Interval-Valued Fuzzy Graphs

Divya,S
(12PMA004)

Thesis Submitted to
Avinashilingam Institute for Home Science and Higher Education for Women,
Coimbatore-641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

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Signature of the Supervisor

Acknowledgement

ACKNOWLEDGEMENT

First and foremost, the author would like to offer her obeisance to the **LORD ALMIGHTY** for all the graceful blessings and showerings.

The author takes immense pleasure in thanking **Dr.T.S.K.MEENAKSHISUNDARAM,** M.A., M.Phil., Ph.D., Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing the conducive infrastructure for the conduct of the research study.

The author would like to thank **Dr. (Tmt.) SHEELA RAMACHANDRAN**, M.Sc., P.G., Dip., Ph.D., Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement and for providing the opportunity to conduct the study successfully.

The author extends her heartfelt thanks to **Dr. (Tmt.) GOWRI RAMAKRISHNAN,** M.Sc., M. Phil., Ph.D., Registrar, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement given by her during the investigation.

The author is very much grateful to express her deep sense of gratitude to **Hony. Col. Dr. (Tmt.) SAROJA PRABHAKARAN,** M.A., Dip. Ed., Ph.D., Former Vice Chancellor, Present Director, Halls of Residence, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for all the necessary support and guidance towards the completion of the study.

The author would like to thank **Dr.P.SANTHANAKRISHNAN**, Ph.D., Director Research and consultancy, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for all the encouragement.

The author is immensely pleased to express her deep sense of gratitude to **Dr. (Tmt).R. PARVATHAM**, M.Sc., Dip. Ed., M. Phil.,Ph.D., Former Dean , Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the right ambiance.

The author extends her heartfelt thanks to **Dr. (Tmt). RACHEL OOMMEN**, M.Sc., Dip. Ed., B.Ed., M. Phil., Ph.D., Former Dean , Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the support extended.

The investigator would like to thank **Dr.(Tmt.)A.PARVATHI,** M.Sc., Dip.Ed., M.Phil., Ph.D., Dean , Faculty of Science, Professor, Head of the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her excellent support, advice and encouragement during the course of the investigation.

The investigator owes her reverential gratitude to her thesis advisor **Dr. (Tmt.) K.SIVAKAMASUNDARI,** M.Sc., M.Phil., (Annamalai), Dip.H.Ed., (Madras), Ph.D., (Avinashilingam), Associate Professor, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her inspiring guidance, innovative ideas, meticulous care, critical suggestions, constant encouragement and patience throughout the completion of this work.

The investigator would like to express sincere thanks to all the **STAFF MEMBERS OF THE DEPARTMENT OF MATHEMATICS** who were responsible for the good finish of this dissertation.

Words fail to express her deep indebtedness to her **BELOVED PARENTS, LOVING SISTERS, BROTHERS, FRIENDS AND ALSO THE GRACEFUL RELATIVES** to their belief in the importance of education. They inspired her with the importance of learning and the clarity of understanding from childhood.

Contents

CONTENTS

Introduction

INTRODUCTION

"As for everything else, so for mathematical theory:

beauty can be perceived but not explained"

-Arthur Cayley

Discrete Mathematics is a branch of mathematics dealing with finite or countable processes and elements. Graph theory is an area of discrete mathematics which studies configuration (called graphs) of networking consisting of a set of nodes (called vertices) interconnecting by lines (called edges). From humble beginning and almost recreational type problems, graph theory has found its calling in the modern world complex systems and especially of the computer. Graph theory and its applications can be found not only in other branches of mathematics, but also in scientific disciplines such as engineering, computer science, operational research, management sciences and the life science.

Presently, science and technology is featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models are based on an extension of the ordinary set theory, namely, fuzzy sets. Zadeh(1965) introduced the notion of fuzzy subset of a set as a method of presenting uncertainty.

Zadeh(1975) introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers between 0 and1.Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control.

The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Rosenfeld in 1975. The fuzzy relations between fuzzy sets were first considered by Rosenfeld and he developed the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Later, Bhattacharya(1987) gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Moderson and Peng (1994). The concept of weak isomorphism, co- weak isomorphism and isomorphism between fuzzy graphs were introduced by K.R.Bhutani(1988).

The complement of a fuzzy graph was defined by Moderson(1998) and further studied by Sunitha and Vijayakumar(2002). Bhutani and Battou(2003) introduced the concept of M-strong fuzzy graphs and studied some properties. The concept of strong arcs in fuzzy graphs was discussed by Bhutani and Rosenfeld in 2003.

Hongmei and Lianhua(2009) gave the definition of interval-valued fuzzy graph. Akram(2011) defined Bipolar fuzzy graphs and studied Interval-valued fuzzy line graphs in 2012. Talebi and Rashmanlou(2012) studied the properties of isomorphism and complement on interval-valued fuzzy graphs. In 2013, Akram and Dudek studied intuitionistic fuzzy hypergraphs with applications.

The main aim of this thesis is to study the Interval-valued fuzzy graphs. The following articles are considered for the study.

1. "Interval-Valued Fuzzy Graphs" by Muhammad Akram and Wieslaw A. Dudek[2011].
2. "Complete Interval-Valued Fuzzy graphs" by Hossein Rashmanlou[2013].
3. "Certain types of Interval-Valued Fuzzy Graphs" by Muhammad Akram, Noura Omair Alshehri and Wieslaw A.Dudek[2013].
4. "Irregular Interval-Valued Fuzzy Graphs" by Madhumangal Pal and Hossein Rashmanlou[2013].

This thesis consists of four chapters including the introductory one.

Chapter I deals with all the preliminary definitions and results on Interval-Valued Fuzzy Graphs. Various fuzzy graphs used in the course of the study are also discussed.

Chapter II deals with Interval-Valued Fuzzy Graphs. Here the operations of Cartesian product, composition, union and join on interval-valued fuzzy graphs are defined and some of their properties are investigated. Isomorphism on interval-valued fuzzy graphs is also studied.

The important results obtained in this chapter are as follows:

- The Cartesian product of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.
- The composition of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.
- The Union of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.
- The Join of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.
- A weak isomorphism preserves the weight of the nodes but not necessarily the weights of the arcs.
- An isomorphism between interval-valued fuzzy graphs is an equivalence relation.

Chapter III deals with Interval-Valued Fuzzy Complete Graphs. The notion of interval valued fuzzy complete graphs is introduced and some properties of self-complementary and self-weak complementary interval-valued fuzzy complete graphs are discussed. Three new operations on interval-valued fuzzy graphs namely direct product, semi strong product and strong product are provided. The sufficient conditions for each one of them to be complete is presented.

