GENERAL SERVICE QUEUEING SYSTEM WITH STATE DEPENDENT ARRIVAL, VACATION AND SERVER BREAKDOWN

By

SARANYA, R.

(11 PM 14)

A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION
FOR WOMEN, COIMBATORE – 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY 2013

GENERAL SERVICE QUEUEING SYSTEM WITH STATE DEPENDENT ARRIVAL, VACATION AND SERVER BREAKDOWN

By

SARANYA, R.
(11 PM 14)

A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR
WOMEN, COIMBATORE – 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS
MAY 2013

Parvathi 6/5/13
SIGNATURE OF THE
HEAD OF THE DEPARTMENT

N. Balamani 6/5/2013
SIGNATURE OF THE
GUIDE

ACKNOWLEDGEMENT

ACKNOWLEDGEMENT

First and foremost, the investigator is extremely thankful to the **LORD ALMIGHTY** for his graces and blessings showered on her.

The author takes immense pleasure in thanking **Thiru T.S.K.MEENAKSHI SUNDARAM**, M.A., M.Phil., Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing the conducive infrastructure for the conduct of the research study.

The author would like to thank **Dr. (Mrs.) SHEELA RAMACHANDRAN** M.Sc., P.G. Dip., Ph.D. (Avinashilingam), Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the support extended by her throughout the study.

The author extends her heartfelt thanks to **Dr. (Tmt.) GOWRI RAMAKRISHNAN**, M.Sc. (Madras), M.Phil., Ph.D. (Avinashilingam), Registrar, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement given by her during the investigation.

My words never fail to express my deep sense of gratitude to **Hony. COL. Dr. (Tmt.) SAROJA PRABHAKARAN,** M.A., Dip.Ed., (Madras), Ph.D., (Mother Teresa), Former vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and The Director, Halls of Residence, Avinashilingam Education Trust Institutions- Hostel for Women, Coimbatore-

641043, for all the necessary support and guidance towards the completion of the study.

The author would like to thank **Dr.P.SANDHANA KRISHNAN**, Ph.D (Tamil Nadu Agricultural University,Coimbatore),Director (research and consultancy), Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for all the encouragement.

The author is immensely pleased to express her deep sense of gratitude to **Dr. (Tmt.) R.PARVATHAM**, M.Sc., Dip.Ed., M.Phil. (Madras), Ph.D. (Avinashilingam), Dean, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore for all the encouragement.

The investigator would like to thank **Dr. (Tmt.) A.PARVATHI**, M.Sc., Dip.Ed., M.Phil. (Madras), Ph.D. (Bharathiar), Professor and Head of the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her excellent advice and valuable guidance extended throughout the study.

The author is deeply indebted to her thesis advisor **Mrs.N.BALAMANI**, M.Sc., M.Phil. (Avinashilingam), Assistant Professor, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the invaluable help, guidance and persistent efforts.

The investigator would like to express her sincere thanks to all the **STAFF MEMBERS OF DEPARTMENT OF MATHEMATICS,** Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, who were responsible for the good finish of this dissertation.

Words fail to express her deep indebtedness to her **LOVING PARENTS, BROTHER, FRIENDS** and **ALL WELL WISHERS** for being the motivating forces behind this dissertation and providing moral support and encouragement in carrying out work.

CONTENT

CONTENT

INTRODUCTION

INTRODUCTION

Queueing theory deals with the analysis of queues (or waiting lines) where customers wait to receive a service. Queueing theory has been used in operations research, manufacturing and systems analysis. Traditional queueing theory problems refer to customers visiting a store, analogous to requests arriving at a device.

A queueing system can be described as customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served.

CHARACTERISTICS OF QUEUEING THEORY

The basic features which characterize a system are

- Arrival Pattern
- Service Pattern
- Queue Discipline
- The Number of Service Channels
- System Capacity

ARRIVAL PATTERN

The arrival pattern describes the way in which the customers arrive and joining the queueing system. Generally, it is not possible to find out or to observe and control the moment of customers arrival for service. Hence, the arrival pattern to a queueing system is measured in terms of the average number of arrivals per some unit of time, mean arrival rate or by the average time between successive arrivals, mean inter-arrival time.

Arrival may be either single or batches of variable or fixed size. An arrival pattern that does not change with time is called a stationary arrival pattern. In this case, a steady state condition occurs. If the arrival pattern is time dependent, then the input process is called non-stationary.

SERVICE PATTERN

The service pattern describes the manner in which service is rendered to the arrivals. Customers may be served either singly or in batches of variable or fixed size. The amount of time which a customer takes to be serviced by the server is called the service time.

QUEUE DISCIPLINE

The queue discipline is the method by which customers are selected from the queue for processing by the service mechanisms (also called servers). The queue discipline is normally **First-In-First-Out** (FIFO), where the customers are processed in the order in which they arrived in the queue such that the head of the queue is always processed next. Most queueing models assume FIFO as the queue discipline. Some other common usage are **Last In First Out** (LIFO), selection for **Service In Random Order** (SIRO) and a variety of priority schemes, where customers are given priorities upon entering the system.

THE NUMBER OF SERVICE CHANNELS

The number of servers refers to the number of parallel nodes, which can service customers simultaneously. In telephone systems describe trunks, tone detectors, tone generators and time slots.

The service channels may be arranged in parallel or in series or combination of both depending on the design of the system's service mechanism.

In parallel channels, a number of channels provide identical service facilities, so that several customers may be served simultaneously. In case of series channels, a customer must pass successively through the ordered channels before service is completed.

A queueing system is called single server model, when the system has one server only and when the system has two or more parallel servers, it is called as multi server model.

SYSTEM CAPACITY

Maximum number of customers in the system can be either finite or infinite. In some situations, only limited numbers of customers are allowed to enter the system unless the number becomes less than the limiting value.

KENDALL’S NOTATION

Kendall introduced a notation that is commonly used to describe or classify the type of a queueing system.

According to the notation, a queueing system is described by the string (A/B/X/Y/Z).

A – Denotes the distribution of inter arrival times.

B – Denotes the distribution of service times.

X – Denotes the number of servers.

Y – Denotes the maximum size of waiting line in the finite case (If $Y=\infty$, then this letter is omitted).

Z – Denotes the queue discipline and if the service discipline is FCFS then Z is omitted.

Special letters used to symbolize the inter-arrival time distribution (A) and service time distribution (B) are as follows

M – Exponential distribution.

D – Deterministic distribution.

G – General (any arbitrary probability distribution).

E_k – Erlang-k distribution.

EXAMPLE

- M/M/1 denotes a single-server queue with Poisson arrival process and exponential service time with infinite buffer and FIFO service ord
- M/G/1 denotes a single server processor sharing queue with Poisson arrivals and generally distributed customer service time requirement.

MEAN QUEUE SIZE

The average number of customers waiting in a queue is called mean queue size.

MEAN SYSTEM SIZE

The average number of customers waiting in the system (including the one in service) is called mean system size.

WAITING TIME

The waiting time of a customer in a queue is the time that a customer has to wait in the queue before starting his service. The waiting time of a customer in queue along with his service time is called waiting time in the system.

QUEUE LENGTH

Queue length is the number of customers actually waiting [may include the one who are being served].

TRANSIENT AND STEADY STATES

Queueing theory analysis involves the study of systems (operating characteristics) over time. A queueing system is said to be in **transient state** when its operating characteristics are dependent of time. A queueing system is said to be in **steady state** when its operating characteristics are independent of time.

SERVER’S VACATION

There are queueing models in which the server on completion of service to the existing customer continues to stay in the empty system, awaiting new arrivals. From practical considerations; it may not always be worthwhile to keep servers unnecessarily idle. In such situations, the server may utilize his idle time in a useful and optimal way to perform additional jobs or for preventive maintenance work and it is termed as server’s vacation.

SINGLE VACATION

After completing service, if the server finds an empty queue, he leaves for a vacation. On retuning form vacation, if the server finds that the number of customers in the queue is not to the required level for service, he may decide to stay back in the system itself waiting for the customers to start the next service. The server will take next vacation only after performing at least one service. This type of server’s vacation is called single vacation.

MULTIPLE VACATION

In some cases, after returning from a vacation if the server does not find sufficient number of customer, then the server leaves for another vacation and repeats his vacations until he finds the sufficient number of customers. This type of server’s vacation is called multiple vacation.

