CHAPTER 7

Chapter 7

Generalized π -closed sets and grill

7.1 Introduction

Choquet (1947) initiated the idea of grills on topological spaces. This concept is a powerful supporting tool like nets and filters for getting a deeper insight for further study of topological spaces. Roy and Mukherjee (2007) and Roy et al.(2008) defined and studied a topology by associating the existing topology and a grill. Dhananjoy Mandal and Mukherjee (2012) defined and studied $\mathcal{G}$-g-closed sets in grill topological spaces. Arockiarani and Karthika (2012) introduced the notion of generalized B-closed sets in grill topology.

In this chapter, we have introduced and developed the notion of gπ -closed sets in terms of grills. The concepts of gπ(θ)-convergence, gπ(θ)-adherence, gπ(θ)-linked and gπ(θ)-conjoint are introduced and their properties are discussed.

7.2 Generalized π- closed sets with respect to a grill

Definition 7.2.1. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X.

Then the mapping $\pi\Phi_{\mathcal{G}}: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, with respect to the grill $\mathcal{G}$ and τ, is defined by

$$\pi\Phi_{\mathcal{G}}(A)= \{x \in X : U \cap A \in \mathcal{G} \text{ for every } \pi\text{-open set } U \text{ containing } x\} \text{ for } A \in \mathcal{P}(X)$$

Theorem 7.2.2. Let (X, τ) be a topological space.

(i) If $\mathcal{G}$ is any grill on X, then $\pi\Phi_{\mathcal{G}}$ is an increasing function in the sense that

$A \subseteq B(\subseteq X)$ implies $\pi\Phi_{\mathcal{G}}(A) \subseteq \pi\Phi_{\mathcal{G}}(B)$,

(ii) $\mathcal{G}_1$ and $\mathcal{G}_2$ are two grills on X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\pi\Phi_{\mathcal{G}1}(A) \subseteq \pi\Phi_{\mathcal{G}2}(A)$, for all $A \subseteq X$.

(iii) For any grill $\mathcal{G}$ on X and any $A \subseteq X$, if $A \notin \mathcal{G}$ then $\pi\Phi_{\mathcal{G}}(A) = \varphi$.

Theorem 7.2.3 Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$,

(i) $\pi\Phi_{\mathcal{G}}(A \cup B) = \pi\Phi_{\mathcal{G}}(A) \cup \pi\Phi_{\mathcal{G}}(B)$,

(ii) $\pi\Phi_{\mathcal{G}}(\pi\Phi_{\mathcal{G}}(A)) \subseteq \pi\Phi_{\mathcal{G}}(A) = \pi cl(\pi\Phi_{\mathcal{G}}(A)) \subseteq \pi cl(A)$

Proof. (i) In view of Theorem 7.2.2(i) it suffices to show that $\pi\Phi_{\mathcal{G}}(A \cup B) \subseteq \pi\Phi_{\mathcal{G}}(A) \cup \pi\Phi_{\mathcal{G}}(B)$. Suppose $x \notin \pi\Phi_{\mathcal{G}}(A) \cup \pi\Phi_{\mathcal{G}}(B)$. Then there are $U_1, U_2 \in \pi\tau(x)$ such that $A \cap U_1 \notin \mathcal{G}$ and $B \cap U_2 \notin \mathcal{G}$ and hence $(A \cap U_1) \cup (B \cap U_2) \notin \mathcal{G}$. Now $U_1 \cap U_2 \in \pi\tau(x)$ and $(A \cup B) \cap (U_1 \cap U_2) \subseteq (A \cap U_1) \cup (B \cap U_2) \notin \mathcal{G}$, and hence $x \notin \pi\Phi_{\mathcal{G}}(A \cup B)$. Therefore $\pi\Phi_{\mathcal{G}}(A \cup B) = \pi\Phi_{\mathcal{G}}(A) \cup \pi\Phi_{\mathcal{G}}(B)$.

(ii) $x \notin \pi cl(A)$, there exists $U \in \pi\tau(x)$ such that $U \cap A = \varphi \notin \mathcal{G} \Rightarrow x \notin \pi\Phi_{\mathcal{G}}(A)$. Thus $\pi\Phi_{\mathcal{G}}(A) \subseteq \pi cl(A)$. Now we shall show that $\pi cl(\pi\Phi_{\mathcal{G}}(A)) \subseteq \pi\Phi_{\mathcal{G}}(A)$. Indeed, $x \in \pi cl(\pi\Phi_{\mathcal{G}}(A))$ and $U \in \pi\tau(x) \Rightarrow U \cap \pi\Phi_{\mathcal{G}}(A) \neq \varphi$. Let $y \in U \cap \pi\Phi_{\mathcal{G}}(A)$, i.e., $y \in U$ and $y \in \pi\Phi_{\mathcal{G}}(A)$. Then $U \cap A \in \mathcal{G}$ and so $x \in \pi\Phi_{\mathcal{G}}(A)$. Thus $\pi cl(\pi\Phi_{\mathcal{G}}(A)) = \pi\Phi_{\mathcal{G}}(A)$.