The important results discussed in this chapter are as follows:

- The composition of two interval-valued fuzzy complete graphs is an interval-valued fuzzy complete graph.
- Two interval-valued fuzzy graphs are isomorphic iff their complements are isomorphic.
- The direct product of two interval-valued fuzzy complete graphs is an interval-valued fuzzy complete graph.
- The semi-strong product of two interval-valued fuzzy complete graphs is an interval-valued fuzzy complete graph.
- The strong product of two interval-valued fuzzy complete graphs is an interval-valued fuzzy complete graph.
- If the direct product of two interval-valued fuzzy graphs is complete, then atleast one of them must be complete.
- If either the semi-strong product or strong product of two interval-valued fuzzy graphs is complete, then atleast one of them must be complete.

Chapter IV deals with certain types of interval-valued fuzzy graphs. Here certain types of interval-valued fuzzy graphs including the balanced interval-valued fuzzy graphs, neighbourly irregular interval-valued fuzzy graphs, neighbourly total irregular interval-valued fuzzy graphs, highly irregular interval-valued fuzzy graphs, and highly total irregular interval-valued fuzzy graphs are proposed. Some interesting properties associated with these new interval-valued fuzzy graphs are investigated, and necessary and sufficient conditions under which neighbourly irregular and highly irregular interval valued fuzzy graphs are equivalent are obtained. The relationship between intuitionistic fuzzy graphs and interval valued fuzzy graphs are also described.

The main results brought out through this chapter are as follows:

- Every complete interval-valued fuzzy graph is totally regular.
- Every regular interval-valued fuzzy graph may not be balanced.
- Any complete interval-valued fuzzy graph is balanced.

- The complement of strictly balanced interval-valued fuzzy graph is strictly balanced.
- In isomorphic interval-valued fuzzy graphs, if one is balanced, then the other is also balanced.
- A complete interval-valued fuzzy graph may not be neighbourly irregular.
- A neighbourly total irregular interval-valued fuzzy graph may not be neighbourly irregular.

Review of Literature

REVIEW OF LITERATURE

Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical subjects like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, operations research, neural networks, planning etc. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research.

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph model'.

Yeh and Bang's(1975) approach for the study of fuzzy graphs were motivated by its applicability to pattern classification and clustering analysis. They worked more with the fuzzy matrix of a fuzzy graph, introduced concepts like vertex connectivity $\Omega(G)$, edge connectivity $\lambda(G)$ and established the fuzzy analogue of Whitney's theorem. They also proved that for any three real numbers a, b, c such that $0 < a \leq b \leq c$, there exists a fuzzy graph G with $\Omega(G) = a, \lambda(G) = b$ and $\delta(G) = c.$ Techniques of fuzzy clustering analysis can also be found in Yeh and Bang's(1975).

The concepts of connectedness and acyclicity levels were introduced for fuzzy graphs and several fuzzy tree definitions which are consistent with cut-level representations were given in Delgado M, et al (1985). Introducing the notion of fuzzy chordal graphs, Craine W.L(1994) had obtained the fuzzy analogue of the Gilmore and Hoffman(1964) characterization of interval graphs and also that of Fulkerson and Gross(1965).

J.N.Moderson and C.S.Peng (1994) have introduced the notions of union, join, cartesian product and composition of fuzzy graphs and had studied some basic properties. Shannon and Atanassov(1994) introduced the notion of an intuitionistic fuzzy graph.

In 2003, Deschrijver and Kerre established the relationships between some extensions of fuzzy sets. Nair(2008) established the definition of perfect and precisely perfect fuzzy graphs. Some operations on intuitionistic fuzzy graphs are discussed in Parvathi(2009).Nagoorgani and Malarvizhi(2009) established the isomorphism properties of strong fuzzy graphs.

Hongmei and Lianhua(2009) gave the definition of interval-valued fuzzy graph. Hawary(2011) defined complete fuzzy graphs and gave three new operations on it. Akram(2011) defined Bipolar fuzzy graphs and studied Interval-valued fuzzy line graphs in 2012. Talebi and Rashmanlou(2012) studied the properties of isomorphism and complement on interval-valued fuzzy graphs.. In 2013, Akram and Dudek studied intuitionistic fuzzy hypergraphs with applications.

Interval-valued fuzzy graph theory is now growing and expanding its applications. The theoretical development in this area is discussed here.

1) COMPLEMENT OF A FUZZY GRAPH

M.S.Sunitha and A.Vijaya Kumar[2002]

The definition of a complement of a fuzzy graph had been modified and some properties of self complementary fuzzy graphs were studied. The automorphism groups of a fuzzy graph and its complement were identified. A relative study of complement and some other operations on fuzzy graphs such as union, join and composition had been identified in this paper.

2) ON REGULAR FUZZY GRAPH

A.Nagoor Gani and R.Radha[2008]

In this paper, regular fuzzy graphs, total degree and totally regular fuzzy graphs were introduced and compared through various examples. A necessary and sufficient condition under which they are equivalent and the characterization of regular fuzzy graph on a cycle had been provided. Some properties of regular fuzzy graphs were studied and are examined for totally regular fuzzy graphs.

3) INTERVAL-VALUED FUZZY RELATION-BASED CLUSTERING WITH ITS APPLICATION TO PERFORMANCE EVALUATION

Yuh-YuanGuha, Miin-Shen Yangb, Rung-WeiPoc, E.Stanley Lee d[2008]

In this paper fuzzy relations were extended to interval-valued fuzzy relations and then interval-valued similarity relations for performance evaluation were constructed. Interval-valued types of fuzzy relation, similarity relation and resolution form were defined and constructed into a hierarchical structure schema. It was shown that both of procedures and results for the partition tree derived from interval-valued and crisp- valued similarity relation matrices had some corresponding relationships and different merits.

4) INTERVAL-VALUED FUZZY SUBSEMIGROUPS AND SUBGROUPS ASSOCIATED BY INTERVAL-VALUED FUZZY GRAPHS

Ju Hongmei and Wang Lianhua[2009]

In this paper the notion of interval-valued fuzzy graphs had been introduced. It was shown how to associate an Interval-Valued Fuzzy sub(semi) group with an Interval-Valued Fuzzy graph in a natural way.

5) INTERVAL-VALUED FUZZY SETS IN SOFT COMPUTING

Humberto Bustince[2010]

The reasons for which, for some specific problems, interval-valued fuzzy sets must be considered a basic component of Soft Computing had been explained in this paper.

