CHAPTER 1

CHAPTER – 1

FUZZY SETS

In this chapter fundamentals of fuzzy sets, fuzzy topological spaces, neighbourhoods of fuzzy sets and compact fuzzy spaces due to Chang [20] are studied.

Section 1.1

Preliminary Definitions and Results of Fuzzy Sets

Definition: 1.1.1

Let X be a non-empty set and I be the unit interval [0, 1]. A fuzzy set in X is a function with domain X and values in I, that is an element of I^X. Let A, B $\in \mathrm{I}^X$. We define the following fuzzy sets

(i) A include B (i.e., $B \subset A$) by $B(x) \leq A(x)$ for every $x \in X$.

(ii) $A \cap B \in \mathrm{I}^X$ by $(A \cap B)(x) = \min\{A(x), B(x)\}$ for each $x \in X$.

(iii) $A \cup B \in \mathrm{I}^X$ by $(A \cup B)(x) = \max\{A(x), B(x)\}$ for each $x \in X$.

(iv) $A' \in \mathrm{I}^X$ by $A'(x) = 1 - A(x)$ for each $x \in X$.

Let Λ be an indexing set and $\{A_\lambda / \lambda \in \Lambda\}$ be a family of fuzzy sets in X. Then their **Union** and **intersection** are defined as follows.

$$(\cup A_\lambda)(x) = \sup\{A_\lambda(x) / \lambda \in \Lambda\}$$
$$(\cap A_\lambda)(x) = \inf\{A_\lambda(x) / \lambda \in \Lambda\}.$$

Definition: 1.1.2

Let $A_1, A_2 \ldots, A_n$ be fuzzy sets in X. The product $A = A_1 \times A_2 \times \ldots \times A_n$ of fuzzy sets is a fuzzy set in X^n defined by $A(x_1, x_2, \ldots, X_n) = \min(A_1(x_1) \ldots, A_n(X_n))$. The ordinary subsets of X can be considered as fuzzy sets by identifying them with their characteristic functions.

Ordinary subsets are referred to as **crisp sets** when they are considered as fuzzy sets. Ordinary topological spaces are referred to as **crisp topological spaces**.

If $A \subset X$, and if we consider A as a fuzzy set then, we mean

$$A(x) = 1, \quad \text{if } x \in A$$
$$= 0, \quad \text{if } x \in A$$

when an ordinary set A is considered as a fuzzy set, we write it as χ_A or A itself.

In view of this, the empty set ϕ and the whole space X can be considered as fuzzy sets by identifying them, with the constant functions **0** and **1** respectively.

Definition: 1.1.5

Let f be a function from X to Y. Let B be a fuzzy set in Y. Then **inverse image of B or preimage of B** written as $f^{-1}(B)$ is a fuzzy set in X defined by $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$.

Let A be a fuzzy set in X. The image of A, written as f(A) is a fuzzy set in Y defined by

$$f(A)(y) = \sup_{z \in f^{-1}(y)} A(z), \quad \text{if } f^{-1}(y) \text{ is non-empty}$$
$$= 0, \quad \text{otherwise,}$$

for all $y \in Y$, where $f^{-1}(y) = \{x / f(x) = y\}$.

Note

$$f(A)\, f(x) = \sup_{y \in f^{-1}(f(x))} A(y)$$
$$\geq A(x), \text{ as } x \in f^{-1}(f(x))$$

Properties: 1.1.6

Let f be a function from X to Y. Then

(a) $f^{-1}(B') = \{f^{-1}(B)\}'$ for any fuzzy set B in Y.

(b) $f(A') \supset \{f(A)\}'$ for any fuzzy set A in X.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where B_1, B_2 are fuzzy sets in Y.

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where A_1 and A_2 are fuzzy sets in S.

(e) $B \supset f\{f^{-1}(B)\}$ for any fuzzy set B in Y.

(f) $A \subset f^{-1}\{f(A)\}$ for any fuzzy set A in X.

(g) Let f be a function from X to Y and g be a function from Y to Z. Then $(g \circ f)^{-1}\{c\} = f^{-1}\{g^{-1}(c)\}$ for any fuzzy set C in Z. Where g$\circ$f is the composition of g and f.

(h) If f is into then $f(f^{-1}(A)) = A$.