Now, $\pi\Phi_{\mathcal{G}}(\pi\Phi_{\mathcal{G}}(A)) \subseteq \pi cl(\pi\Phi_{\mathcal{G}}(A)) = \pi\Phi_{\mathcal{G}}(A) \subseteq \pi cl(A)$.

Definition 7.2.4 Let $\mathcal{G}$ be a grill on a space X. Then the map $\pi\psi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by $\pi\psi(A) = A \cup \pi\Phi_{\mathcal{G}}(A)$, for all $A \in \mathcal{P}(X)$.

Remark 7.2.5 The above map satisfies Kuratowski's closure axioms.

Definition 7.2.6. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on a space X. There exists a unique topology $\tau_{\mathcal{G}}\pi$ on X given by $\tau_{\mathcal{G}}\pi = \{U \subset X : \pi\psi(X \setminus U) = X \setminus U\}$

Note 7.2.7 For any grill $\mathcal{G}$ on a space X and any A ⊆ X,

$\pi\psi$ (A) = A ∪ $\pi\Phi_{\mathcal{G}}$(A) = $\tau_{\mathcal{G}}$ π-cl(A).

Lemma 7.2.8. For any grill $\mathcal{G}$ on a space X and any A,B ⊆ X,

$\pi\Phi_{\mathcal{G}}(A) \setminus \pi\Phi_{\mathcal{G}}(B) = \pi\Phi_{\mathcal{G}}(A \setminus B) \setminus \pi\Phi_{\mathcal{G}}(B)$.

Corollary 7.2.9. Let $\mathcal{G}$ be a grill on a space X, and suppose A, B ⊆ X with B $\notin \mathcal{G}$. Then $\pi\Phi_{\mathcal{G}}(A \cup B) = \pi\Phi_{\mathcal{G}}(A) = \pi\Phi_{\mathcal{G}}(A \setminus B)$.

Definition 7.2.10 Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then a subset A of X is said to be gπ -closed with respect to the grill $\mathcal{G}$ ($\mathcal{G}$- gπ -closed) if $\pi\Phi_{\mathcal{G}}$(A) ⊆ U whenever A ⊆ U and U is open in X.

A subset A of X is said to be $\mathcal{G}$- gπ -open if X\ A is $\mathcal{G}$- gπ -closed.

Theorem 7.2.11 For a topological space (X, τ) and a grill $\mathcal{G}$ on X,

(a) For any subset A in X , $\pi\Phi_{\mathcal{G}}$(A) is $\mathcal{G}$- gπ -closed

(b) Any non member of $\mathcal{G}$ is $\mathcal{G}$- gπ -closed.

(c) Every $\mathcal{G}$- gπ -closed set is $\mathcal{G}$- g -closed.

(d) Every gπ -closed set is $\mathcal{G}$- gπ -closed.

Proof: (a) Let A be a subset in X Then $\pi\Phi_{\mathcal{G}}(\pi\Phi_{\mathcal{G}}(A)) \subseteq \pi\Phi_{\mathcal{G}}(A) \subseteq U$, whenever U is open in X. $\pi\Phi_{\mathcal{G}}$(A) is $\mathcal{G}$- gπ -closed.

(b) Let A$\notin \mathcal{G}$ then $\pi\Phi_{\mathcal{G}}(A) = \varphi \subseteq U$, whenever U is open in X. A is $\mathcal{G}$- gπ -closed..

(c) Let A be a $\mathcal{G}$- gπ -closed and A⊆ U and U is open in X. Since A is $\mathcal{G}$- gπ closed, $\pi\Phi_{\mathcal{G}}(A) \subseteq \Phi_{\mathcal{G}}(A) \subseteq U$, A is $\mathcal{G}$- g -closed. Thus every $\mathcal{G}$- gπ closed set is $\mathcal{G}$- g -closed.

(d) Let A be a gπ -closed set and U be an open set in X such that A ⊆ U then πcl(A) ⊆ U, As $\pi\Phi_{\mathcal{G}}$ (A) ⊆ πcl(A) ⊆ U, A is $\mathcal{G}$- gπ-closed. Thus every gπ -closed set is $\mathcal{G}$- gπ -closed.