6) INTERVAL-VALUED $(\in, \in V_{q\tilde{m}})$ – FUZZY SUBQUASIGROUPS

Muhammad Akram and Wieslaw A. Dudek[2010]

In this paper, the notion of interval-valued $(\in, \in V_{q\tilde{m}})$ – fuzzy subquasigroups and some of their properties were introduced . Interval-valued

$(\in, \in V_{q\tilde{m}})$ – fuzzy subquasigroups had been characterized by their level subsets. The implication-based such new fuzzy subquasigroups were also established.

7) INTERVAL-VALUED FUZZY GRAPHS

Muhammad Akram and Wieslaw A.Dudekb[2012]

The Cartesian product, composition, union and join on interval-valued fuzzy graphs were defined and some of their properties were investigated. The notion of interval valued fuzzy complete graphs were introduced and some properties of self-complementary and self-weak complementary interval-valued fuzzy complete graphs were presented.

8) GENERALIZED OPERATIONS ON FUZZY GRAPHS

D.Venugopalam, Nagamurthi Kumar, M.Vijaya Kumar[2013]

The Cartesian product, Composition, Union, Join on Interval-valued fuzzy graphs had been discussed. The notion of Interval-valued fuzzy complete graph was introduced.

9) SOME REMARKS ON COMPLEMENT OF FUZZY GRAPHS

K.R.Sandeep Narayan and M.S.Sunitha[2013]

In this paper, the structures of complement of many important fuzzy graphs such as Fuzzy cycles, Blocks had been studied.

10) CERTAIN TYPES OF INTERVAL-VALUED FUZZY GRAPHS

Muhammad Akram, Noura Omair Alshehri and Wieslaw A. Dudek[2013]

In this paper, certain types of interval-valued fuzzy graphs including balanced interval-valued fuzzy graphs, neighbourly irregular interval-valued fuzzy graphs, neighbourly total irregular interval-valued fuzzy graphs, highly irregular interval-valued fuzzy graphs, and highly total irregular interval-valued fuzzy graphs had been proposed. Some interesting properties associated with these new interval-valued fuzzy graphs are investigated. The necessary and sufficient conditions under which neighbourly irregular and highly irregular interval valued fuzzy graphs were obtained and also the relationship between intuitionistic fuzzy graphs and interval valued fuzzy graphs were described.

11) COMPLETE INTERVAL-VALUED FUZZY GRAPHS

Hossein Rashmanlou, Young Bae Jun[2013]

In this paper, three new operations on interval-valued fuzzy graphs; namely direct product, semi strong product and strong product were defined. Likewise, sufficient conditions for each one of them to be complete are given. It was shown that if any of these products is complete, then at least one factor was a complete interval-valued fuzzy graph.

12) DOMINATION IN INTERVAL-VALUED FUZZY GRAPHS

Pradip Debnath[2013]

The concept of domination in interval-valued fuzzy graphs was introduced. Order of an interval-valued fuzzy graph had been defined and its relation with domination number had been established. The characterization for minimal dominating set was provided and the relations between independent sets and dominating sets were found out. Further, the notion of total dominating set had been introduced and some important results are proved.

13) IRREGULAR INTERVAL-VALUED FUZZY GRAPHS

Madhumangal Pal and Hossein Rashmanlou[2013]

In this paper, irregular interval-valued fuzzy graphs and their various classifications were defined and the size of regular interval-valued fuzzy graphs was derived. The relation between highly and neighbourly irregular interval-valued fuzzy graphs had been established and some basic theorems related to the stated graphs had also been presented.

14) ANTIPODAL INTERVAL-VALUED FUZZY GRAPHS

Hossein Rashmanlou, Madhumangal Pal[2013]

The concept of antipodal interval - valued fuzzy graph and self median interval-valued fuzzy graph of the given interval-valued fuzzy graph had been introduced. The isomorphism properties of antipodal interval - valued fuzzy graphs were investigated.

15) **ISOMETRY ON INTERVAL-VALUED FUZZY GRAPHS**

Hossein Rashmanlou, Madhumangal Pal[2013]

In this paper, isometry on interval–valued fuzzy graphs were defined. It was shown that isometry on interval–valued fuzzy graphs is an equivalence relation.

16) BALANCED INTERVAL-VALUED FUZZY GRAPHS

Hossein Rashmanlou, Madhumangal Pal[2013]

In this paper, the notion of ring sum of product interval–valued fuzzy graphs and tensor product of interval–valued fuzzy graphs had been discussed. It was shown that the tensor product of two product interval–valued fuzzy graphs is a product interval–valued fuzzy graph. Three independent theorems based on ring sum, join and isomorphism of product of interval–valued fuzzy graphs were provided. Finally, balanced and strictly balanced interval-valued fuzzy graphs were defined and their properties had been investigated.

17) CONNECTIONS BETWEEN INTERVAL-VALUED FUZZY GRAPHS AND FUZZY GRAPHS WITH (S,T)-NORMS

H.HEDAYATI AND Z.JAFARI[2013]

Based on the concept of the interval-valued intuitionistic fuzzy sets, the notion of interval-valued intuitionistic fuzzy graphs with respect to t-norm T and s-norm S had been introduced and their characteristic properties were described. It was shown how to associate interval-valued intuitionistic fuzzy sub(semi)groups with interval-valued intuitionistic fuzzy graphs in a natural way.

Chapter - I

CHAPTER I

PRELIMINARY DEFINITIONS AND RESULTS

SECTION 1.1

GRAPHS AND FUZZY GRAPHS

Definition 1.1.1:

A **Graph** is an ordered pair G^*=(V,E), where V is the set of vertices of G^* and E is the set of edges of G^*.

Definition 1.1.2:

Two vertices x and y in a graph G^* are said to be **adjacent** in G^* if {x,y} is in edge of G^*

Definition 1.1.3:

A **Simple graph** is a graph without loops and multiple edges.

Definition 1.1.4:

A **Complete graph** is a simple graph in which every pair of distinct vertices is connected by an edge.

Remark 1.1.5:

The complete graph on n vertices has n vertices and n(n-1)/2 edges.

Definition 1.1.6:

By a **Complementary graph** $\overline{G^*}$ of a simple graph G^*, we mean a graph having the same vertices as G^* and such that two vertices are adjacent in $\overline{G^*}$ if and only if they are not adjacent in G^*.

Definition 1.1.7:

An **isomorphism of graphs** G_1^* and G_2^* is a bijection between the vertex sets of G_1^* and G_2^* such that any two vertices v_1 and v_2 of G_1^* are adjacent in G_1^* if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G_2^*.