REVIEW OF LITERATURE

Queueing system with server vacations have been studied by numerous researchers including Levy and Yechiali (1975), Doshi (1986) and Tian and Zhang (2006).Takagi (1994) studied M/G/1/N queue with server vacation and exhaustive service and obtained the distribution of the unfinished work , the virtual waiting time and the real waiting time ,etc.

The steady state behavior of an arbitrary service time queue with deterministic server vacation was analyzed by Madan (1999).The queueing system with setup and vacation was considered by Choudhury (2000). Further, Choudhury (2002) examined a queueing system with two different vacation times under multiple vacation policy.

Zhang et al (2005) analyzed an M/M/1/N queue with balking, reneging and server vacations and derived the matrix form solution of the steady state probabilities and formulated a cost model to determine the optimal service rate. Boxma et al (2008) studied the length of a vacation and steady state work load distribution both for single and multiple vacations.

In real life situations, a queueing system might suddenly breakdown and hence the server will not be able to provide service unless the system is repaired. Wang (1995) proposed an N-policy M/M/1 queueing system with server breakdowns and obtained analytic closed form solution. Srivastava and Jain (1999) analyzed an optimal N-policy model for single Markovian queue with breakdown repair and state dependent arrival rate and obtained steady state for various operational characteristic and optimal value of N under a linear cost structure.

Ke (2003) proposed M/G/1 queueing system with server vacation, start up and breakdowns and obtained the system total expected cost function per unit time under optimal control mechanism. Single server queueing system with the homogeneous breakdowns and deterministic repair times was analyzed by Madan (2003).Wang (2004) worked on the M/G/1 queueing system with second optimal service and server breakdowns. Ke and Pearn (2004) studied the management policy of M/M/1 queueing service system with heterogeneous arrivals under N policy in which the server is characterized by breakdowns and vacations and derived the distribution of the system size and mean queue length.

Thangraj and Vanitha (2010) discussed the single server model with two stages of heterogeneous service with different service time distribution subject to random breakdowns and compulsory service vacation with arbitrary vacation periods. Gray et al (2000, 2004) studied the vacation queueing model with service breakdown under the assumption of different arrival rates and obtained formulas for queue length distribution and the mean queue length for M/M/1.

The operating characteristic for the heterogeneous batch arrival queue with start up and breakdown and performed a sensitivity analysis among the optimal value of N, specific values of system parameters , and the cost elements of M^XM/1queue were studied by Ke and Wang (2003).

Jain and Agarwal (2009) analyzed the $M^X/M/1$ queueing system with multiple types of server breakdown under N-policy and provided the numerical results to demonstrate the effects of various parameters on the system performance characteristics. Maraghi et al (2009) have studied batch arrival queueing system with random breakdowns and Bernoulli schedule random vacation having general vacation time.

PROFILE OF THE WORK

In chapter two “Finite capacity queueing system with vacations and server breakdowns” presented by Ghimire and Ritu Basnet (2011) is analyzed.

The paper entitled “Analysis of M/G/1 queueing model with state dependent arrival and vacation” studied by Charan Jeet Singh, Madhu Jain and Binay Kumar (2012) is considered in chapter three.

CHAPTER-II

CHAPTER-II

FINITE CAPACITY QUEUEING SYSTEM WITH VACATIONS AND SERVER BREAKDOWNS

A single server finite capacity queueing system with state dependent Poisson arrivals, multiple vacations and server breakdown is analyzed in this chapter .The service times and repair times are exponentially distributed. Using partial generating function method ,queue length distribution, average queue length, average number of customers in the system, average waiting time for a customer in queue and in the system are derived.

MODEL DESCRIPTION

Customers arrive in the system in Poisson fashion at rate λ_0 during vacation, faster rate λ_f during active service and slower rate $\lambda_s \geq 0$ during the breakdown. Customers are served exponentially with the rate μ .The server completely stops serving customers during a vacation and start serving whenever number of customers $N(\geq 1)$ in the system. Once service starts, there can be an interruption due to server breakdown at rate b and it is immediately repaired exponentially with the rate r. As soon as the repair process completes, the server starts to serve the same interrupted customer.

DEFINITIONS AND NOTATIONS

$p(0,i)$ = The steady state probability that there are i customers in the queue and server is on vacation, $0 \le i \le N$.

$p(1,i)$ = The steady state probability that there are i customers in the system during active service, $1 \le i \le N$.

$p(2,i)$ = The steady state probability that there are i customers in the system during repair process, $1 \le i \le N$

EQUATIONS GOVERNING THE SYSTEM

According to the mathematical model mentioned above, the system has the following set of steady state equations.

$$\lambda_0 p(0,0) = \mu p(1,1) \qquad (1)$$

$$(\lambda_0 + v)p(0,i) = \lambda_0 p(0,N-1), \qquad 1 \le i < N \qquad (2)$$

$$v p(0,N) = \lambda_0 p(0,N-1) \qquad (3)$$

$$(\lambda_f + \mu + b)p(1,1) = v p(0,1) + \mu p(1,2) + r p(2,1) \qquad (4)$$

$$(\lambda_f + \mu + b)p(1,i) = \lambda_f p(1,i-1) + v p(o,i) + \mu p(1,i+1) + r p(2,i), \qquad 2 \le i < N \qquad (5)$$

$$(\mu + b)p(1,N) = \lambda_f p(1,N-1) + v p(o,N) + r p(2,N) \qquad (6)$$

$$(\lambda_s + r)p(2,1) = b p(1,1) \qquad (7)$$

$$(\lambda_s + r)p(2,i) = b p(1,i) + \lambda_s p(2,i-1), \qquad 2 \le i < N \qquad (8)$$

$$r p(2,N) = b p(1,N) + \lambda_s p(2,N-1) \qquad (9)$$

GENERATING FUNCTIONS OF QUEUE LENGTH

Denote the probability generating functions,

$$\left.\begin{aligned} F_0(z) &= \sum_{i=0}^{N} p(0,i)z^i \\ F_f(z) &= \sum_{i=1}^{N} p(1,i)z^i \\ F_s(z) &= \sum_{i=1}^{N} p(2,i)z^i \end{aligned}\right\} \qquad (10)$$

From equation (1), we get

$$p(1,1) = \frac{\lambda_0}{\mu} p(0,0) \qquad (11)$$

From equation (4), we have

$$p(0,N) = \frac{\lambda_0}{v} p(0,N-1) \qquad (12)$$

Substituting equation (11) in to (7), we obtain

$$(\lambda_s + r)p(2,1) = \frac{b\lambda_0}{\mu} p(0,0)$$

$$p(2,1) = \frac{b\lambda_0}{\mu(\lambda_s + r)} p(0,0) \qquad (13)$$

From equation (2), we get

$$\begin{aligned} p(0,i) &= \frac{\lambda_0}{\lambda_0 + v} p(0,i-1), && 1 \le i < N \\ &= \rho_o^i p(0,0), && 1 \le i < N \qquad (14) \end{aligned}$$

Where $\rho_0 = \dfrac{\lambda_0}{\lambda_0 + v}$

Using equations (12) and (14) in equation (10), we obtain

$$F_0(z) = \sum_{i=1}^{N} p(0,i)z^i$$

$$= p(0,0)+p(0,1)z+p(0,2)z^2+.......+p(0,N-1)z^{N-1}+p(0,N)z^N$$

$$= p(0,0)+\rho_0^1\,p(0,0)\,z+\rho_0^2\,p(0,0)\,z^2+.......+\rho_0^{N-1}\,p(0,0)\,z^{N-1}+p(0,N)z^N$$

$$= p(0,0)\left\{1+\rho_0^1 z+\rho_0^2 z^2+.......+\rho_0^{N-1}\,z^{N-1}\right\}+p(0,N)z^N$$

$$= p(0,0)\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}\right\}+\frac{\lambda_0}{v}p(0,N-1)z^N$$

$$= p(0,0)\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}\right\}+\frac{\lambda_0\,\rho_0^{N-1}}{v}p(0,0)z^N$$

$$= p(0,0)\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}+\frac{\lambda_0\rho_0^{N-1}}{v}z^N\right\} \qquad (15)$$

Multiplying equation (5) by z^i, and summing for i = 2, 3 …N-1 and using the generating function defined in equation (10) we get,

$$\left\{\mu+b-\frac{\mu}{z}-\lambda_f(z-1)\right\}F_f(z) = (\lambda_f+\mu+b)p(1,1)z-\frac{\mu}{z}\left\{p(1,1)z+p(1,2)z^2\right\}+r\,F_s(z)$$

$$+v\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}+\frac{\lambda_0\rho_0^{N-1}}{v}z^N-1-\rho_0 z\right\}p(0,0)-rp(2,1)z$$