Definition 7.2.12 Let X be a space and (φ≠) A⊆ X. Then [A]={B⊆ X: A ∩ B≠ φ} is a grill on X, called the principal grill generated by A.

Proposition 7.2.13 In the case of [X] principal grill generated by X, it is known that $\tau = \tau_{[X]}$ so that any [X]-gπ-closed set becomes simply a gπ-closed set and vice-versa.

Theorem 7.2.14 Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X .If a subset A of X is $\mathcal{G}$- gπ-closed then $\tau_{\mathcal{G}}$ π cl(A) ⊆ U whenever A ⊆ U and U is open.

Proof: Let A be a $\mathcal{G}$- gπ -closed set and U be a open in X such that A⊆ U then $\pi\Phi_{\mathcal{G}}$ (A) ⊆U$^{\bullet}$ ⇒A ∪ $\pi\Phi_{\mathcal{G}}$ (A) ⊆ U ⇒ $\tau_{\mathcal{G}}$ π cl(A) ⊆ U. Thus $\tau_{\mathcal{G}}$ π -cl(A) ⊆ U whenever A⊆ U and U is open.

Theorem 7.2.15 Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X .If a subset A of X is $\mathcal{G}$- gπ -closed then for all x ∈ $\tau_{\mathcal{G}}$ π cl(A) , π cl({x}) ∩ A≠ φ.

Proof: Let x ∈ $\tau_{\mathcal{G}}$ π cl(A). If π cl({x}) ∩ A = φ ⇒ A ⊆ X \ π cl({x}) then by theorem 7.2.14 $\tau_{\mathcal{G}}$ π cl(A) ⊆ X\ π cl({x}) which is a contradiction to our assumption that x ∈ $\tau_{\mathcal{G}}$ π cl(A). Therefore π cl({x}) ∩ A≠ φ.

Theorem 7.2.16 Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X .If a subset A of X is $\mathcal{G}$- gπ-closed then $\tau_{\mathcal{G}}$ π cl(A) \ A contains no non-empty closed set of (X, τ). Moreover $\pi\Phi_{\mathcal{G}}$ (A) \ A contains no non-empty closed set of (X, τ).

Proof: Let F be a closed set contained in $\tau_{\mathcal{G}}$ π cl(A) \ A and let x ∈ F. Then F∩ A = φ and π cl({x}) ∩ A = φ. This is a contradiction to the fact that π cl({x}) ∩ A≠ φ . Therefore $\tau_{\mathcal{G}}$ π cl(A) \ A contains no non-empty closed set of

(X, τ). Since $\pi\Phi_{\mathcal{G}}$ (A) \ A = $\tau_{\mathcal{G}}$ π cl(A) \ A, $\pi\Phi_{\mathcal{G}}$ (A) \ A contains no non-empty closed set of (X, τ).

Theorem 7.2.17 Let $\mathcal{G}$ be grill on a space (X, τ) and A be a $\mathcal{G}$- gπ-closed set. Then the following are equivalent

(a) A is $\tau_{\mathcal{G}}$ π -closed.

(b) $\tau_{\mathcal{G}}$π cl(A) \ A is closed in (X, τ)

(c) $\pi\Phi_{\mathcal{G}}$ (A) \ A is closed in (X, τ)

Proof:

(a)⇒ (b)Let A be $\tau_{\mathcal{G}}$ π -closed. Then $\tau_{\mathcal{G}}$π cl(A) \ A = φ and so $\tau_{\mathcal{G}}$π cl(A) \ A is a closed set

(b)⇒ (c) since $\tau_{\mathcal{G}}$π cl(A) \ A = $\pi\Phi_{\mathcal{G}}$ (A) \ A, $\tau_{\mathcal{G}}$π cl(A) \ A is closed in (X, τ)

(c)⇒ (a)Let $\pi\Phi_{\mathcal{G}}$ (A) \ A be closed in (X, τ) since A is $\mathcal{G}$- gπ -closed by Theorem 7.2.16, $\pi\Phi_{\mathcal{G}}$ (A) \ A = φ so A is $\tau_{\mathcal{G}}$ π -closed.

Theorem 7.2.18 For any subset A of a space (X, τ) and a grill $\mathcal{G}$ on X. If A is $\mathcal{G}$- gπ -closed then A ∪(X\ $\pi\Phi_{\mathcal{G}}$ (A)) is $\mathcal{G}$- gπ closed.