Remark 1.1.8:

Isomorphic graphs are denoted by $G_1^* \cong G_2^*$.

Definition 1.1.9:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two simple graphs , we can construct several new graphs. The first construction called the **Cartesian product** of G_1^* and G_2^* gives a graph $G_1^* \times G_2^* = (V, E)$ with $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2) | x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | x_1 y_1 \in E_1, z \in V_2\}$$

Definition 1.1.10:

The **Composition of graphs** G_1^* and G_2^* is the graph $G_1^*[G_2^*] = (V_1 \times V_2, E^\circ)$, where $E^\circ = E \cup \{(x_1, x_2)(y_1, y_2) | x_1 y_1 \in E_1, x_2 \neq y_2\}$ and

$$E = \{(x, x_2)(x, y_2) | x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | x_1 y_1 \in E_1, z \in V_2\}$$

Remark 1.1.11:

Note that $G_1^*[G_2^*] \neq G_2^*[G_1^*]$.

Definition 1.1.12:

The number of vertices, the cardinality of V, is called the **order of graph** and denoted by $|V|$.

Definition 1.1.13:

The number of edges, the cardinality of E, is called the **size of graph** and denoted by $|E|$.

Definition 1.1.14:

A **path** in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2 \ldots, v_{n-1}, e_n, v_n$.

Remark 1.1.15:

The path graph with n vertices is denoted by P_n .A path is sometimes denoted by $P_n : v_0 v_1 \ldots\ldots. v_n (n > 0)$.

Definition 1.1.16:

The **length** of a path P_n in G is n.

Definition 1.1.17:

A path $P_n : v_0 v_1 \ldots\ldots. v_n$ in G^* is called **a cycle** if $v_0 = v_n$ and $n \geq 3$.

Remark 1.1.18:

The path graph P_n has $n-1$ edges and can be obtained from a cycle graph, c_n, by removing any edge.

Definition 1.1.19:

An undirected graph G^* is **connected** if there is a path between each pair of distinct vertices.

Definition 1.1.20:

The **neighbourhood** of a vertex V in a graph G^* is the induced subgraph of G^* consisting of all vertices adjacent to V and all edges connecting two such vertices. It is denoted by $N(v)$.

Definition 1.1.21:

The **degree** $\deg(v)$ of vertex v is the number of edges incident on v or equivalently, $\deg(v) = |N(v)|$.

Definition 1.1.22:

The set of neighbors, called a **(open) neighborhood N(v)** for a vertex v in a graph G^*, consists of all vertices adjacent to v but not including v; that is, $N(v) = \{u \in V | vu \in E\}$

Definition 1.1.23:

The set of neighbors, called **a closed neighborhood N[v]** for a vertex v in a graph G^*, consists of all vertices adjacent to v including v; that is, $N[v] = N(v) \cup \{v\}$.

Definition 1.1.24:

A **regular graph** is a graph where each vertex has the same number of neighbors, hat is, all the vertices have the same closed neighbourhood degree.

Definition 1.1.25: A connected graph is **highly irregular** if each of its vertices is adjacent only to vertices with distinct degrees. Equivalently, a graph

G^* is highly irregular if every two vertices of G^* connected by a path of length 2 have distinct degrees.

Definition 1.1.26:

A connected graph is said to be **neighbourly irregular** if no two adjacent vertices of G^* have the same degree. Equivalently, a connected graph G^* is called neighbourly irregular if every two adjacent vertices of G^* have distinct degree.

Definition 1.1.27:

A **fuzzy graph** with V as the underlying set is a pair $G:(\sigma,\mu)$ where $\sigma:V\to[0,1]$ is a fuzzy subset and $\mu:V\times V\to[0,1]$ is a fuzzy relation on σ such that $\mu(x,y)\le\sigma(x)\wedge\sigma(y)$ for all $x,y\in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^*:(\sigma^*,\mu^*)$ where $\sigma^*=\sup p(\sigma)=\{x\in V:\sigma(x)>0\}$ and $\mu^*=\sup p(\mu)=\{(x,y)\in V\times V:\mu(x,y)>0\}$. $H=(\sigma',\mu')$ is a fuzzy subgraph of G if there exists $X\subseteq V$ such that, $\sigma':X\to[0,1]$ is a fuzzy subset and $\mu':X\times X\to[0,1]$ is a fuzzy relation on σ' such that $\mu(x,y)\le\sigma(x)\wedge\sigma(y)$ for all $x,y\in X$.

Definition 1.1.28:

A fuzzy graph $G:(\sigma,\mu)$ is **complete** if $\mu(x,y)=\sigma(x)\wedge\sigma(y)$ for all $x,y\in V$.

Definition 1.1.29:

Two fuzzy graphs $G_1:(\sigma_1,\mu_1)$ with the crisp graph $G_1^*:(V_1,E_1)$ and $G_2:(\sigma_2,\mu_2)$ with the crisp graph $G_2^*:(V_2,E_2)$ are **isomorphic** if there exists a bijection $h:V_1\to V_2$ such that $\sigma_1(x)=\sigma_2(h(x))$ and $\mu_1(x,y)=\mu_2(h(x),h(y))$ for all $x,y\in V_1$.

Definition 1.1.30:

The **semi strong product of two fuzzy graphs** $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with the crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the fuzzy graph $G_1 \bullet G_2 : (\sigma_1 \bullet \sigma_2, \mu_1 \bullet \mu_2)$ with the crisp graph $G^* : (V_1 \times V_2, E)$ where $E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \bigcup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$,

$(\sigma_1 \bullet \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$,

$(\mu_1 \bullet \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$ and

$(\mu_1 \bullet \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)$.

Definition 1.1.31:

The **strong product of two fuzzy graphs** $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with the crisp graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the fuzzy graph $G_1 \otimes G_2 : (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ with the crisp graph $G^* : (V_1 \times V_2, E)$ where

$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \bigcup \{(u_1, w)(u_2, w) : w \in V_2, (u_1, u_2) \in E_1\}$$
$$\bigcup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$(\sigma_1 \otimes \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$,

$(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$,

$(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) = \sigma_2(w) \wedge \mu_1(u_1, u_2)$ and

$(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2)$.