Section 1.2

Fuzzy Topological Spaces

Definition: 1.2.1 (Chang [20])

A **fuzzy topology** on a set X in a collection of fuzzy sets in X satisfying the following axioms:

(i) $\phi, X \in \delta$

(ii) $A, B \in \delta \Rightarrow A \cap B \in \delta$

(iii) $A_\lambda \in \delta$ for $\lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in \delta$

The pair (X,δ) is referred to as a **Fuzzy topological space**.

Definition: 1.2.2 (Lowen [45])

Let X be a non-empty set $\delta \subset I^X$ is a fuzzy topology on X iff

(1) for every $\alpha \in [0, 1]$, the constant function $\alpha \in \delta$

(2) for every $A, B \in \delta$, $A \cap B \in \delta$

(3) for every $(A_j)_{j \in J} \subset \delta$, $\bigcup_{j \in J} A_j \in \delta$

The pair (X,δ) is referred to as **fts** in short.

Definition: 1.2.3

If (X,δ) is a fuzzy topological space, members of δ are called **open fuzzy sets**. A fuzzy set A is called a **closed fuzzy set** iff $A' \in \delta$.

Definition: 1.2.4

The **closure** and **interior** of a fuzzy set $A \in I^X$ are defined respectively as

$$\overline{A} = \cap \{B / B \supset A, B' \in \delta\}$$

$$A^{\circ} = \cup \{B / B \supset A, B \in \delta\}$$

Remark: 1.2.5

$\overline{A}$ is the smallest closed fuzzy set larger than A and that A° is the largest open fuzzy set smaller than A.

Definition: 1.2.6

Let δ be a fuzzy topology on a set X. A subfamily $\mathcal{B}$ of δ is a **base for δ** iff each member of δ can be expressed as the union of member of $\mathcal{B}$.

Definition: 1.2.7

Let (X,δ), (Y,δ') be two fts's. A mapping f of (X,δ) into (Y,δ') is **fuzzy continuous** iff for each open fuzzy set V in δ', the inverse image $f^{-1}(V)$ is in δ.

Definition: 1.2.8

Let (X, δ), (Y, δ') be two fts's. A mapping f of (X, δ) into (Y, δ') is called **fuzzy open** iff for each open fuzzy set V in δ, the image $f(V)$ is open in δ'.

Definition: 1.2.9

A bijective mapping f of a fts (X,δ) into (Y,δ') is a **fuzzy homeomorphism** iff it is fuzzy continuous and fuzzy open.

Theorem: 1.2.10

Let f be a fuzzy continuous (resp. fuzzy open) mapping of a fts (X, δ) into a fts (Y, δ') and g be a fuzzy continuous (resp. fuzzy open) mapping of (Y, δ') into a fts (Z, δ''). Then the composition $g \circ f$ is a fuzzy continuous (resp. fuzzy open) mapping of (X, δ) into (Z, δ'').

Definition: 1.2.11

Given two fuzzy topologies δ_1, δ_2 on the same set X, δ_1 is said to be finer than δ_2 (and that δ_2 is **coarser** than δ_1) if the identity mapping of (X, δ_1) into (X, δ_2) is fuzzy continuous.

Definition: 1.2.12

Let f be a mapping of a set X into a set Y and δ' be a fuzzy topology on Y. The coarser fuzzy topology δ on X for which f in fuzzy continuous is called the **inverse image under f** of δ'. Then open fuzzy sets in X are the inverse image of δ' open fuzzy sets in Y.

Definition: 1.2.13

Let f be a mapping of a set X into a set Y, and let δ' be a fuzzy topology on X. The finest fuzzy topology δ' on Y for which f is fuzzy continuous is called the **image under f** of δ. A fuzzy set B in T is U-open iff $f^{-1}(B)$ is a δ-open fuzzy set in S.

Definition: 1.2.14

Let A be a fuzzy set in X and δ a fuzzy topology on X. Then **the induced fuzzy topology** on A is the family of fuzzy subsets of A which are the intersections with A of δ-open fuzzy sets in X. The induced fuzzy topology is denoted by δ_A and the pair (A, δ_A) is called **fuzzy subspace** of (X, δ).

Definition: 1.2.15

Let δ be a fuzzy topology on a set X and δ_A the induced fuzzy topology on a fuzzy subset A of X. A sub family $\mathcal{B}'$ of δ_A is a base for δ_A iff each member of δ_A is expressed as the union of members of $\mathcal{B}'$.