Proof: Let A∪ (X\ $\pi\Phi_{\mathcal{G}}$ (A)) ⊆ U, where U is open in X. Then X \ U⊆ X \ (A∪ (X\ $\pi\Phi_{\mathcal{G}}$ (A)))= $\pi\Phi_{\mathcal{G}}$ (A)\A. Since A is $\mathcal{G}$- gπ-closed, by Theorem 7.2.15, we have X\U= φ, i.e.,X=U. Since X is the only open set containing A∪ (X\ $\pi\Phi_{\mathcal{G}}$ (A)), A∪ (X\ $\pi\Phi_{\mathcal{G}}$ (A)) is $\mathcal{G}$- gπ-closed.

Theorem 7.2.19 For any subset A of a space (X, τ) and a grill $\mathcal{G}$ on X, the following are equivalent

(a) A∪ (X\ $\pi\Phi_{\mathcal{G}}$ (A)) is $\mathcal{G}$- gπ -closed

(b) $\pi\Phi_{\mathcal{G}}$ (A)\A is $\mathcal{G}$- gπ -open.

Proof: Follows from the fact that X \ ($\pi\Phi_{\mathcal{G}}$ (A)\A)=A ∪ (X\ $\pi\Phi_{\mathcal{G}}$ (A)).

Theorem 7.2.20 Let(X, τ) be a space, $\mathcal{G}$ be a grill on X and A, B be subsets of X such that A ⊆ B ⊆ $\tau_{\mathcal{G}}$ π cl(A). If A is $\mathcal{G}$- gπ-closed, then B is $\mathcal{G}$- gπ -closed.

Proof: Let B ⊆ U, where U is open in X. since A is $\mathcal{G}$- gπ -closed, $\pi\Phi_{\mathcal{G}}$ (A) ⊆ U ⇒ $\tau_{\mathcal{G}}$ π cl(A) ⊆ U. Now, A ⊆ B⊆ $\tau_{\mathcal{G}}$ π -cl(A)

⇒ $\tau_{\mathcal{G}}$ π -cl(A)⊆ $\tau_{\mathcal{G}}$ π -cl(B) ⊆ $\tau_{\mathcal{G}}$ π -cl(A). Thus $\tau_{\mathcal{G}}$ π -cl(B) ⊆ U and hence B is $\mathcal{G}$- gπ -closed.

Theorem 7.2.20 Let $\mathcal{G}$ be a grill on a space (X, τ). Then a subset A of X is $\mathcal{G}$- gπ -open iff F ⊆ $\tau_{\mathcal{G}}$ π -int (A) whenever F⊆ A and F is closed.

Proof : Let A be $\mathcal{G}$- gπ -open and F ⊆ A ,where F is closed in (X, τ) .Then X\A ⊆ X\F ⇒ $\pi\Phi_{\mathcal{G}}$ (X\A) ⊆ X\F ⇒ $\tau_{\mathcal{G}}$ πcl(X\A) ⊆ X\F ⇒ F ⊆ $\tau_{\mathcal{G}}$ π -int(A).

Conversely, X\A⊆ U where U is open in (X, τ) ⇒ X\U ⊆ $\tau_{\mathcal{G}}$ π - int(A) ⇒ $\tau_{\mathcal{G}}$ π cl(X\A) ⊆ U. Thus (X\A) is $\mathcal{G}$- gπ -closed and hence A is $\mathcal{G}$- gπ -open.

7.3 Grills: gπ(θ)-convergence and gπ(θ)-adherence

Definition 7.3.1 A grill $\mathcal{G}$ on a topological space X is said to:

(a) gπ(θ)-adhere at x ∈ X if for each U ∈ gπO(x) and each G ∈ $\mathcal{G}$, gπclU ∩ G ≠ φ,

(b) gπ(θ)-converge to a point x ∈ X if for each U ∈ gπO(x), there is some G ∈ $\mathcal{G}$ such that G ⊆ gπcl(U) (in this case we shall also say that $\mathcal{G}$ is gπ(θ)-convergent to x).

Remark 7.3.2 It at once follows that a grill $\mathcal{G}$ is gπ(θ)-convergent to a point x ∈ X iff $\mathcal{G}$ contains the collection {gπcl(U) : U ∈ gπO(x)}.

Definition 7.3.3. A filter $\mathcal{F}$ on a topological space X is said to gπ(θ)-adhere at x ∈ X (gπ(θ)-converge to x ∈ X) if for each F ∈ $\mathcal{F}$ and each U ∈ gπO(x), F ∩ gπcl(U) ≠ φ (resp. to each U ∈ gπO(x), there corresponds F ∈ $\mathcal{F}$ such that F ⊆ gπcl(U)).