Definition 1.1.32:

The direct product of two fuzzy graphs $G_1:(\sigma_1,\mu_1)$ with crisp graph $G_1^*=(V_1,E_1)$ and $G_2:(\sigma_2,\mu_2)$ with crisp graph $G_2^*=(V_2,E_2)$, where we assume that $V_1\bigcap V_2=\phi$, is defined to be the fuzzy graph $G_1\sqcap G_2:(\sigma_1\sqcap\sigma_2,\mu_1\sqcap\mu_2)$ with crisp graph $G^*:(V_1\times V_2,E)$ where
$E=\{(u_1,v_1)(u_2,v_2):(u_1,u_2)\in E_1,(v_1,v_2)\in E_2\}$

$(\sigma_1\sqcap\sigma_2)(u,v)=\sigma_1(u)\wedge\sigma_2(v),$ for all $(u,v)\in V_1\times V_2$ and

$(\mu_1\sqcap\mu_2)((u_1,v_1)(u_2,v_2))=\mu_1(u_1,u_2)\wedge\mu_2(v_1,v_2).$

Definition 1.1.33:

Partial fuzzy subgraph $\xi'=(V,\tau,v)$ of ξ is such that $\tau(v)\le\sigma(v)$ for all $v\in V$ and $\mu(u,v)\le V(u,v)$ for all $u,v\in V.$

Definition 1.1.34:

Fuzzy subgraph $\xi''=(P,\sigma',\mu')$ of ξ is such that $P\subseteq V,\sigma'(u)=\sigma(u)$ for all $u\in P,\mu'(u,v)=\mu(u,v)$ for all $u,v\in P.$

Definition 1.1.35:

The **degree of vertex** u is $d(u)=\sum_{(u,v)\in\xi}\mu(u,v).$

Definition 1.1.36:

The **minimum degree** of ξ is $\delta(\xi)=\wedge\{d(u)|u\in V\}.$

Definition 1.1.37:

The **maximum degree** of ξ is $\Delta(\xi)=\vee\{d(u)|u\in V\}.$

Definition 1.1.38:

The **total degree** of a vertex $u\in V$ is $td(u)=d(u)+\sigma(u).$

SECTION 1.2

FUZZY SETS AND INTERVAL-VALUED FUZZY SETS

Definition 1.2.1:

A **fuzzy set** A on a set X is characterized by a mapping $m: X \rightarrow [0,1]$, called the membership function.

Remark 1.2.2:

A fuzzy set is denoted as $A = (X, m)$.

Definition 1.2.3:

A **fuzzy subset** μ on a set X is a map $\mu : X \rightarrow [0,1]$.

Definition 1.2.4:

A map $\nu : X \times X \rightarrow [0,1]$ is called a **fuzzy relation** on X if $\nu(x, y) \leq \min(\mu(x), \mu(y))$ for all $x, y \in X$.

Definition 1.2.5:

A **fuzzy binary relation** on X is a fuzzy subset μ on $X \times X$.

Definition 1.2.6:

A fuzzy relation μ is **symmetric** if $\nu(x, y) = \nu(y, x)$ for all $x, y \in X$.

Definition 1.2.7:

An **interval number D** is an interval $[a^-, b^+]$ with $0 \le a^- \le b^+ \le 1$. The interval [a,a] is identified with the number $a \in [0,1]$. D[0,1] denotes the set of all interval numbers.

For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$, we define

- $r\min(D_1, D_2) = r\min([a_1^-, b_1^+],[a_2^-, b_2^+]) = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$
- $r\max(D_1, D_2) = r\max([a_1^-, b_1^+],[a_2^-, b_2^+]) = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^-.a_2^-, b_1^+ + b_2^+ - b_1^+.b_2^+]$
- $D_1 \le D_2 \Leftrightarrow a_1^- \le a_2^-$ and $b_1^+ \le b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$
- $D_1 < D_2 \Leftrightarrow D_1 \le D_2$ and $D_1 \ne D_2$
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+], where 0 \le k \le 1.$

Then, $(D[0,1], \le, \vee, \wedge)$ is a complete lattice with [0,0] as the least element and [1,1] as the greatest.

Definition 1.2.8:

The **interval-valued fuzzy set A** in V is defined by $A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in V\}$, where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \le \mu_A^+(x)$ for all $x \in V$. For any two interval-valued sets $A = [\mu_A^-(x), \mu_A^+(x)]$ and $B = [\mu_B^-(x), \mu_B^+(x)]$ in V we define:

- $A \bigcup B = \{(x, \max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))) : x \in V\}$
- $A \bigcap B = \{(x, \min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))) : x \in V\}$

Definition 1.2.9:

If $G^* = (V, E)$ is a graph, then by an interval-valued fuzzy relation B on a set E we mean an interval-valued fuzzy set such that $\mu_B^-(xy) \le \min(\mu_A^-(x), \mu_A^-(y))$, $\mu_B^+(xy) \le \min(\mu_A^+(x), \mu_A^+(y))$, for all $xy \in E$.

Definition 1.2.10:

An interval-valued fuzzy relation R in a universe $X \times Y$ is a mapping $R : X \times Y \to D[0,1]$ such that $R(x, y) = [R^-(x, y), R^+(x, y)] \in D[0,1]$ for all pairs $(x, y) \in X \times Y$.

Chapter - II

CHAPTER II

INTERVAL VALUED FUZZY GRAPHS

SECTION 2.1

OPERATIONS ON INTERVAL VALUED FUZZY GRAPHS

Definition 2.1.1:

By an **interval-valued fuzzy graph** of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$ where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on E.

Example 2.1.2:

Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$ .Let A be an interval-valued fuzzy set of V and let B be an interval-valued fuzzy set of $E \subseteq V \times V$ defined by

A=<($\frac{x}{0.2}$, $\frac{y}{0.3}$, $\frac{z}{0.4}$),($\frac{x}{0.4}$, $\frac{y}{0.5}$, $\frac{z}{0.5}$)>, B=<($\frac{xy}{0.1}$, $\frac{yz}{0.2}$, $\frac{zx}{0.1}$),($\frac{xy}{0.3}$, $\frac{yz}{0.4}$, $\frac{zx}{0.4}$)>.

We know that $A = [\mu_A^-, \mu_A^+]$ and $B = [\mu_B^-, \mu_B^+]$.