(16)

Multiplying equation (8) by z^i, summing for i = 2, 3…... N-1 and using equation (10), we obtain

$$\sum_{i=2}^{N-1}(\lambda_s+r)p(2,i)z^i = \sum_{i=2}^{N-1}b\,p(1,i)z^i+\sum_{i=2}^{N-1}\lambda_s\,p(2,i-1)z^i$$

$$(\lambda_s + r)\sum_{i=2}^{N-1} p(2,i)z^i \quad = \quad b\sum_{i=2}^{N-1} p(1,i)z^i + \lambda_s \sum_{i=2}^{N-1} p(2,i-1)z^i$$

$$(\lambda_s + r)F_s(z) \quad = \quad bF_f(z) + \lambda_s z F_s(z)$$

$$(\lambda_s + r - z\lambda_s)F_s(z) \quad = \quad bF_f(z)$$

$$F_s(z) \quad = \quad \frac{b}{(\lambda_s + r - z\lambda_s)}F_f(z) \tag{17}$$

Substituting equations (17) into (16), we get

$$\frac{(z-1)Q(z)}{z(\lambda_s + r - z\lambda_s)}F_f(z) \quad = \quad (\lambda_f + \mu + b)p(1,1)z - \frac{\mu}{z}\left\{p(1,1)z + p(1,2)z^2\right\}$$
$$+ v\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z} + \frac{\lambda_0 \rho_0^{N-1}}{v}z^N - 1 - \rho_0 z\right\}p(0,0) - rp(2,1)z \tag{18}$$

where

$$\varphi(z) \quad = \quad \lambda_f \lambda_s z^2 - (\lambda_f \lambda_s + \lambda_f r + b\lambda_s + \mu\lambda_s)z + \mu(\lambda_s + r) \tag{19}$$

Substituting z = 1 in equation (18), we obtain

$$(\lambda_f + \mu + b)p(1,1) - \mu\{p(1,1) + p(1,2)\} - rp(2,1) + v\left\{\frac{1-\rho_0{}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\}p(0,0) = 0$$

$$\mu p(1,2) \quad = \quad (\lambda_f + \mu + b)p(1,1) - rp(2,1) - \mu p(1,1) + v\left\{\frac{1-\rho_0{}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\}p(0,0)$$

$$\mu p(1,2) \quad = \quad (\lambda_f + b)p(1,1) - rp(2,1) + v\left\{\frac{1-\rho_0{}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\}p(0,0)$$

$$= \quad (\lambda_f + b)p(1,1) - \frac{rb}{(\lambda_s + r)}p(1,1) + v\left\{\frac{1-\rho_0{}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\}p(0,0)$$

$$\mu p(1,2) \;=\; \left(\lambda_f + b - \frac{r\,b}{(\lambda_s + r)}\right)p(1,1) + v\left\{\frac{1-{\rho_0}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\} p(0,0) \tag{20}$$

Substituting equation (20) in (18), we get

$$\frac{(z-1)Q(z)}{z(\lambda_s + r - z\lambda_s)}F_f(z) \;=\; (\lambda_f + \mu + b)p(1,1)z + v\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z} + \frac{\lambda_0\rho_0^{N-1}}{v}z^N - 1 - \rho_0 z\right\}p(0,0)$$

$$-\,\frac{\mu}{z}\,p(0,0) - r\,p(2,1)z - \frac{1}{z}\left[\lambda_f + b - \frac{rb}{(\lambda_s + r)}\right]p(1,1)$$

$$-\,\frac{v}{z}\left\{\frac{1-{\rho_0}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\}z^2\,p(0,0)$$

$$=\; (\lambda_f + \mu + b)p(1,1)z + v\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z} + \frac{\lambda_0\rho_0^{N-1}}{v}z^N - 1 - \rho_0 z\right\}p(0,0)$$

$$-\frac{1}{z}\left(\lambda_f + b - \frac{rb}{(\lambda_s + r)}\right)p(1,1) - r\,p(2,1)z - \frac{v}{z}\left\{\frac{1-{\rho_0}^N}{1-\rho_0} + \frac{\lambda_0\rho_0^{N-1}}{v} - 1 - \rho_0\right\}z^2\,p(0,0) \tag{21}$$

Consider the following term in equation (21) and simplifying we get,

$$\left[v\,\frac{\lambda_0}{v}\,\rho_0^{N-1}z^N - \frac{v}{z}\,\frac{\lambda_0}{v}\,\rho_0^{N-1}z^2\right]p(0,0) \;=\; \lambda_0\,\rho_0^{N-1}\,[z^N - z]p(0,0) \tag{22}$$

Similarly,

$$-v\,p(0,0) - v\left[\frac{-1}{2}\right]z^2\,p(0,0) \;=\; v\,[z-1]\,p(0,0) \tag{23}$$

$$v\left[\frac{1-(\rho_0 z)^N}{1-\rho_0 z} - \frac{v}{z}\left(\frac{1-{\rho_0}^N}{1-\rho_0}\right)z^2\right]p(0,0) \;=\; v\left[\frac{1-(\rho_0^N z^N)}{1-\rho_0 z} - z\left(\frac{1-{\rho_0}^N}{1-\rho_0}\right)\right]p(0,0)$$

$$= v\left[\frac{1-(\rho_0^N z^N)(1-\rho_0)-z(1-\rho_0^N)(1-\rho_0 z)}{(1-\rho_0 z)(1-\rho_0)}\right]p(0,0)$$

$$= v\left[\frac{1-\rho_0^N z^N-\rho_0+\rho_0^{N+1}z^N-z-\rho_0 z^2+\rho_0^N z-\rho_0^{N+1}z^2}{(1-\rho_0 z)(1-\rho_0)}\right]p(0,0)$$

$$= v\left[\frac{\rho_0^{N+1}[z^N-z^2]-\rho_0^N[z^N-z]+\rho_0[z^2-1]-(z-1)}{(1-\rho_0 z)(1-\rho_0)}\right]p(0,0) \tag{24}$$

$$-v\rho_0 z\,p(0,0)-\frac{1}{z}(-v\rho_0 z^2)p(0,0) = [-v\rho_0 z+v\rho_0 z]\,p(0,0)$$

$$=0 \tag{25}$$

$$(\lambda_f+\mu+b)p(1,1)z-\frac{\mu}{z}\,p(1,1)z-\frac{1}{z}\left[\lambda_f+b-\frac{rb}{(\lambda_s+r)}\right]z^2p(1,1)-rp(2,1)z$$

$$= (\lambda_f+\mu+b)p(1,1)z-\frac{\mu}{z}\,p(1,1)z-\frac{1}{z}\left[\lambda_f+b-\frac{rb}{(\lambda_s+r)}\right]z^2\,p(1,1)-\left[\frac{rb}{(\lambda_s+r)}\right]z\,p(1,1)$$

$$= (\lambda_f+\mu+b)p(1,1)z-\frac{\mu}{z}\,p(1,1)z-\left[\lambda_f+b-\frac{rb}{(\lambda_s+r)}\right]zp(1,1)-z\left[\frac{rb}{(\lambda_s+r)}\right]p(1,1)z$$

$$= \lambda_f\,p(1,1)z+\mu p(1,1)z+bp(1,1)z-\mu p(1,1)-\left[\lambda_f+b-\frac{rb}{(\lambda_s+r)}\right]p(1,1)z-\frac{rb}{(\lambda_s+r)}\,p(1,1)z$$

$$= \mu p(1,1)\,z-\mu p(1,1)$$

$$= \mu[z-1]\,p(1,1) \tag{26}$$

Substituting equations (22) to (26) in (21) and simplifying we get

$$\frac{(z-1)Q(z)}{z(\lambda_s + r - z\lambda_s)}F_f(z) \;=\; \mu[z-1]p(1,1) + \lambda_0\,\rho_0^{N-1}[z^N - z]p(0,0) + v(z-1)p(0,0)$$

$$+\,v\left\{\frac{\rho_0^{N+1}[z^N - z^2] - \rho_0^N[z^N - z] + \rho_0[z^2 - 1] - (z-1)}{(1-\rho_0 z)(1-\rho_0)}\right\}p(0,0) \tag{27}$$

$$\frac{Q(z)}{(\lambda_s + r - z\lambda_s)}F_f(z) \;=\; (\lambda_0 + v)\,z\,p(0,0) + \lambda_0\,\rho_0^{N-1}\varphi(z)p(0,0)$$

$$+\,v\left\{\frac{\rho_0^{N+1}\varphi(z) - \rho_0^{N+1}z^2 - \rho_0^N\varphi(z) + \rho_0 z^2 + \rho_0 z - z}{(1-\rho_0 z)(1-\rho_0)}\right\}p(0,0) \tag{28}$$

where $\varphi(z) \;=\; z^N + z^{N-1} + ... + z^2$ (29)

