i \mid i \in I\}$ be an open cover of Y. Since f is λ_g^δ-continuous, $\{f^{-1}(V_i) \mid i \in I\}$ is a λ_g^δ-open cover of X. Since X is λ_g^δ-Lindelof, there exists a countable subcover $\{f^{-1}(V_1), f^{-1}(V_2), ..., f^{-1}(V_n), ...\}$ of $\{f^{-1}(V_i) \mid i \in I\}$. Since f is surjective, $\{V_1, V_2, ..., V_n, ...\}$ is a countable subcover of Y. Hence Y is Lindelof. □

Theorem 7.5.5. *A surjective, quasi λ_g^δ-continuous image of a Lindelof space is λ_g^δ-Lindelof.*

Proof. Let $f : X \to Y$ be a surjective, quasi λ_g^δ-continuous function and $\{V_i \mid i \in I\}$ be a λ_g^δ-open cover of Y. Since f is quasi λ_g^δ-continuous, $\{f^{-1}(V_i) \mid i \in I\}$ is an open cover of X. Since X is Lindelof, there exists a countable subcover $\{f^{-1}(V_1), f^{-1}(V_2), ...,$ $f^{-1}(V_n), ...\}$ of $\{f^{-1}(V_i) \mid i \in I\}$. Since f is surjective, $\{V_1, V_2, ..., V_n, ...\}$ is a countable subcover of Y and hence Y is λ_g^δ-Lindelof. □

7.6 $\lambda_g^\delta G_i$ - Axioms

Definition 7.6.1. A topological space (X, τ) is called a ***$\lambda_g^\delta G_1$-space*** if for any point $p \in X$ and any connected subset M of X with $p \notin M$, there exist λ_g^δ-open sets U and V such that $p \in U, M \subseteq V, U \cap M = \phi$ and $\{p\} \cap V = \phi$.

Example 7.6.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}\}$. Then (X, τ) is a $\lambda_g^\delta G_1$-space as for a connected set $M = \{a, b\}$ with $c \notin \{a, b\}$, there exist λ_g^δ-open sets $U = \{c\}$ and $V = \{a, b\}$ such that $c \in \{c\}, \{a, b\} \subseteq \{a, b\}, \{c\} \cap \{a, b\} = \phi$.

Theorem 7.6.3. *If every connected subset of X is λ_g^δ-closed then for any two disjoint connected subsets M and N of X, there exist λ_g^δ-open sets U and V such that $M \subseteq U$, $N \subseteq V$, $U \cap N = \phi$ and $M \cap V = \phi$.*

Proof. Let M and N be any two disjoint connected subsets of X. Then by hypothesis, M and N are λ_g^δ-closed. This implies $X \setminus M$ and $X \setminus N$ are λ_g^δ-open sets containing N and M respectively, as M and N are disjoint. Now let $U = X \setminus N$ and $V = X \setminus M$. Then $N \cap U = V \cap M = \phi$. □

Theorem 7.6.4. *If for any two disjoint connected subsets M and N of X, there exist λ_g^δ-open sets U and V such that $M \subseteq U$, $N \subseteq V$, $U \cap N = \phi$ and $V \cap M = \phi$ then X is a $\lambda_g^\delta G_1$-space.*

Proof. Follows from the fact that every singleton is connected and taking $M = \{p\}$. □

Definition 7.6.5. Let X be a topological space and Q be its subspace. Then a subset A of Q is λ_g^δ-open in Q if A can be written as $A = Q \cap K$ where K is λ_g^δ-open in X.

Theorem 7.6.6. *Every δ-open subspace Q of a $\lambda_g^\delta G_1$-space X is $\lambda_g^\delta G_1$.*

Proof. Let A be a connected subset in Q. Then A is connected in X as well. Let

$q \in Q \subseteq X$ such that $q \notin A$. Then by hypothesis, there exist λ_g^δ-open sets U and V such that $q \in U$, $A \subseteq V$, $U \cap A = \phi$ and $\{q\} \cap V = \phi$. By the definition of subspace topology, $Q \cap U$ and $Q \cap V$ are λ_g^δ-open sets in Q such that $q \in Q \cap U$, $A \subseteq Q \cap V$ and $(Q \cap U) \cap A = \{q\} \cap (Q \cap V) = \phi$. Hence Q is a $\lambda_g^\delta G_1$-space. □

Theorem 7.6.7. *A bijective, continuous and λ_g^δ-irresolute image of a $\lambda_g^\delta G_1$-space is a $\lambda_g^\delta G_1$-space.*

Proof. Let $\alpha : X \to Y$ be a bijective, continuous function and M be a connected subset in X such that $p \notin M$. Then $\alpha(M)$ is connected in Y. Since α is one to one and onto, $\alpha(p) \notin \alpha(M)$. Now since Y is $\lambda_g^\delta G_1$, there exist λ_g^δ-open sets U and V in Y such that $\alpha(p) \in U, \alpha(M) \subseteq V$ and $U \cap \alpha(M) = \{\alpha(p)\} \cap V = \phi$. Since α is λ_g^δ-irresolute, $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are λ_g^δ-open sets in X with $p \in \alpha^{-1}(U)$, $M \subseteq \alpha^{-1}(V)$ and $\alpha^{-1}(U) \cap M = \{p\} \cap \alpha^{-1}(V) = \phi$. Hence X is a $\lambda_g^\delta G_1$-space. □

Definition 7.6.8. A topological space (X, τ) is called ***$\lambda_g^\delta G_2$-space*** if for every connected set F and a point $p \notin F$, there exist λ_g^δ-open sets U and V such that $p \in U$, $F \subseteq V$ and $U \cap V = \phi$.

Example 7.6.9. Let X and τ be defined as in Example 7.6.2. Then (X, τ) is a $\lambda_g^\delta G_2$-space as for a connected set $F = \{a, b\}$ with $c \in \{a, b\}$, there exist λ_g^δ-open sets $U = \{c\}$ and $V = \{a, b\}$ such that $c \in \{c\}, \{a, b\} \subseteq \{a, b\}, \{c\} \cap \{a, b\} = \phi$.

Theorem 7.6.10. *Every $\lambda_g^\delta G_2$-space is a $\lambda_g^\delta T_2$-space.*

Proof. Let X be a $\lambda_g^\delta G_2$-space with $p \neq q \in X$. Then $p \notin \{q\}$, which is a connected set. By hypothesis, there exist λ_g^δ-open sets U and V such that $p \in U$, $\{q\} \subseteq V$ and $U \cap V = \phi$. Therefore there exist λ_g^δ-open sets U and V such that $p \in U$, $q \in V$. Hence X is a $\lambda_g^\delta T_2$-space. □

Theorem 7.6.11. *A δ-open subspace of a $\lambda_g^\delta G_2$-space is $\lambda_g^\delta G_2$.*

Theorem 7.6.12. *If a topological space X is $\lambda_g^\delta G_2$ then for any point $p \in X$ and any connected subset M not containing p, $\lambda_g^\delta cl(U) \cap M = \phi$, where U is a λ_g^δ-open set* *containing p.*

Proof. Let M be a connected subset of X such that $p \notin M$. Since X is a $\lambda_g^\delta G_2$-space, there exist disjoint, λ_g^δ-open sets U and V such that $p \in U, M \subseteq V$ and $U \cap V = \phi$. This implies $U \subseteq X \setminus V$ and hence $\lambda_g^\delta cl(U) \subset \lambda_g^\delta cl(X \setminus V) = X \setminus V$, as $X \setminus V$ is λ_g^δ-closed. Further $\lambda_g^\delta cl(U) \cap M = \phi$, as $M \subseteq V$. □