$\mu_A^-(x) = 0.2$ $\mu_A^+(x)$ =0.4 $\mu_A^-(y)$ =0.3 $\mu_A^+(y)$ =0.5 $\mu_A^-(z)$ =0.4 $\mu_A^+(z)$ =0.5

$\mu_B^-(xy)$ =0.1 $\mu_B^+(xy)$ =0.3 $\mu_B^-(yz)$ =0.2 $\mu_B^+(yz)$ =0.4 $\mu_B^-(zx)$ =0.1 $\mu_B^+(zx)$ =0.4

$E = \{xy, yz, zx\}$

$\mu_B^-(xy) \le \min(\mu_A^-(x)\mu_A^-(y))$ $\mu_B^+(xy) \le \min(\mu_A^+(x)\mu_A^+(y))$

$0.1 \le \min(0.2,0.3) = 0.2$ $0.3 \le \min(0.4,0.5) = 0.3$

$\mu_B^-(yz) \le \min(\mu_A^-(y)\mu_A^-(z))$ $\mu_B^+(yz) \le \min(\mu_A^+(y)\mu_A^+(z))$

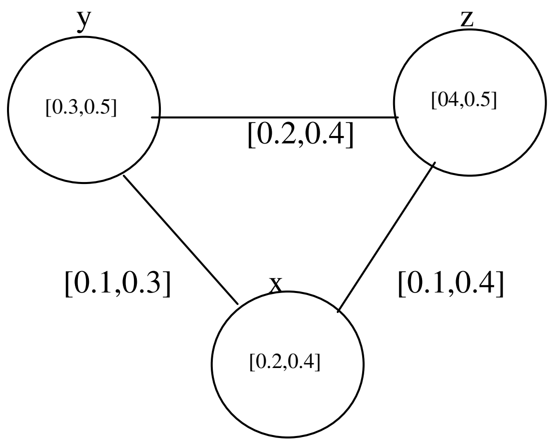
$0.2 \le \min(0.3,0.4) = 0.3$ $0.4 \le \min(0.5,0.5) = 0.5$

$\mu_B^-(zx) \le \min(\mu_A^-(z)\mu_A^-(x))$

$0.1 \le \min(0.4,0.2) = 0.2$

$\mu_B^+(zx) \le \min(\mu_A^+(z)\mu_A^+(x))$

$0.4 \le \min(0.5,0.4) = 0.4$



Thus we have $G = (A, B)$ is an interval-valued fuzzy graph of G^*.

Definition 2.1.3:

The **Cartesian product** $G_1 \times G_2$ **of two interval-valued fuzzy graphs** $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is defined as a pair $(A_1 \times A_2, B_1 \times B_2)$ such that

i. $\begin{cases} (\mu_{A_1}^- \times \mu_{A_2}^-)(x_1, x_2) = \min(\mu_{A_1}^-(x_1), \mu_{A_2}^-(x_2)) \\ (\mu_{A_1}^+ \times \mu_{A_2}^+)(x_1, x_2) = \min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(x_2)) \end{cases}$ for all $(x_1, x_2) \in V$

ii. $\begin{cases} (\mu_{B_1}^- \times \mu_{B_2}^-)(x, x_2)(x, y_2) = \min(\mu_{A_1}^-(x), \mu_{B_2}^-(x_2 y_2)) \\ (\mu_{B_1}^+ \times \mu_{B_2}^+)(x, x_2)(x, y_2) = \min(\mu_{A_1}^+(x), \mu_{B_2}^+(x_2 y_2)) \end{cases}$ for all $x \in V_1$ and $x_2 y_2 \in E_2$,

iii. $\begin{cases} (\mu_{B_1}^- \times \mu_{B_2}^-)(x_1, z)(y_1, z) = \min(\mu_{B_1}^-(x_1 y_1), \mu_{A_2}^-(z)) \\ (\mu_{B_1}^+ \times \mu_{B_2}^+)(x_1, z)(y_1, z) = \min(\mu_{B_1}^+(x_1 y_1), \mu_{A_2}^+(z)) \end{cases}$ for all $z \in V_2$ and $x_1 y_1 \in E_1$.

Example 2.1.4:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$, $E_2 = \{cd\}$ .Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ where

$A_1 = <(\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.4}, \frac{b}{0.5})>$, $\quad B_1 = <(\frac{ab}{0.1}, \frac{ab}{0.2}>$,

$A_2 = <(\frac{c}{0.1}, \frac{d}{0.2}), (\frac{c}{0.4}, \frac{d}{0.6})>$, $\quad B_1 = <(\frac{cd}{0.1}, \frac{cd}{0.3}>$. Then,

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a,c)(a,d)) = \min(\mu_{A_1}^-(a), \mu_{B_2}^-(cd)) = \min(0.2, 0.1) = 0.1$$

$$(\mu_{B_1}^+ \times \mu_{B_2}^+)((a,c)(a,d)) = \min(\mu_{A_1}^+(a), \mu_{B_2}^+(cd)) = \min(0.4, 0.3) = 0.3$$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a,c)(b,c)) = \min(\mu_{B_1}^-(ab), \mu_{A_2}^-(c)) = \min(0.1, 0.1) = 0.1$$

$$(\mu_{B_1}^+ \times \mu_{B_2}^+)((a,c)(b,c)) = \min(\mu_{B_1}^+(ab), \mu_{A_2}^+(c)) = \min (0.2, 0.4) = 0.2$$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a,d)(b,d)) = \min(\mu_{B_1}^-(ab), \mu_{A_2}^-(d)) = \min(0.1, 0.2) = 0.1$$

$$(\mu_{B_1}^+ \times \mu_{B_2}^+)((a,d)(b,d)) = \min(\mu_{B_1}^+(ab), \mu_{A_2}^+(d)) = \min (0.2, 0.6) = 0.2$$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((b,c)(b,d)) = \min(\mu_{A_1}^-(b), \mu_{B_2}^-(cd)) = \min(0.3, 0.1) = 0.1$$

$$(\mu_{B_1}^+ \times \mu_{B_2}^+)((b,c)(b,d)) = \min(\mu_{A_1}^+(b), \mu_{B_2}^+(cd)) = \min(0.5, 0.3) = 0.3$$