From equation (28), we obtain

$$F_f(z) \;=\; \frac{(\lambda_s + r - z\lambda_s)\lambda_0}{Q(z)(1-\rho_0 z)}\left[(1-\rho_0^N)z^2 - (z-1)\,(\,z + \rho_0^N\varphi(z)\,)\,\right]p(0,0) \tag{30}$$

For $\lambda_s > 0$, discriminant Δ of the quadratic expression (19) satisfies

$$\lambda_s^2 b^2 + \lambda_s^2\mu^2 + \lambda_f^2\lambda_s^2 + \lambda_f^2 r^2 - 2\lambda_f\lambda_s^2\mu - 2\lambda_f\lambda_s\mu r + 2\lambda_f^2\lambda_s r$$

$$=\; \lambda_s^2 b^2 + (\lambda_f\lambda_s + \lambda_f r - \lambda_s\mu)^2 > 0$$

So the equation $Q(z) \;=\; 0$ has two distinct real roots given by

$$z_1, z_2 \;=\; \frac{(\lambda_f\lambda_s + \lambda_f r + \lambda_s\mu + b\lambda_s) \pm \sqrt{(\lambda_f\lambda_s + \lambda_f r + \lambda_s\mu + b\lambda_s)^2 - 4\lambda_f\lambda_s\mu(r + \lambda_s)}}{2\lambda_f\lambda_s}$$

Hence product and sum of the roots are

$$z_1 z_2 = \frac{\mu(r+\lambda_s)}{\lambda_f \lambda_s}$$

$$z_1 + z_2 = \frac{\lambda_f \lambda_s + \lambda_f r + \lambda_s \mu + b \lambda_s}{\lambda_f \lambda_s}$$

In order for the steady state queue length distribution to exist, both roots of equation $Q(z)=0$ must be greater than 1.

Since in $Q(z)$, the co efficient of z^2 is positive, the two roots of $Q(z)=0$ will be greater than 1.if $Q(1)>0$ and $Q'(1)<0$ .Since $Q(1) = r\mu - b\lambda_s - \lambda_f r$

We assume that $r\mu > b\lambda_s + \lambda_f r$ or $\frac{b\lambda_s}{r\mu} + \frac{\lambda_f}{\mu} < 1$ (31)

The above equation implies that $\mu > \lambda_f$, so if (30) holds, then

$$Q'(1) = \lambda_s(\lambda_f - \mu) - b\lambda_s - \lambda_f r < 0$$

Thus, if we assume that (31) holds, then the roots z_1 and z_2 of $Q(z)=0$ will be greater than 1.

From equation (29), we have for z = 1

$$\varphi(1) = N - 1 \quad (32)$$

$$\varphi'(1) = \frac{N(N+1)}{2} - 1 \quad (33)$$

Let F (z) denote the probability generating function of the queue size irrespective of the state of the system.

$$F(z) = F_0(z) + F_s(z) + F_f(z) \quad (34)$$

Substituting equation (15), (17) and (30) in equation (34), we get

$$F(z) \;=\; \left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}+\frac{\lambda_0\,\rho_0^{N-1}}{v}z^N\right\}p(0,0)+\left[\frac{b}{\lambda_s+r-z\lambda_s}\right]F_f(z)$$

$$+\frac{(\lambda_s+r-z\lambda_s)\lambda_0}{Q(z)(1-\rho_0 z)}\left[(1-\rho_0^N)z^2-(z-1)\left\{z+\rho_0^N\varphi(z)\right\}\right]p(0,0)$$

$$F(z) \;=\; \left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}+\frac{\lambda_0\,\rho_0^{N-1}}{v}z^N\right\}p(0,0)$$

$$+\frac{(b+\lambda_s+r-z\lambda_s)\lambda_0}{Q(z)(1-\rho_0 z)}\left[(1-\rho_0^N)z^2-(z-1)\left\{z+\rho_0^N\varphi(z)\right\}\right]p(0,0) \tag{35}$$

$$\text{Since}\quad \rho_0 \;=\; \frac{\lambda_0}{\lambda_0+v}$$

$$\frac{1}{\rho_0} \;=\; \frac{\lambda_0+v}{\lambda_0} \;=\; 1+\frac{v}{\lambda_0}$$

$$\frac{v}{\lambda_0} \;=\; 1-\frac{1}{\rho_0} \;=\; \frac{\rho_0-1}{\rho_0}$$

$$\frac{\lambda_0}{v} \;=\; \frac{\rho_0-1}{\rho_0} \tag{36}$$

Using equation (36) in (35) we get

$$F(z) \;=\; \frac{\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}+\frac{\rho_0\,\rho_0^{N-1}z^N}{1\text{-}\rho_0}\right\}Q(z)(1-\rho_0 z)p(0,0) + (b+\lambda_s+r-z\lambda_s)\lambda_0\,[(1-\rho_0^N)z^2-(z-1)(z+\rho_0^N\,\varphi(z))]p(0,0)}{Q(z)(1-\rho_0 z)}$$

$$F(z) = \frac{\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z} + \frac{\rho_0^N z^N}{1-\rho_0}\right\} Q(z)(1-\rho_0 z)p(0,0) + (b+\lambda_s+r-z\lambda_s)\lambda_0\,[(1-\rho_0^N)z^2-(z-1)\,(z+\rho_0^N\varphi(z))]\,p(0,0)}{Q(z)(1-\rho_0 z)} \qquad (37)$$

Using the normalizing condition $F(z)=1$, we obtain

$$\frac{\left\{\frac{1-\rho_0{}^N}{1-\rho_0} + \frac{\rho_0^N}{1-\rho_0}\right\} Q(1)(1-\rho_0)\,p(0,0) + (b+r)\lambda_0\left[(1-\rho_0^N)-0\right]p(0,0)}{Q(1)(1-\rho_0)} = 1 \qquad (38)$$

Substituting $Q(1) = r\mu - \lambda_s b - \lambda_f r$ in (38) we get,

$$\frac{\left(\frac{1}{1-\rho_0}\right)(r\mu-\lambda_s b-\lambda_f r)(1-\rho_0)p(0,0) + (b+r)\lambda_0(1-\rho_0^N)p(0,0)}{(r\mu-\lambda_s b-\lambda_f r)(1-\rho_0)} = 1$$

$$\frac{\left[(r\mu-\lambda_s b-\lambda_f r)+(b+r)\lambda_0(1-\rho_0^N)\right]p(0,0)}{(1-\rho_0)(r\mu-\lambda_s b-\lambda_f r)} = 1$$

$$p(0,0) = \frac{(1-\rho_0)(r\mu-\lambda_s b-\lambda_f r)}{(r\mu-\lambda_s b-\lambda_f r)+(b+r)\lambda_0(1-\rho_0^N)} \qquad (39)$$

Now, assuming $\lambda_s > 0$ and substituting $\alpha = \frac{1}{z_1}$ and $\beta = \frac{1}{z_2}$

Equation (39) becomes,

$$p(0,0) = \frac{\mu(r+\lambda_s)(1-\alpha)(1-\beta)(1-\rho_0)}{(r\mu-\lambda_s b-\lambda_f r)+(b+r)\lambda_0(1-\rho_0^N)} \qquad (40)$$

Using equation (40) in (37), we obtain

$$F(z) = R(z)\frac{(1-\alpha)(1-\beta)(1-\rho_0)}{(1-\alpha z)(1-\beta z)(1-\rho_0 z)} \tag{41}$$

Where

$$R(z) = \frac{\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z} + \frac{\rho_0^N z^N}{1-\rho_0}\right\}Q(z)(1-\rho_0 z) + (b+r)\lambda_0\left[(1-\rho_0^N)z^2 - (z-1)(z+\rho_0^N\varphi(z))\right]}{(r\mu - \lambda_s b - \lambda_f r) + (b+r)\lambda_0(1-\rho_0 z)} \tag{42}$$

Put z=1 in (42) and simplifying we get

$R(1) = 1$

PARTICULAR CASE

No customer admitted in the queue during a repair process.