Definition: 1.2.16

If (A, δ_A), (B, δ'_B) are fuzzy subspaces of fts (X, δ) and (Y, δ') respectively, and f is a mapping of (X, δ) into (Y, δ'), then we say f is a **mapping of (A, δ_A) into (B, δ'_B) iff f(A) $\subset$ B.**

Definition: 1.2.17

Let (A, δ_A), (B, δ'_B) are fuzzy subspaces of fts (X, δ) and (Y, δ') respectively. Then a mapping f of (A, δ_A) into (B, δ'_B)) is **relatively fuzzy continuous** iff for each open fuzzy set V′ is δ'_B, the intersection $f^{-1}(V') \cap A$ is in δ_A. Conversely, f is **relatively fuzzy open** iff for each open fuzzy set U′ in δ_A the image f(U′) is in δ'_B.

Definition: 1.2.18

A bijective mapping f of a fuzzy subspace (A, δ_A) of (X, δ) into a fuzzy subspace (B, δ'_B) of (X, δ') is a **relatively fuzzy homeomorphism**, iff f(A) = B and f is relatively fuzzy continuous and relatively fuzzy open.

Theorem: 1.2.19

Let (A, δ_A), (B, δ'_B) be fuzzy subspaces of fts's (X, δ) and (Y, δ') respectively and let f be a fuzzy continuous mapping of (X, δ) into (Y, δ') such that $f(A) \subset B$. Then f is a relatively fuzzy continuous mapping of (A, δ_A) into (B, U_B).

Section 1.3
Neighbourhoods of Fuzzy Sets

Definition: 1.3.1

A fuzzy set U in a fts (X, T) is a **neighbourhood of a fuzzy set** A iff there exists an open fuzzy set 0 such that $A \subset 0 \subset U$.

Theorem: 1.3.2

A fuzzy set A is open iff for each fuzzy set B contained in A; A is a neighbourhood of B.

Proof

Given fuzzy set A is open then a fuzzy set A in a fts (X, T) is a neighbourhood of a fuzzy set B iff there exist an open fuzzy set 0 such that $B \subset 0 \subset A$. Therefore A is a neighbourhood of B.

Conversely, since $A \subset A$, there exist an open fuzzy set 0 such that $A \subset 0 \subset A$. Hence, A = 0 and A is open. Therefore, the neighbourhood system of a fuzzy set is the family of all neighbourhood's of the fuzzy set.

Hence the theorem.

Theorem: 1.3.3

If U is the collection of all neighbourhoods of a fuzzy set, then finite intersection of members of U belong to U and each fuzzy set which contains a member of U belongs to U.

Proof

If R and S are neighbourhoods of a fuzzy set A, then there exist neighbourhoods R_0 and S_0 contained in R and S, respectively. Then $R \cap S$ contains the open neighbourhood $R_0 \cap S_0$ and is hence a neighbourhood of A. Thus, the intersection of two members of U is a member of U. Hence, if a fuzzy set R contains a neighbourhood of A it contains an open neighbourhood of A and consequently is itself a neighbourhood.

Hence the theorem.

Theorem: 1.3.4

Let A be a fuzzy set in a fts (X, T). Then A^0 is open and is the largest open fuzzy set contained in A. The fuzzy set A is open iff $A = A^0$.

Proof

By Definition 1.3.4, Clearly, A^0 is itself an interior fuzzy set of A.

Hence there exist an open fuzzy set 0 such that $A^0 \subset 0 \subset A$. But 0 is an interior fuzzy set of A, hence $0 \subset A^0$. Hence $A^0 = 0$. Thus, A^0 is open and is the largest open fuzzy set contained in A.If A is open, then $A \subset A^0$, for A is an interior fuzzy set of A. Hence, $A = A^0$. Hence the theorem.

Definition: 1.3.5: Sequences of Fuzzy Sets

A **sequence of fuzzy sets**, say $\{A_n, n = 1, 2,\ldots\}$ is eventually contained in a fuzzy set A iff there is an integer m such that, if $n \geq m$, then $A_n \subset A$.

The sequence is frequently contained in A iff for each integer m there is an integer n such that $n \geq m$ and $A_n \subset A$.

If the sequence is in a fts (X, T), then we say that the sequence converges to a fuzzy set A iff it is eventually contained in each neighbourhood of A.