Now we see that $G_1 \times G_2$ is an interval-valued fuzzy graph of $G_1^* \times G_2^*$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)((a,c)(a,d)) \leq \min(\mu_{A_1}^- \times \mu_{A_2}^-(a,c), \mu_{A_1}^- \times \mu_{A_2}^-(a,d))$$

$$0.1 \le \min(\min(\mu^-_{A_1}(a), \mu^-_{A_2}(c)), \min(\mu^-_{A_1}(a), \mu^-_{A_2}(d)))$$

$$0.1 \le \min(\min(0.2, 0.1), \min(0.2, 0.2)$$

$$0.1 \le \min(0.1, 0.2) = 0.1$$

$$(\mu^+_{B_1} \times \mu^+_{B_2})((a,c)(a,d)) \le \min(\mu^+_{A_1} \times \mu^+_{A_2}(a,c), \mu^+_{A_1} \times \mu^+_{A_2}(a,d))$$

$$0.3 \le \min(\min(\mu^+_{A_1}(a), \mu^+_{A_2}(c)), \min(\mu^+_{A_1}(a), \mu^+_{A_2}(d)))$$

$$0.3 \le \min(\min(0.4, 0.4), \min(0.4, 0.6)$$

$$0.3 \le \min(0.4, 0.4) = 0.4$$

$$(\mu^-_{B_1} \times \mu^-_{B_2})((a,c)(b,c)) \le \min(\mu^-_{A_1} \times \mu^-_{A_2}(a,c), \mu^-_{A_1} \times \mu^-_{A_2}(b,c))$$

$$0.1 \le \min(\min(\mu^-_{A_1}(a), \mu^-_{A_2}(c)), \min(\mu^-_{A_1}(b), \mu^-_{A_2}(c)))$$

$$0.1 \le \min(\min(0.2, 0.1), \min(0.3, 0.1)$$

$$0.1 \le \min(0.1, 0.1) = 0.1$$

$$(\mu^+_{B_1} \times \mu^+_{B_2})((a,c)(b,c)) \le \min(\mu^+_{A_1} \times \mu^+_{A_2}(a,c), \mu^+_{A_1} \times \mu^+_{A_2}(b,c))$$

$$0.2 \le \min(\min(\mu^-_{A_1}(a), \mu^-_{A_2}(c)), \min(\mu^-_{A_1}(b), \mu^-_{A_2}(c)))$$

$$0.2 \le \min(\min(0.4, 0.4), \min(0.5, 0.4)$$

$$0.2 \le \min(0.4, 0.4) = 0.4$$

$$(\mu^-_{B_1} \times \mu^-_{B_2})((a,d)(b,d)) \le \min(\mu^-_{A_1} \times \mu^-_{A_2}(a,d), \mu^-_{A_1} \times \mu^-_{A_2}(b,d))$$

$$0.1 \le \min(\min(\mu^-_{A_1}(a), \mu^-_{A_2}(d)), \min(\mu^-_{A_1}(b), \mu^-_{A_2}(d)))$$

$$0.1 \le \min(\min(0.4, 0.4), \min(0.5, 0.4)$$

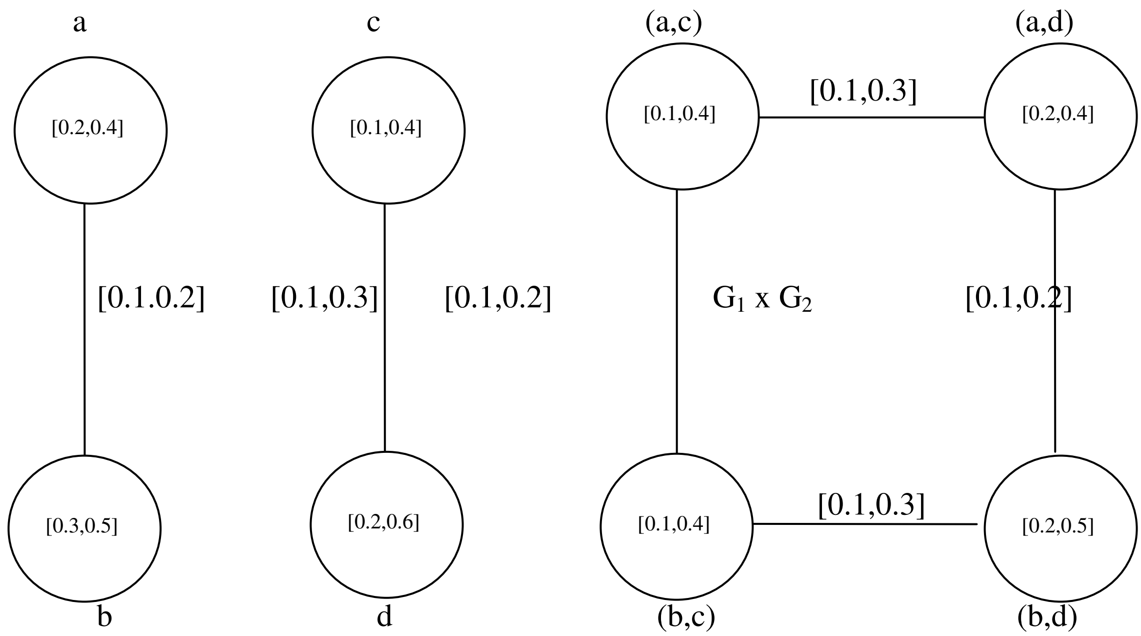
$$0.1 \le \min(0.4, 0.4) = 0.4$$

$$(\mu^+_{B_1} \times \mu^+_{B_2})((a,d)(b,d)) \le \min(\mu^+_{A_1} \times \mu^+_{A_2}(a,d), \mu^+_{A_1} \times \mu^+_{A_2}(b,d))$$

$$0.2 \le \min(\min(\mu^{+}_{A_1}(a), \mu^{+}_{A_2}(d)), \min(\mu^{+}_{A_1}(b), \mu^{+}_{A_2}(d)))$$

$$0.2 \le \min(\min(0.4, 0.6), \min(0.5, 0.6))$$

$$0.2 \le \min(0.4, 0.5) = 0.4$$



$$(\mu^{-}_{B_1} \times \mu^{-}_{B_2})((b,c)(b,d)) \le \min(\mu^{-}_{A_1} \times \mu^{-}_{A_2}(b,c), \mu^{-}_{A_1} \times \mu^{-}_{A_2}(b,d))$$

$$0.1 \le \min(\min((\mu^{-}_{A_1}(b), \mu^{-}_{A_2}(c)), \min(\mu^{-}_{A_1}(b), \mu^{-}_{A_2}(d)))$$

$$0.1 \le \min(\min(0.3, 0.1), \min(0.3, 0.2)$$

$$0.1 \le \min(0.1, 0.2) = 0.4$$

$$(\mu^{+}_{B_1} \times \mu^{+}_{B_2})((b,c)(b,d)) \le \min(\mu^{+}_{A_1} \times \mu^{+}_{A_2}(b,c), \mu^{+}_{A_1} \times \mu^{+}_{A_2}(b,d))$$

$$0.3 \le \min(\min((\mu^{+}_{A_1}(b), \mu^{+}_{A_2}(c)), \min(\mu^{+}_{A_1}(b), \mu^{+}_{A_2}(d)))$$

$$0.3 \le \min(\min(0.5, 0.4), \min(0.5, 0.6)$$

$$0.3 \le \min(0.4, 0.5) = 0.4$$

Thus we see that $G_1 \times G_2$ is an interval-valued fuzzy graph of $G_1^* \times G_2^*$.