Definition 7.3.4. If $\mathcal{G}$ is a grill (or a filter) on a space X, then the section of $\mathcal{G}$, denoted by sec$\mathcal{G}$, is given by sec$\mathcal{G}$ = {A ⊆ X : A ∩ G ≠ φ; for all G ∈ $\mathcal{G}$}

Theorem.7.3.5

(a) For any grill (filter) $\mathcal{G}$ on a space X, sec $\mathcal{G}$ is a filter (resp. grill) on X.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are respectively a filter and a grill on a space X with $\mathcal{F} \subseteq \mathcal{G}$, then there is an ultra filter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{G}$.

We note at this stage that unlike the case of filters, the notion of gπ(θ)-adherence of a grill is strictly stronger than that of gπ(θ)-convergence.

Theorem.7.3.6 If a grill G on a topological space X, gπ(θ)-adheres at some point x ∈ X, then $\mathcal{G}$ is gπ(θ)-convergent to x.

Proof: Let a grill $\mathcal{G}$ on X and $\mathcal{G}$ be gπ(θ)-adhere at x ∈ X. Then for each U ∈ gπO(x) and each G ∈ $\mathcal{G}$, gπcl(U) ∩G ≠ φ and hence gπcl(U) ∈ sec$\mathcal{G}$, for each U ∈ gπO(x), and therefore X\gπcl(U) ∉ $\mathcal{G}$. Then gπcl(U) ∈ $\mathcal{G}$ (as $\mathcal{G}$ is a grill and X ∈ $\mathcal{G}$), for each U ∈ gπO(x). Hence $\mathcal{G}$ must gπ(θ)-converge to x.

The following example shows that a gπ(θ)-convergent grill need not gπ(θ) adhere at any point of the space even if the space is finite.

Example.7.3.7 Let X = {a, b,c}, τ = {X, φ, {a},{b}, {a, b}} and

$\mathcal{G}$ = {{b}, {c}, {a, b}, {a, c}, {b, c},X}. Then it can be easily seen that $\mathcal{G}$ is a grill on X and τ is a topology on X such that gπ O(X) = τ . Now cl({a}) = {a, c}, cl({b}) = {b,c}, cl({a, b}) = X and cl({a, c}) = {a, c}. So, {cl(U) : U ∈ gπ O(x)} = {{a,c}, {b,c},X} ⊆ $\mathcal{G}$, ∀ x ∈ X. It can be verified that the grill $\mathcal{G}$ is gπ(θ)-convergent to each x ∈ X, but does not gπ(θ)-adhere at any x ∈ X.

Notation.7.3.8 Let X be a topological space. Then for any x ∈ X, we adopt the following notations:

(a). $\mathcal{G}$(gπ(θ), x) = {A ⊆ X : x ∈ gπ(θ)-clA};

(b). secG(gπ(θ), x) = {A ⊆ X : A ∩ G ≠ φ, for all G ∈ $\mathcal{G}$(gπ(θ); x)}.

Theorem.7.3.9 A grill $\mathcal{G}$ on a space X, gπ(θ)-adheres to a point x ∈ X iff $\mathcal{G}$ ⊆ $\mathcal{G}$(gπ(θ), x).

Proof: A grill $\mathcal{G}$ on a space X gπ(θ)-adheres at x ∈ X

⇒ gπcl(U) ∩ G ≠ φ, for all U ∈ gπO(x) and all G ∈ $\mathcal{G}$

⇒ x ∈ gπ(θ)-cl(G), for all G ∈ $\mathcal{G}$

⇒ G ∈ $\mathcal{G}$(gπ(θ), x), for all G ∈ $\mathcal{G}$

⇒ $\mathcal{G}$ ⊆ $\mathcal{G}$(gπ(θ), x).

Conversely, let $\mathcal{G}$ ⊆ $\mathcal{G}$(gπ(θ), x). Then for all G ∈ $\mathcal{G}$, x ∈ gπ(θ)-cl(G), so that for all U ∈ gπO(x) and for all G ∈ $\mathcal{G}$, gπcl (U) ∩ G ≠ φ. Hence $\mathcal{G}$ is gπ(θ)-adheres at x.

Theorem.7.3.7 A grill $\mathcal{G}$ on a topological space X is gπ(θ)-convergent to a point x of X iff sec$\mathcal{G}$ (gπ(θ), x) ⊆ $\mathcal{G}$.