In this case $\lambda_s = 0$

Then, equation (19) becomes

$$Q(z) = r\mu(1-\rho_f z) \qquad \text{Where} \quad \rho_f = \frac{\lambda_f}{\mu} < 1 \tag{43}$$

Substituting (43) into equation (37), we get

$$F(z) = \frac{\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z} + \frac{\rho_0^N}{1-\rho_0}z^N\right\}r\mu(1-\rho_f z)(1-\rho_0 z)p(0,0) + (b+r)\lambda_0\left[(1-\rho_0^N)z^2 - (z-1)(z+\rho_0^N\varphi(z))\right]p(0,0)}{r\mu(1-\rho_f z)(1-\rho_0 z)}$$

Again using the normalizing condition $F(1) = 1$, we obtain

$$\frac{\left[\frac{1-\rho_0{}^{N}}{1-\rho_0}+\frac{\rho_0^N}{1-\rho_0}\right] r\mu(1-\rho_f)(1-\rho_0)p(0,0)+(b+r)\lambda_0(1-\rho_0^N)p(0,0)}{r\mu\,(1-\rho_f)(1-\rho_0)} = 1$$

$$\frac{\left[\frac{1}{1-\rho_0}\right] r\mu(1-\rho_f)(1-\rho_0)p(0,0)+(b+r)\lambda_0(1-\rho_0^N)p(0,0)}{r\mu(1-\rho_f)(1-\rho_0)} = 1$$

$$\frac{[r\mu(1-\rho_f)+(b+r)\lambda_0(1-\rho_0^N)]p(0,0)}{r\mu(1-\rho_f)(1-\rho_0)} = 1$$

$$p(0,0) = \frac{r\mu(1-\rho_f)(1-\rho_0)}{[r\mu(1-\rho_f)+(b+r)\lambda_0(1-\rho_0^N)]} \tag{45}$$

$$F(z) = R(z)\,\frac{(1-\rho_f)(1-\rho_0)}{(1-\rho_f z)(1-\rho_0 z)} \tag{46}$$

$$R(z) = \frac{\left\{\frac{1-(\rho_0 z)^N}{1-\rho_0 z}+\frac{\rho_0^N}{1-\rho_0}z^N\right\} r\mu(1-\rho_f z)(1-\rho_0 z) + (b+r)\,\lambda_0\left[(1-\rho_0^N)z^2-(z-1)(z+\rho_0^N\varphi(z))\right]}{(r\mu-\lambda_s b-\lambda_f r)+(b+r)\lambda_0(1-\rho_0^N)} \tag{47}$$

Equations (41) and (46) are the queue length distribution for $\lambda_s > 0$ and $\lambda_s = 0$ respectively.

If $\lambda_s = 0$. Expression (31) becomes the necessary and sufficient condition for the queue length distribution to exist; it gives the utilization factor for M/M/1 queue which is independent from breakdown and repair rates.

THE AVERERAGE QUEUE SIZE WHEN $\lambda_s > 0$

The mean number of customers in the queue under steady state is given by

$$L_q = \lim_{z\to 1} \frac{d}{dz} F(z) = F'(1)$$

$$L_q = \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_s(\lambda_f - \mu - b) + \lambda_0 \left\{ b + r - \lambda_s(1-\rho_0^{N}) \right\} - \lambda_f r - \left\{ \dfrac{\rho_0^{N+1}}{1-\rho_0}(\mu r - \lambda_s b - \lambda_f r) \right\}}{(r\mu - \lambda_s b - \lambda_f r) + \lambda_0 (b+r)(1-\rho_0^{N})} \tag{48}$$

THE AVERAGE NUMBER OF CUSTOMERS IN THE SYSTEM

$$L_s = L_q + \frac{\lambda_0}{\mu} + \frac{\lambda_f}{\mu} + \frac{\lambda_s}{\mu} \tag{49}$$

THE AVERAGE WAITING TIMES PER CUSTOMER IN THE QUEUE AND IN THE SYSTEM

$$W_q = \frac{L_q}{\lambda_0} + \frac{L_q}{\lambda_f} + \frac{L_q}{\lambda_s} \tag{50}$$

$$W_s = L_s + \frac{1}{\mu} \tag{51}$$

THE AVERAGE QUEUE SIZE WHEN $\lambda_s = 0$

Using the value of F (z) given in equation (46), we obtain

$$L_q = \lim_{z\to 1} \frac{d}{dz} F(z) = F'(1)$$

$$L_q = \frac{\rho_f}{1-\rho_f} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_0(b+r) - r\mu\rho_f - \frac{\rho_0^{N+1}}{1-\rho_0}\left(r\mu(1-\rho_f)\right)}{r\mu\,(1-\rho_f) + \lambda_0(b+r)(1-\rho_0 z)} \tag{52}$$

THE AVERAGE NUMBER OF CUSTOMERS IN THE SYSTEM

$$L_s = L_q + \frac{\lambda_0}{\mu} + \frac{\lambda_f}{\mu} \tag{53}$$

THE AVERAGE WAITING TIMES IN THE QUEUE AND IN THE SYSTEM

$$W_q = \frac{L_q}{\lambda_0} + \frac{L_q}{\lambda_f} \tag{54}$$

$$W_s = L_s + \frac{1}{\mu} \tag{55}$$

SPECIAL CASES

The system capacity is infinite, (i.e.) $N = \infty$

Case a: when $\lambda_s > 0$

$$L_q = \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_s(\lambda_f - \mu - b) + \lambda_0(b + r - \lambda_s) - \lambda_f r}{(r\mu - \lambda_s b - \lambda_f r) + \lambda_0(b+r)} \tag{56}$$

Case b: when $\lambda_s = 0$

$$L_q = \frac{\rho}{1-\rho} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_0(b+r) - r\mu\rho}{r\mu(1-\rho) + \lambda_0(b+r)} \tag{57}$$

CHAPTER-III

CHAPTER-III

ANALYSIS OF M/G/1 QUEUEING MODEL WITH STATE DEPENDENT ARRIVAL AND VACATION

The steady state behavior of a single server M/G/1 queueing model with vacation and varying arrival rates is considered in this chapter. Using supplementary variable technique the steady state solution as well as the probability generating functions for the number of units in the queue, the average number of units in the queue and in the system and average waiting time are obtained.

MODEL DESCRIPTION

The units arrive at the system according to Poisson fashion with state dependent rates. Let B(v) and b(v) respectively be the distribution function and the density function of the essential service time. It is assumed that service time of the units follow general distribution. Let $\mu(x)dx$ be the conditional probability of completion of the service of the unit during the interval $(x, x + dx)$ with elapsed time x so that,

$$\mu(x) = \frac{b(x)}{1 - B(x)} \tag{1}$$

and

$$b(v) = \mu(v)\, e^{-\int_0^v \mu(x)dx} \tag{2}$$

The server may decide to take a vacation of fixed length $d(>0)$ at the completion of each service with probability p or may continue to be available in the system for the next service with probability $1-p$ .The customers are served according to the first come first served rule. The inter arrival times, service times and the vacation times are independent of each other.

NOTATIONS AND DEFINITIONS

λ_1 = Mean arrival rate of the units in idle state

λ_2 = Mean arrival rate of the units in busy state

λ_3 = Mean arrival rate of the units in vacation state

$W_n(t, x)$ = Probability of n units in the system at time t when the server is busy in rendering service to the unit with elapsed service time lying between x and $x + dx$.

$V_n(t)$ = Probability of n units in the queue at time t when the server is on vacation.

$Q(t)$ = Probability that there are no units in the system and the server is in idle state at time t.

$P_q(z)$ = Probability generating function of the number of units in the system.

K_r = Probability of r arrivals during a vacation period.