Definition: 1.3.6

Let N be a map from the set of non-negative integers to the set of non-negative integers. Then the sequence $\{B_i, i = 1, 2, \ldots\}$ is a subsequence of a sequence $\{A_n, n = 1, 2, \ldots\}$ iff there is a map N such that $B_i = A_{N(i)}$ and for each integer m there is an integer n such that $N(i) \geq m$ whenever $i \geq n$.

Definition: 1.3.7

A fuzzy set A in a fts (X, T) is a **cluster fuzzy set** of a sequence of fuzzy sets iff the sequence is frequently contained in every neighbourhood of A.

Theorem: 1.3.8

If the neighbourhood system of each fuzzy set in a fts (X, T) is countable, then:

(a) A fuzzy set A is open iff each sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ which converges to a fuzzy set B contained in A is eventually contained in A.

(b) If A is a cluster set of sequence $\{A_n, n = 1, 2, \ldots\}$ of fuzzy sets, then there is a subsequence of the sequence converging to A.

Proof

(a) Given a fuzzy set A is open if each sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ which converges to a fuzzy set B contained in A is eventually contained in A. To prove, A is eventually contained in A. Since A is open, A is a neighbourhood of B. Hence, $\{A_n, n = 1, \ldots\}$ is eventually contained in A. Conversely, for each $B \subset A$, let $U_1, \ldots, U_n, \ldots$ be the neighbourhood system of B.

Let $V_n = \bigcap_{i=1}^{n} \{U_i\}$. Then $V_1, \ldots, V_n, \ldots$ is a sequence which is eventually contained in each neighbourhood of B. (i.e.,) $V_1, \ldots, V_n, \ldots$ converges to B. Hence, there is an m such that for $n \geq m$, $V_n \subset A$. Then V_n are neighbourhoods of B. Therefore, by theorem 1.3.2, A is open.

Hence the proof (a).

(b) Given A is a cluster fuzzy set of a sequence $\{A_n, n = 1, 2, \ldots\}$ of fuzzy sets. To prove, there is a subsequence of the sequence converging to A.

Let $R_1, \ldots, R_n, \ldots$ be a neighbourhood system of A. Let $S_n = \bigcup_{i=1}^{n} \{R_i\}$.

Then $S_1, \ldots, S_n, \ldots$ is a sequence such that $S_{n+1} \subset S_n$ for each n.

For every non-negative integer i, choose N(i) such that $N(i) \geq i$ and $A_{N(i)} \subset S_i$.

Then surely $\{A_{N(i)}, i= 1, 2, \ldots\}$ is a subsequence of the sequence $\{A_n, n = 1, 2, \ldots\}$. Clearly this subsequence converges to A.

Hence the proof (b).Hence the theorem.

Section 1.4

Fuzzy Continuous Function

Definition: 1.4.1

A function f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, U) is **F-continuous** iff the inverse of each U-open fuzzy set is T-open. Clearly if f is an F-continuous function on X to Y and g is an F-continuous function on Y to Z, then the composition $g \circ f$ is an F-continuous function X to Z, for $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ for each fuzzy set V in Z, and using the F-continuity of g and f it follows that if V is open so is $(g \circ f)^{-1}(V)$.

Theorem: 1.4.2

If X and Y are fts's and f is a function on X to Y, then the condition below are related as follows (a) and (b) are equivalent (c) and (d) are equivalent ; (a) implies (c) and (d) implies (e).

(a) The function f is F-continuous

(b) The inverse of every closed fuzzy set is closed.

(c) For each fuzzy set A in X, the inverse of every neighbourhood of f[A] is a neighbourhood of A.

(d) For each fuzzy set A in X and each neighbourhood V of f(A), there is a neighbourhood W of A such that $f(W) \subset V$.

(e) For each sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ in X which converges to a fuzzy set A in X, the sequence $\{f(A_n), n = 1, 2, \ldots\}$ converges to f(A).

Proof

(a) ⇔ (b)

The function f is F-continuous iff the inverse of every closed fuzzy set is closed. Given f is F-continuous.

To prove, the inverse of every closed fuzzy set is closed.

This is an immediate consequence of the fact that $f^{-1}(B') = \{f^{-1}(B)\}'$ for every fuzzy set B in Y.

(a) ⇔ (c)

Given f is F-continuous. To prove, the inverse of every neighbourhood of f(A) is a neighbourhood of A.