Proof: Let $\mathcal{G}$ be a grill on X, gπ(θ)-converging to x ∈ X. Then for each U ∈ gπO(x) there exists G ∈ $\mathcal{G}$ such that G ⊆ gπcl(U), and hence

gπcl(U) ∈ $\mathcal{G}$ for each U ∈ gπO(x)..(1)

Let B ∈ sec$\mathcal{G}$(gπ(θ), x). Then X \ B $\notin$ $\mathcal{G}$(gπ(θ), x) $\Rightarrow$ x $\notin$ gπ(θ)-cl (X \ B) $\Rightarrow$ there exists U ∈ gπO(x) such that gπcl(U) ∩ (X \ B) = φ $\Rightarrow$ gπcl(U) $\subseteq$ B, where U ∈gπO(x) $\Rightarrow$ B ∈ $\mathcal{G}$ (by (1)).

Conversely, let if possible, $\mathcal{G}$ be not to gπ(θ)-converge to x. Then for some U ∈ gπO(x), gπclU $\notin$ $\mathcal{G}$ and hence gπcl(U) $\notin$ sec$\mathcal{G}$(gπ(θ), x). Thus for some A ∈ $\mathcal{G}$(gπ(θ), x),

A ∩ gπcl(U) = φ..(2)

But A ∈ $\mathcal{G}$(gπ(θ), x) $\Rightarrow$ x ∈ gπ(θ)-clA $\Rightarrow$ gπcl(U) ∩ A ≠ φ, contradicting (2). Hence the result.

Definition.7.3.11 A grill $\mathcal{G}$ on a space X is said to be:

(a) gπ(θ)-linked if for any two members A,B ∈ $\mathcal{G}$,

gπ(θ)-cl(A) ∩ gπ(θ)-cl(B) ≠ φ.

(b) gπ(θ)-conjoint if for every finite subfamily $A_1, A_2, \ldots A_n$ of $\mathcal{G}$,

gπint [$\bigcap_{i=1}^{n}$ gπ(θ)-cl(A_i)] ≠ φ:

Remark 7.3.12 It follows from the definitions that every gπ(θ)-conjoint grill is gπ(θ)-linked. That the converse is false is exhibited by the following example.

Example.7.3.12 Let X = {a, b, c}, τ = {X, φ, {a},{b}, {a, b}} and

$\mathcal{G}$ = {{a}, {b}, {a, b}, {a, c}, {b, c},X}. Then it can be easily seen that $\mathcal{G}$ is a grill on X and τ is a topology on X such that gπ O(X) = τ. Now gπ(θ)-cl({a}) = {a, c}, gπ(θ)-cl({b}) = gπ(θ)-cl({b,c})={b,c}, gπ(θ)-cl({a, b}) = X and gπ(θ)-cl({a, c}) = {a, c}. It is obvious that $\mathcal{G}$ is a gπ(θ)-linked grill on X but it is not gπ(θ)-conjoint as gπint [$\bigcap_{A \in G}$ gπ(θ)-cl(A)] = gπint {c} = φ.

7.4 gπ-closed space and grills

Definition.7.4.1 A non-empty subset A of a topological space X is called gπ-closed relative to X if for every cover $\mathcal{U}$ of A by gπ -open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup \{g\pi cl(U) : U \in \mathcal{U}_0\}$. If, in addition, A = X, then X is called a gπ-closed space.

Theorem 7.4.2 For any topological space X, the following are equivalent.

(a) X is gπ-closed.

(b) Every maximal filterbase gπ(θ)-converges to some point of X.

(c) Every filterbase gπ(θ)-adheres at some point of X.

(d) For every family $\{U_\alpha \mid \alpha \in \Delta\}$ of gπ-closed subsets such that $\cap\{U_\alpha \mid \alpha \in \Delta\} = \varphi$, there exists a finite subset Δ_0 of Δ such that $\cap\{ g\pi\ int(U_\alpha) \mid \alpha \in \Delta_0\} = \varphi$.

Proof: (a)⇒(b). Let $\mathcal{F}$ be a maximal filterbase on X. Suppose that $\mathcal{F}$ does not gπ(θ)-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not gπ(θ) adheres at any point of X. For each $x \in X$, there exists $F_x \in \mathcal{F}$ and gπ-open set U_x containing x such that $F_x \cap g\pi\ cl\ (U_x) = \varphi$. Now $\{U_x \mid x \in X\}$ is a gπ open cover for X. By (a), there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $X = \cup\{ g\pi\ cl(U_{xi}) \mid 1 \leq i \leq n\}$. Since $\mathcal{F}$ is a filterbase, there exists $F \in \mathcal{F}$ such that $F \subset \cap F_{xi}$. Now, $\cap F_{xi} =(\cap F_{xi}) \cap (\cup\ g\pi\ cl\ (U_{xi})) = \cup((\cap F_{xi}) \cap g\pi\ cl\ (U_{xi})) = \varphi$. Hence $F = \varphi$, a contradiction.