Where

$$K_r = \frac{e^{-\lambda_3 d}(\lambda_3 d)^r}{r!} \quad , r = 0, 1, 2\ldots..$$

EQUATIONS GOVERNING THE SYSTEM

The model is governed by the following set of differential-difference equations.

$$\frac{d}{dx}W_n(x) + (\lambda_2 + \mu(x))W_n(x) = \lambda_2 W_{n-1}(x) , \qquad n \ge 1 \qquad (3)$$

$$\frac{d}{dx}W_0(x) + (\lambda_2 + \mu(x))W_0(x) = 0 \qquad (4)$$

$$\lambda_1 Q = (1-p)\int_0^\infty W_0(x)\mu(x)dx + V_0 K_0 \qquad (5)$$

$$V_n = p\int_0^\infty W_n(x)\mu(x)dx , \qquad n \ge 0 \qquad (6)$$

With the boundary conditions,

$$W_n(0) = (1-p)\int_0^\infty W_{n+1}(x)\,\mu(x)dx + V_0 K_{n+1} + V_1 K_n + \ldots\ldots.. + V_{n+1} K_0 , \qquad n \ge 1 \qquad (7)$$

$$W_0(0) = (1-p)\int_0^\infty W_1(x)\mu(x)dx + V_0 K_1 + V_1 K_0 + \lambda_1 Q \qquad (8)$$

GENERATING FUNCTIONS OF THE QUEUE LENGTH

Define the probability generating functions,

$$W(x,\ z) = \sum_{n=0}^{\infty} W_n(x) z^n \tag{9}$$

$$W(z) = \sum_{n=0}^{\infty} W_n z^n \tag{10}$$

$$V(z) = \sum_{n=0}^{\infty} V_n z^n \tag{11}$$

Multiplying equation (3) by z^n, adding with equation (4) summing over n from 0 to ∞ using the generating functions defined in equations (9), (10) and (11) we get,

$$\frac{d}{dx} \sum_{n=0}^{\infty} W_n(x) z^n + (\lambda_2 + \mu(x)) \sum_{n=0}^{\infty} W_n(x) z^n = \lambda_2 \sum_{n=0}^{\infty} W_{n-1}(x) z^n$$

$$\frac{d}{dx} W(x,\ z) + (\lambda_2 + \mu(x))\ W(x,\ z) = \lambda_2\, z \sum_{n=0}^{\infty} W_{n-1}(x) z^{n-1}$$

$$\frac{d}{dx} W(x,\ z) + (\lambda_2 + \mu(x))\ W(x,\ z) = \lambda_2\, z\, W(x,\ z)$$

$$\frac{d}{dx} W(x,\ z) + (\lambda_2 + \mu(x) - \lambda_2 z)\ W(x,\ z) = 0 \tag{12}$$

Multiplying equation (6) by z^n, summing over n and using equations (9) to (11), we have

$$\sum_{n=0}^{\infty} V_n z^n = p \int_0^{\infty} \sum_{n=0}^{\infty} W_n(x)\, \mu(x)\, z^n dx$$

$$V(z) = p\int_0^\infty W(x,z)\mu(x)dx \tag{13}$$

$$\sum_{n=0}^\infty K_n z^n = \sum_{n=0}^\infty \frac{e^{-\lambda_3 d}(\lambda_3 d)^n z^n}{n!}$$

$$= e^{-\lambda_3 d}\sum_{n=0}^\infty \frac{(\lambda_3 d\, z)^n}{n!}$$

$$= e^{-\lambda_3 d(1-z)} \tag{14}$$

Performing similar operations on equations (7) and (8), we obtain

$$\begin{aligned} zW(0,z) &= (1-p)\int_0^\infty W(x,z)\mu(x)dx + V(z)\, e^{-\lambda_3 d(1-z)} - K_0 V_0 \\ &\quad + \lambda_1 Q z - (1-p)\int_0^\infty W_0(x)\mu(x)dx \end{aligned} \tag{15}$$

Using equation (5) in (15), we obtain

$$\begin{aligned} zW(0,z) &= (1-p)\int_0^\infty W(x,z)\mu(x)dx + V(z)\, e^{-\lambda_3 d(1-z)} + \lambda_1 Q z - \lambda_1 Q \\ &= (1-p)\int_0^\infty W(x,z)\mu(x)dx + V(z)\, e^{-\lambda_3 d(1-z)} + \lambda_1 Q(z-1) \end{aligned}$$

$$W(0,z) = \frac{(1-p)\int_0^\infty W(x,z)\mu(x)dx + V(z)\, e^{-\lambda_3 d(1-z)} + \lambda_1 Q(z-1)}{z} \tag{16}$$

Solving the differential equation (12), we obtain

$$W(x,z) = c_1 e^{-\int(\lambda_2 - \lambda_2 z)dx} e^{-\int_0^x \mu(t)dt}$$

Taking x = 0

Then $c_1 = W(0,z)$ and hence

$$W(x,z) = W(0,z)\, e^{-\lambda_2(1-z)x} e^{-\int_0^x \mu(t)dt} \tag{17}$$

Integrating equation (17) with respect to x, we get

$$\int_0^\infty W(x,z)\,dx = W(0,z) \int_0^\infty e^{-\lambda_2(1-z)x} e^{-\int_0^x \mu(t)dt} dx$$

$$= W(0,z) \int_0^\infty e^{-\lambda_2(1-z)x} e^{-\int_0^x \frac{b(t)}{1-B(t)}dt} dx$$

$$= W(0,z) \int_0^\infty e^{-\lambda_2(1-z)x} e^{\log[1-B(t)]} dx$$

$$= W(0,z) \int_0^\infty e^{-\lambda_2(1-z)x} \left[1-B(t)\right] dx$$

$$W(z) = W(0,z)\left[\frac{1-\overline{b}(\lambda_2 - \lambda_2 z)}{\lambda_2 - \lambda_2 z}\right] \tag{18}$$

Where $\overline{b}(\lambda_2 - \lambda_2 z) = \int_0^\infty e^{-\lambda_2(1-z)x}\, b(x)\,dx$

Multiplying both sides of equation (17) by $\mu(x)$ and integrating with respect to x from 0 to ∞, we get

$$\int_0^\infty W(x,z)\,\mu(x)dx = W(0,z)\int_0^\infty e^{-\lambda_2(1-z)x}\, e^{-\int_0^x \mu(t)dt}\,\mu(x)dx$$

$$= W(0,z)\int_0^\infty e^{-\lambda_2(1-z)x}\, b(x)\,dx$$

$$= W(0,z)\,\bar{b}(\lambda_2 - \lambda_2 z) \qquad (19)$$

Using equation (16) in (19) and solving for $W(0, z)$, we obtain

$$W(0,z) = \frac{V(z)\,e^{-\lambda_3 d(1-z)} + \lambda_1 Q(z-1)}{z - \bar{b}(\lambda_2 - \lambda_2 z)(1-p)} \qquad (20)$$

Substituting equation (20) in (18) we get

$$W(z) = \left[\frac{V(z)e^{-\lambda_3 d(1-z)} + \lambda_1 Q(z-1)}{z - \bar{b}(\lambda_2 - \lambda_2 z)(1-p)}\right]\left[\frac{1 - \bar{b}(\lambda_2 - \lambda_2 z)}{\lambda_2 - \lambda_2 z}\right] \qquad (21)$$

Using equation (19) in (13) we get,

$$V(z) = p\,W(0,z)\bar{b}(\lambda_2 - \lambda_2 z) \qquad (22)$$

Substituting equation (20) in (22)

$$V(z)\,[z - \bar{b}(\lambda_2 - \lambda_2 z)(1-p)] = p[V(z)\,e^{-\lambda_3 d(1-z)} + \lambda_1 Q(z-1)]\bar{b}(\lambda_2 - \lambda_2 z)$$

$$V(z)\,[z-\bar{b}(\lambda_2-\lambda_2 z)+p\,\bar{b}(\lambda_2-\lambda_2 z)]-p\,[V(z)\,e^{-\lambda_3 d(1-z)}\,\bar{b}(\lambda_2-\lambda_2 z)]$$

$$=\ p\lambda_1 Q(z-1)\bar{b}(\lambda_2-\lambda_2 z)$$

$$V(z)\ =\ \frac{p\lambda_1 Q(z-1)\bar{b}(\lambda_2-\lambda_2 z)}{z-\bar{b}(\lambda_2-\lambda_2 z)+p\,\bar{b}(\lambda_2-\lambda_2 z)(1-e^{-\lambda_3 d(1-z)})} \qquad (23)$$

Substituting equation (23) in (21), we get

$$W(z)\ =\ \left[\frac{p\lambda_1 Q(z-1)\bar{b}(\lambda_2-\lambda_2 z)\,e^{-\lambda_3 d(1-z)}}{z-\bar{b}(\lambda_2-\lambda_2 z)+p\bar{b}(\lambda_2-\lambda_2 z)(1-e^{-\lambda_3 d(1-z)})}+\lambda_1 Q(z-1)\right]$$

$$\times\left[\frac{1}{z-\bar{b}(\lambda_2-\lambda_2 z)(1-p)}\right]\left[\frac{1-\bar{b}(\lambda_2-\lambda_2 z)}{\lambda_2-\lambda_2 z}\right]$$

$$W(z)\ =\ \frac{\lambda_1 Q[\bar{b}(\lambda_2-\lambda_2 z)-1]}{\lambda_2\,[z-\bar{b}(\lambda_2-\lambda_2 z)+p\bar{b}(\lambda_2-\lambda_2 z)(1-e^{-\lambda_3 d(1-z)})]} \qquad (24)$$

Let $P_q(z)$ be the probability generating function of the queue length, whether the server is on vacation or available in the system.