If f is F-continuous, A is a fuzzy set in X, and V is a neighbourhood of f(A), then V contains an open neighbourhood W of f[A]. Since $f[A] \subset W \subset V$, $f^{-1}(f(A)) \subset f^{-1}(W) \subset f^{-1}(V)$. But $A \subset f^{-1}(f(A))$ and $f^{-1}(W)$ is open. Consequently, $f^{-1}(V)$ is a neighbourhood of A.

(c) ⇔ (d)

Given the inverse of every neighbourhood of f(A) is a neighbourhood of A.

To prove, for each fuzzy set A in X and each neighbourhood V of f(A), there is a neighbourhood W of A such that $f(W) \subset V$. Since $f^{-1}(V)$ is a neighbourhood of A, we have $f(W) = f(f^{-1}(V)) \subset V$, where $W = f^{-1}(V)$.

(d) ⇔ (c)

Given for each fuzzy set A in X and each neighbourhood V of f(A), there is a neighbourhood W of A such that $f(W) \subset V$.
To prove, the inverse of every neighbourhood of f(A) is a neighbourhood of A.

Let V be a neighbourhood of f(A). Then there is a neighbourhood W of A such that $f(W) \subset V$. Hence, $f^{-1}(f(W)) \subset f^{-1}(V)$. Furthermore, since $W \subset f^{-1}(f(W))$, $f^{-1}(V)$ is a neighbourhood of A.

(d) ⇔ (e)

Given for each fuzzy set A in X and each neighbourhood of f(A), there is a neighbourhood W of A such that $f(W) \subset V$.

To prove, for each sequence of fuzzy sets $\{A_n, n = 1, 2, \ldots\}$ in X which converges to a fuzzy set A in X, the sequence $\{f(A_n), n = 1, 2, \ldots\}$ converges to f(A). If V is a neighbourhood of f(A), there is a neighbourhood W of A such that $f(W) \subset V$. Since $\{A_n, n = 1, 2, \ldots\}$ is eventually contained in W. (i.e.,) there is an m such that for $n \geq m$, $A_n \subset W$, we have $f(A_n) \subset f(W) \subset V$ for $n \geq m$.Therefore $\{f(A_n), n = 1, 2, \ldots\}$ converges to f(A).
Hence the theorem.

Section 1.5
Compact Fuzzy Spaces

Definition: 1.5.1

A family A of fuzzy sets is a cover of a fuzzy set B iff $B \subset \cup \{A / A \in A\}$. It is an open cover iff each member of A is an open fuzzy set. A subcover of A is a subfamily of A which is also a cover.

Definition: 1.5.2

A fts (X, T) is **compact** iff each open cover has a finite subcover.

Definition: 1.5.3

A family A of fuzzy sets has the finite intersection property iff the intersection of the members of each finite subfamily of A is nonempty.

Theorem: 1.5.4

A fts is compact iff each family of closed fuzzy sets which has the finite intersection property has a non-empty intersection.

Proof

If A is a family of fuzzy sets in a fts (X, T), then A is a cover of X

$\Leftrightarrow \cup\{A / A \in A\} \quad = \quad X$ or

$\Leftrightarrow \{\cup\{A / A \in A)\}' = \quad X' = \phi$ or

$\Leftrightarrow \cap\{A' / A \in A\} \quad = \quad \phi$ by the Demorgan's law.

Hence, the fuzzy space X is compact iff each family of open fuzzy sets in X such that no finite subfamily covers X, fails to be a cover, and this is true iff each family of closed fuzzy sets which possesses the finite intersection property has a nonempty intersection. Hence the theorem.

Theorem: 1.5.5

Le f be an F-continuous function carrying the compact fts X onto the fts Y. Then Y is compact.

Proof

Let $\mathcal{B}$ be an open cover of Y. Then, since

$$\mu_{\bigcup\limits_{B\in\mathcal{B}} f^{-1}(B)}(x) = \sup_{B\in\mathcal{B}} \{\mu_{f^{-1}(B)}(x)\}$$

$$= \sup_{B\in\mathcal{B}} \{\mu_B(f(x))\} = 1 \qquad \forall x \in X,$$

The family of all fuzzy sets of the form $f^{-1}(B)$, for B in $\mathcal{B}$, is an open cover of X which has a finite sub cover. Since f is onto, $f(f^{-1}(B)) = B$ for any fuzzy set B in Y. Thus, the family of images of members of the sub cover is a finite subfamily of $\mathcal{B}$ which covers Y and consequently Y is compact.

Hence the theorem.