(b)⇒(c). Let $\mathcal{F}$ be a filterbase on X. Then there exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subset \mathcal{F}_0$. By (b), $\mathcal{F}_0$ is gπ(θ)-converges to some point $x \in X$. For every $F \in \mathcal{F}$ and every $U \in$ gπ open set containing x, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset g\pi\ cl\ (U)$. Now $F_0 \subset g\pi cl\ (U)$ implies that $F_0 \cap F \subset g\pi cl\ (U) \cap F$. Since $\mathcal{F}_0$ is a filterbase, $F \cap \mathcal{F}_0 \neq \varphi$ and so $F \cap g\pi\ cl\ (U) \neq \varphi$. Hence $\mathcal{F}$ is gπ(θ)-adheres at x.

(c)⇒(d). Let $\{U_\alpha \mid \alpha \in \Delta\}$ be a family of gπ -closed subsets of X such that $\cap\{U_\alpha \mid \alpha \in \Delta\} = \varphi$. Let $\mathcal{I}$ be the family of all finite subsets of Δ. Assume that $A_I = \cap\{ g\pi\ int\ (U_\alpha) \mid \alpha \in I\} \neq \varphi$ for every $I \in \mathcal{I}$. Then the family $\mathcal{F} = \{A_I \mid I \in \mathcal{I}\}$ is a filterbase on X. By hypothesis, $\mathcal{F}$ is gπ(θ)-adheres at some point x ∈ X. Since $\{X - U_\alpha \mid \alpha \in \Delta\}$ is a gπ-open cover of X, $x \in X - U_\beta$ for some β ∈ Δ. Since $g\pi\ cl\ (X - U_\beta) \cap g\pi\ int\ (U_\beta) = \varphi$, a contradiction to the fact that $\mathcal{F}$ is gπ(θ)-adheres at x ∈ X. Thus $\cap\{ g\pi\ int\ (U_\alpha) \mid \alpha \in I\} = \varphi$ for some $I \in \mathcal{I}$.

(d)⇒(a). Let $\{U_\alpha \mid \alpha \in \Delta\}$ be a cover of X by gπ -open sets of X. Then $\{X - U_\alpha \mid \alpha \in \Delta\}$ is a family of gπ -closed subsets of X such that $\cap\{X-U_\alpha \mid \alpha \in \Delta\} = \varphi$. By (d), there exists a finite subset Δ_0 of Δ such that $\cap\{ g\pi\ int\ (X-U_\alpha) \mid \alpha \in \Delta_0\} = \varphi$. Hence $X-\cap\{ g\pi\ int\ (X-U_\alpha) \mid \alpha \in \Delta_0\} = X$ which implies that $\cup\{ g\pi\ cl\ (U_\alpha) \mid \alpha \in \Delta_0\} = X$. Hence X is gπ -closed.

Theorem.7.4.3 A topological space X is gπ-closed iff every grill on X is gπ(θ)-convergent in X.

Proof: Let $\mathcal{G}$ be any grill on a gπ-closed space X. Then by Theorem 7.3.5, sec$\mathcal{G}$ is a filter on X. Let B ∈ sec$\mathcal{G}$, then $X \setminus B \notin \mathcal{G}$ and hence $B \in \mathcal{G}$ (as $\mathcal{G}$ is a grill). Thus sec$\mathcal{G} \subseteq \mathcal{G}$. Then by Theorem 7.3.5(b), there exists an ultra filter $\mathcal{U}$ on X such that sec $\mathcal{G} \subseteq \mathcal{U} \subseteq \mathcal{G}$. Now as X is gπ-closed, in view of Theorem 7. 4.2 the ultra filter $\mathcal{U}$ is gπ(θ)-convergent to some point x ∈ X. Then for each U ∈ gπO(x), there exists $F \in \mathcal{U}$ such that F ⊆ gπcl(U). Consequently, gπcl(U) $\in \mathcal{U} \subseteq \mathcal{G}$, i.e., gπcl(U) $\in \mathcal{G}$, for each U ∈ gπO(x). Hence $\mathcal{G}$ is gπ(θ)-convergent to x.