Then $\quad P_q(z)=V(z)+W(z) \qquad (25)$

Substituting equations (23) and (24) in equation (25)

$$P_q(z) = \left[\frac{p\lambda_1 Q(z-1)\bar{b}(\lambda_2-\lambda_2 z)}{z-\bar{b}(\lambda_2-\lambda_2 z)+p\,\bar{b}(\lambda_2-\lambda_2 z)(1-e^{-\lambda_3 d(1-z)})}\right]$$

$$+\left[\frac{\lambda_1 Q[\bar{b}(\lambda_2-\lambda_2 z)-1]}{\lambda_2\,[z-\bar{b}(\lambda_2-\lambda_2 z)+p\bar{b}(\lambda_2-\lambda_2 z)(1-e^{-\lambda_3 d(1-z)})]}\right] \qquad (26)$$

The unknown constant Q can be obtained by the normalizing condition,

$$P_q(1)+Q=1 \qquad (27)$$

We see that for z=1, $P_q(z)$ in equation (26) is indeterminate of the form $\frac{0}{0}$.

Applying L'Hopitals rule on equation (26) and using the fact that $\bar{b}(0) = 1$,

$-\bar{b}'(0) = E(v) = \frac{1}{\mu}$ and $\bar{b}''(0) = E(v^2)$ where $E(v^2)$ is the second moment of

the service time.

$$V(1) = \lim_{z\to 1} V(z)$$

$$= \frac{p\lambda_1 Q\,\bar{b}(\lambda_2-\lambda_2 z)+p\lambda_1 Q(z-1)\bar{b}(\lambda_2-\lambda_2 z)(-\lambda_2)}{\begin{array}{l}1-\bar{b}'(\lambda_2-\lambda_2 z)(-\lambda_2)+p\bar{b}'(\lambda_2-\lambda_2 z)(-\lambda_2)(1-e^{-\lambda_3 d(1-z)})\\ +p\bar{b}(\lambda_2-\lambda_2 z)(e^{-\lambda_3 d(1-z)}(-\lambda_3 d))\end{array}}$$

$$= \frac{p\lambda_1 Q\,\bar{b}(0)}{1-\frac{\lambda_2}{\mu}+0+p(-\lambda_3 d)}$$

$$V(1) = \frac{\mu p \lambda_1 Q}{\mu - \lambda_2 - p\lambda_3 \mu d}$$

$$W(1) = \lim_{z \to 1} W(z)$$

$$W(1) = \frac{\lambda_1 Q}{\mu - \lambda_2 - p\lambda_3 \mu d}$$

$$P_q(1) = \frac{\mu p \lambda_1 Q + \lambda_1 Q}{\mu - \lambda_2 - p\lambda_3 \mu d}$$

$$= \frac{Q \lambda_1 (1 + p\mu)}{\mu - \lambda_2 - p\lambda_3 \mu d} \tag{28}$$

Therefore adding Q to equation (28) and equating to 1 and simplifying we get,

$$Q = \frac{\mu - \lambda_2 - p\lambda_3 \mu d}{\mu - \lambda_2 - p\lambda_3 \mu d + \lambda_1 (1 + p\mu)}$$

$$Q = 1 - \frac{\lambda_1 (1 + p\mu)}{\mu(1 - p\lambda_3 d) + \lambda_1 (1 + p\mu) - \lambda_2} \tag{29}$$

Substituting equation (29) in (26), we obtain

$$P_q(z) = \left[\frac{\frac{\lambda_1}{\lambda_2}(\bar{b}(\lambda_2 - \lambda_2 z) - 1) + (p\lambda_1 (z - 1)\bar{b}(\lambda_2 - \lambda_2 z))}{z - \bar{b}(\lambda_2 - \lambda_2 z) + p\bar{b}(\lambda_2 - \lambda_2 z)(1 - e^{-\lambda_3 d(1 - z)})} \right]$$

$$\times \left[1 - \frac{\lambda_1 (1 + p\mu)}{\mu(1 - p\lambda_3 d) + \lambda_1 (1 + p\mu) - \lambda_2} \right] \tag{30}$$

The utilization factor ρ of the system is given by

$$\rho = 1-Q = \frac{\lambda_1(1+p\mu)}{\mu(1-p\lambda_3 d)+\lambda_1(1+p\mu)-\lambda_2} \tag{31}$$

Where $\rho < 1$ is the stability condition under which the steady state results exist.

Let $P(z)$ be the probability generating function of the number of units in the system. Then

$$P(z) = Q + zP_q(z)$$

$$= \left[\frac{z\left(1-\frac{\lambda_1}{\lambda_2}\right)+\bar{b}(\lambda_2-\lambda_2 z)\left\{p(1-e^{-\lambda_3 d(1-z)})-1+p\lambda_1 z(z-1)+\frac{\lambda_1}{\lambda_2}z\right\}}{z-\bar{b}(\lambda_2-\lambda_2 z)+p\bar{b}(\lambda_2-\lambda_2 z)(1-e^{-\lambda_3 d(1-z)})}\right]$$

$$\times\left[1-\frac{\lambda_1(1+p\mu)}{\mu(1-p\lambda_3 d)+\lambda_1(1+p\mu)-\lambda_2}\right] \tag{32}$$

THE EXPECTED NUMBER OF UNITS IN THE QUEUE

Let L_q denote expected number of units in the queue. Then

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z) = P_q'(1)$$

Let $N(z)$ be the numerator and $D(z)$ be the denominator of $P_q(z)$

Since $N(1) = D(1) = 0$ by applying L'Hopitals rule, we get

$$L_q = \lim_{z\to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2}$$

$$= \lim_{z\to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \tag{33}$$

From the expressions of $N(z)$ and $D(z)$, we have

$$N'(1) = \left[\lambda_1\left(p + \frac{1}{\mu}\right)\right]\left[1 - \frac{\lambda_1(1+p\mu)}{\mu(1-p\lambda_3 d) + \lambda_1(1+p\mu) - \lambda_2}\right] \tag{34}$$

$$N''(1) = \left[\frac{2p\,\lambda_1\lambda_2}{\mu} + \lambda_1\lambda_2\,E(v^2)\right]\left[1 - \frac{\lambda_1(1+p\mu)}{\mu(1-p\,\lambda_3 d) + \lambda_1(1+p\mu) - \lambda_2}\right] \tag{35}$$

$$D'(1) = 1 - \frac{\lambda_2}{\mu} - \lambda_3 pd \tag{36}$$

$$D''(1) = -\left[\lambda_2^2 E(v^2) + \frac{2p\,\lambda_1\lambda_3\,d}{\mu} + p\,\lambda_3^2\,d^2\right] \tag{37}$$

Using equations (34) to (37) in (33), we obtain

$$L_q = \left[\frac{\lambda_1\lambda_2\,E(v^2)\,[\,(1-\lambda_3 pd) + \lambda_2 p\,] + \dfrac{2p\lambda_1\lambda_2}{\mu}\left(1 - \dfrac{\lambda_2}{\mu}\right) + \dfrac{2\,p\lambda_1\lambda_2\lambda_3 d}{\mu^2} + p^2\lambda_1\lambda_3^2 d^2 + \dfrac{p\,\lambda_1\lambda_3^2 d^2}{\mu}}{2\left(1 - \dfrac{\lambda_2}{\mu} - \lambda_3 pd\right)^2}\right]$$

$$\times\left[1 - \frac{\lambda_1(1+p\mu)}{\mu(1-p\lambda_3 d) + \lambda_1(1+p\mu) - \lambda_2}\right] \tag{38}$$

THE EXPECTED NUMBER OF UNITS IN THE SYSTEM

If L denotes number of units in the system.Using little's formula, we have

$$L = L_q + \rho \tag{39}$$

Where L_q has been found in equation (38) and $\rho = 1 - Q$ (40)

THE EXPECTED WAITING TIME

Let W_q and W denote the expected waiting time in the queue and in the system respectively.

Then using little's formula we obtain,

$$W_q = \frac{L_q}{\lambda_{eff}} \tag{41}$$

$$W = \frac{L}{\lambda_{eff}} \tag{42}$$

Where $\lambda_{eff} = \lambda_1 Q + \lambda_2 W(1) + \lambda_3 V(1)$

SPECIAL CASES

Case a: No server vacation

In this case, p=0, then equation (38) becomes

$$L_q = \frac{\lambda_1 \lambda_2 E(v^2)}{2\left(1 - \frac{\lambda_1}{\mu}\right)^2} \left[\frac{\mu - \lambda_2}{\mu + \lambda_1 - \lambda_2} \right] \tag{43}$$

The result (43) reduces to the average queue length for M/G/1 model with state dependent rates.

Case b: No server vacation and uniform arrival rates

In this case p=0 and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ then equation (38) becomes

$$L_q = \frac{\lambda^2 E(v^2)}{2\left(1 - \frac{\lambda}{\mu}\right)} \tag{44}$$

The result (44) leads to the average queue length for M/G/1 queue.

Case c: service time is K-Erlangian

In this case $E(v^2) = \dfrac{k+1}{k\mu^2}$

Then equation (38) becomes

$$L_q = \left[\frac{\lambda_1\lambda_2\left(\frac{K+1}{K\mu^2}\right)(1 - \lambda_3 pd + \lambda_2 p) + \frac{2p\,\lambda_1\lambda_2}{\mu}\left(1 - \frac{\lambda_2}{\mu}\right) + \frac{2p\,\lambda_1\lambda_2\lambda_3 d}{\mu^2} + p^2\,\lambda_1\lambda_3^2 d^2 + \frac{p\,\lambda_1\lambda_3^2 d^2}{\mu}}{2\left(1 - \frac{\lambda_2}{\mu} - \lambda_3 pd\right)^2} \right]$$

$$\times \left[1 - \frac{\lambda_1(1 + p\mu)}{\mu(1 - p\lambda_3 d) + \lambda_1(1 + p\mu) - \lambda_2} \right] \tag{45}$$

Equation (45) provides the average queue length for $M/E_K/1$ deterministic vacation queueing model.

Case d: Service time is exponential

In this case, we get $E(v^2) = \frac{2}{\mu^2}$

then from equation (38) we obtain,

$$L_q = \left[\frac{\frac{2\lambda_1\lambda_2}{\mu^2}\left(\frac{K+1}{K\mu^2}\right)(1-\lambda_3 pd+\lambda_2 p)+\frac{2p\lambda_1\lambda_2}{\mu}\left(1-\frac{\lambda_2}{\mu}\right)+\frac{2p\lambda_1\lambda_2\lambda_3 d}{\mu^2}+p^2\lambda_1\lambda_3^2 d^2+\frac{p\lambda_1\lambda_3^2 d^2}{\mu}}{2\left(1-\frac{\lambda_2}{\mu}-\lambda_3 pd\right)^2} \right]$$

$$\times \left[1-\frac{\lambda_1(1+p\mu)}{\mu(1-p\lambda_3 d)+\lambda_1(1+p\mu)-\lambda_2} \right] \qquad (46)$$

The result (46) agrees with the results given in M/M/1 deterministic vacation queuing model.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

In this dissertation single server queueing system with state dependent Poisson arrivals, server vacations and random breakdowns are considered.

In chapter two, finite capacity queueing system with vacations and server breakdowns are considered.

Infinite capacity queueing system with deterministic server vacation is analyzed in chapter three.

For both models explicity expressions for expected number of customers in the queue and in the system, average waiting times for a customer in the queue and in the system are obtained.

BIBLIOGRAPHY

---

[1] **Boxma, O., Mandjes, M. and Kella, O.,** (2008), “On a queueing model with service interruptions”, Probability in the Engineering and Informational Sciences, Vol. 22, pp.537-555.

[2] **Charan Jeet Singh, Madhu Jain and Binary Kumar,** (2012), “Analysis of M/G/1 queueing model with state dependent arrival and vacation”, Journal of Industrial Engineering International, 8(2), 1-8.

[3] **Choudhury, G.,** (2000), “An $M^X/M/1$ queueing system with a setup period and vacation period ”, QUESTA ,Vol.36, pp.23-28.

[4] **Choudhury, G.,** (2002), “Some aspects of M/G/M queue with two different vacation times under multiple vacation policy”, Stoch Anal Appl 20(5), 901-909.

[5] **Doshi, B. T.,** (1986), “Single server queues with vacation a survey”, Queueing System, Vol.1, pp.29-66.

[6] **Ghimire, R. P., and Ritu Basnet.,** (2011), “The finite capacity queueing system with vacations and server breakdowns”, Department of Mathematics, School of Science, Kathmandu University, Dhulikhel(Nepal), Vol.24, pp.387-394.

[7] **Gray, W. J., Wang, P. P. and scott, M.,** (2000), “A vacation queueing model with service breakdowns”, Applied Mathematical Modeling, Vol.24, PP. 391-400.

[8] **Gray, W. J., Wang, P. P. and scott, M.,** (2004), “A queueing model with multiple types of server breakdowns”, Quality Technology and Quantitative Management, Vol.1, pp.245-255.

[9] **Jain, M and Agrawal, P. K.,** (2009), “Optimal policy for bulk queue with multiple types of server breakdowns”, International Journal of Operational Research, Vol.4, pp.35-54.

[10] **Ke, J. C.,** (2003), “The optimal control of an M/G/1 queueing system with server vacations, startup and breakdowns”, Computers and Industrial Engineering, Vol.44, pp.567-579.

[11] **Ke, J. C. and Pearn, W. L.,** (2004), “Optimal management policy for heterogeneous arrival queueing systems with server breakdowns and vacations”, Quality Technology and Quantitative Management, Vol.1, pp.149-162.

[12] **Ke, J. C. and Wang, K. H.,**(2003), “Analysis of operating characteristics for the heterogeneous batch arrival queue with server startup and breakdowns”, RAIRO Operations Research ,Vol.37, pp.157-177.

[13] **Levy, Y. and Yechiali, U.,** (1975), “Utilization of idle time in an M/G/M queueing system”, Management Science, Vol.22, pp.202-211.

[14] **Madan, K. C.,** (1999), “An M/G/M queue with optional deterministic server vacations”, Metron, 57(3-4), 83-95.

[15] **Madan, K. C.,** (2003), “An M/G/M type queue with time homogeneous breakdowns and deterministic repair times”, Soochow J Mathematics, 29(1), 103-110.

[16] **Maraghi, F. A., Madan KC, Dowman KD.,** (2009), "Batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general time distribution ", Int J Int Manag Sci,20,55-70.

[17] **Srivastava, V. and Jain, M.,** (1999), "Optimal N-policy for single server Markovian queue with breakdown repair and state dependent arrival rate", Journal of Operational Research Society of India (OPSEARCH), Vol.36, pp.75

[18] **Takagi, H.,** (1994), "M/G/1/N queues with server vacations and exhaustive service", Operations Research, Vol.42, pp.926-939.

[19] **Tian, N. and Zhang, Z. G.,** (2006), "Vacation queueing model theory and applications", Springer, New York.

[20] **Thangraj, V. and Vanitha, S.,** (2010), " M/G/M queue with two stage heterogeneous service compulsory server vacation and random breakdowns", Int J Contemporary Mathematical Sci, 5(7), 307-322.

[21] **Wang, K. H.,** (1995), "Optimal operation of Markovian queueing system with a removable and non reliable server", Microelectronics Reliability, Vol.35, pp.1131-1136.

[22] **Wang, J.,** (2004), "An M/G/M queue with second optional service and server breakdowns", Computers and Mathematics with Applications, 47, 1713-1723.

[23] **Zhang, Y.,Yue, D. and Yue, W.,** (2005), " Analysis of an M/M/1/N queue with balking, reneging and server vacations ", Proceedings of the Fifth International Symposium, pp.37-47.