Conversely, let every grill on X be gπ(θ)-convergent to some point of X. By virtue of Theorem 7.4.2 it is enough to show that every ultra filter on X is gπ(θ)-converges in X,which is immediate from the fact that an ultra filter on X is also a grill on X.

Theorem7.4.4. A subset A of a topological space X is gπ-closed relative to X iff every grill $\mathcal{G}$ on X with $A \in \mathcal{G}$, gπ(θ)-converges to a point in A.

Proof: Let A be gπ-closed relative to X and $\mathcal{G}$ a grill on X satisfying A ∈ $\mathcal{G}$ such that $\mathcal{G}$ does not gπ(θ)-converge to any a ∈ A. Then to each a ∈ A, there corresponds some U_a ∈ gπO(a) such that gπcl(U_a) ∉ $\mathcal{G}$. Now {U_a : a ∈ A} is a cover of A by gπ open sets of X. Then A ⊆ $\bigcup_{i=1}^{n}$ gπcl(U_{ai}) = U (say) for some positive integer n. Since $\mathcal{G}$ is a grill, U ∉ $\mathcal{G}$ and hence A ∉ $\mathcal{G}$, which is a contradiction.

Conversely, let A be not gπ-closed relative to X. Then for some cover $\mathcal{V}$ = {U_α : α ∈ Δ} of A by gπ open sets of X, $\mathcal{F}$ = {A\ $\bigcup_{\alpha\in\Delta 0}$ gπcl(U_α) : Δ_0 is a finite subset of Δ} is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a grill on X with A ∈ $\mathcal{F}^*$ (as each F of $\mathcal{F}$ is a subset of A). Now for each x ∈ A, there must exist β∈ Δ such that x ∈ U_β, as $\mathcal{V}$ is a cover of A. Then for any G ∈ $\mathcal{F}^*$, G ∩ (A\ gπcl(U_β) ≠ φ, so that G ⊄ gπcl(U_β), for all G ∈ $\mathcal{G}$. Hence $\mathcal{F}^*$ cannot gπ(θ)-converge to any point of A. The contradiction proves the desired result.

Theorem.7.4.5. Let X be any topological space such that every grill $\mathcal{G}$ on X, with the property that $\bigcap_{i=1}^{n}$ gπ(θ)-clG_i ≠ φ for every finite subfamily {G_1,G_2,...G_n} of $\mathcal{G}$, gπ(θ)-adheres in X, then X is a gπ-closed space.

Proof: Let $\mathcal{V}$ be any ultrafilter on X. Then $\mathcal{V}$ is a grill on X and also for each finite sub collection {U_1,U_2,...U_n} of $\mathcal{V}$, $\bigcap_{i=1}^{n}$ gπ(θ)-clU_i ⊇ $\bigcap_{i=1}^{n}$ clU_i ≠ φ, so that $\mathcal{V}$ is a grill on X with the given condition. Hence by hypothesis, $\mathcal{V}$ is gπ(θ)-adheres. Consequently, by Theorem 7.4.2, the space X is gπ-closed.

Theorem.7.4.6 For any A ⊆ X, gπ(θ)-clA = ∩{gπcl(U) : A ⊆ U ∈ gπO(X)}.

Theorem.7.4.7 In a gπ-closed space X, every gπ(θ)-conjoint grill gπ(θ)-adheres in X.

Proof. Let $\mathcal{G}$ be a gπ(θ)-conjoint grill on a gπ-closed space X. Since gπ(θ)- cl(A) is gπ-closed for every A ⊂ X, { gπ(θ)-cl (A) | A ∈ $\mathcal{G}$} is a collection of gπ-closed sets in X. Since $\mathcal{G}$ is gπ(θ)-conjoint, for any finite subfamily {A_1,A_2, ...,A_n} of $\mathcal{G}$, gπ int (∩{gπ(θ)- cl(A_i) |1 ≤ i ≤ n}) ≠ φ and so ∩{ gπ int (gπ(θ)-cl (A_i)) |1 ≤ i ≤ n}≠ φ. Hence by Theorem 7.4.2, ∩{ gπ(θ)-cl (A) |A ∈ $\mathcal{G}$} ≠ φ and so there exists x ∈ X such that x ∈ gπ(θ)-cl (A) for every A ∈ $\mathcal{G}$ which implies that A ∈ $\mathcal{G}$(gπ(θ), x) for all A ∈ G which in turn implies that $\mathcal{G}$ ⊂ $\mathcal{G}$(gπ(θ), x). Therefore, $\mathcal{G}$ gπ(θ)-adheres at x, by Theorem 7.3.9.