Proposition 2.1.5:

The **Cartesian product of** $G_1 \times G_2$ **=** $(A_1 \times A_2, B_1 \times B_2)$ **of the two interval-valued fuzzy graphs** G_1^* **and** G_2^* is an interval-valued fuzzy graph of $G_1^* \times G_2^*$.

Proof:

We verify the conditions only for $B_1 \times B_2$ because the conditions for $A_1 \times A_2$ are obvious.

Let $x \in V_1$ and $x_2 y_2 \in E_2$

$$(\mu_{B_1}^- \times \mu_{B_2}^-)(x, x_2)(x, y_2) = \min(\mu_{A_1}^-(x), \mu_{B_2}^-(x_2 y_2))$$

$$\leq \min(\mu_{A_1}^-(x), \min(\mu_{A_2}^-(x_2), \mu_{A_2}^-(y_2)))$$

$$= \min(\min((\mu_{A_1}^-(x), \mu_{A_2}^-(x_2))\min(\mu_{A_1}^-(x), \mu_{A_2}^-(y_2)))$$

$$= \min((\mu_{A_1}^- \times \mu_{A_2}^-)(x, x_2), (\mu_{A_1}^- \times \mu_{A_2}^-)(x, y_2))$$

$$(\mu_{B_1}^+ \times \mu_{B_2}^+)(x, x_2)(x, y_2) = \min(\mu_{A_1}^+(x), \mu_{B_2}^+(x_2 y_2))$$

$$\leq \min(\mu_{A_1}^+(x), \min(\mu_{A_2}^+(x_2), \mu_{A_2}^+(y_2)))$$

$$= \min(\min((\mu_{A_1}^+(x), \mu_{A_2}^+(x_2))\min(\mu_{A_1}^+(x), \mu_{A_2}^-(y_2)))$$

$$= \min((\mu_{A_1}^+ \times \mu_{A_2}^+)(x, x_2), (\mu_{A_1}^+ \times \mu_{A_2}^+)(x, y_2))$$

Similarly for $z \in V_2$ and $x_1 y_1 \in E_1$ we have

$$(\mu_{B_1}^- \times \mu_{B_2}^-)(x_1, z)(y_1, z) = \min(\mu_{B_1}^-(x_1 y_1), \mu_{A_2}^-(z))$$

$$\leq \min(\min(\mu_{A_1}^-(x_1), \mu_{A_1}^-(y_1)), \mu_{A_2}^-(z))$$

$$= \min(\min(\mu_{A_1}^-(x_1), \mu_{A_2}^-(z)), \min(\mu_{A_1}^-(y_1), \mu_{A_2}^-(z)))$$

$$= \min((\mu_{A_1}^- \times \mu_{A_2}^-)(x_1, z), (\mu_{A_1}^- \times \mu_{A_2}^-)(y_1, z)).$$

$$(\mu_{B_1}^+ \times \mu_{B_2}^+)(x_1, z)(y_1, z) = \min(\mu_{B_1}^+(x_1 y_1), \mu_{A_2}^+(z))$$

$$\leq \min(\min(\mu_{A_1}^+(x_1), \mu_{A_1}^+(y_1)), \mu_{A_2}^+(z))$$

$$= \min(\min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(z)), \min(\mu_{A_1}^+(y_1), \mu_{A_2}^+(z)))$$

$$= \min(((\mu_{A_1}^+ \times \mu_{A_2}^+)(x_1, z), (\mu_{A_1}^+ \times \mu_{A_2}^+)(y_1, z)).$$

This completes the proof.

Definition 2.1.6:

The **Composition** $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ of two interval-valued fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

i. $$\begin{cases} (\mu_{A_1}^- \circ \mu_{A_2}^-)(x_1, x_2) = \min(\mu_{A_1}^-(x_1), \mu_{A_2}^-(x_2)) \\ (\mu_{A_1}^+ \circ \mu_{A_2}^+)(x_1, x_2) = \min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(x_2)) \quad \text{for all } (x_1, x_2) \in V \end{cases}$$

ii. $$\begin{cases} (\mu_{B_1}^- \circ \mu_{B_2}^-)(x, x_2)(x, y_2) = \min(\mu_{A_1}^-(x), \mu_{B_2}^-(x_2 y_2)) \\ (\mu_{B_1}^+ \circ \mu_{B_2}^+)(x, x_2)(x, y_2) = \min(\mu_{A_1}^+(x), \mu_{B_2}^+(x_2 y_2)) \quad \text{for all } x \in V_1 \text{ and} \end{cases}$$

$x_2 y_2 \in E_2$,

iii. $$\begin{cases} (\mu_{B_1}^- \circ \mu_{B_2}^-)(x_1, z)(y_1, z) = \min(\mu_{B_1}^-(x_1 y_1), \mu_{A_2}^-(z)) \\ (\mu_{B_1}^+ \circ \mu_{B_2}^+)(x_1, z)(y_1, z) = \min(\mu_{B_1}^+(x_1 y_1), \mu_{A_2}^+(z)) \quad \text{for all } z \in V_2 \text{ and} \end{cases}$$

$x_1 y_1 \in E_1$.

iv. $\begin{cases} (\mu_{B_1}^- \circ \mu_{B_2}^-)((x_1,x_2)(y_1,y_2)) = \min(\mu_{A_2}^-(x_2),\mu_{A_2}^-(y_2),\mu_{B_1}^-(x_1y_1)) \\ (\mu_{B_1}^+ \circ \mu_{B_2}^+)((x_1,x_2)(y_1,y_2)) = \min(\mu_{A_2}^+(x_2),\mu_{A_2}^+(y_2),\mu_{B_1}^+(x_1y_1)) \end{cases}$

for all $(x_1,x_2)(y_1,y_2) \in E^\circ - E$.

Example 2.1.7:

Let $G_1^* = (V_1,E_1)$ and $G_2^* = (V_2,E_2)$ be graphs such that $V_1 = \{a,b\}$, $V_2 = \{c,d\}$, $E_1 = \{ab\}$, $E_2 = \{cd\}$ .Consider two interval-valued fuzzy graphs $G_1 = (A_1,B_1)$ and $G_2 = (A_2,B_2)$ defined by

$A_1 = <(\frac{a}{0.2},\frac{b}{0.3}),(\frac{a}{0.5},\frac{b}{0.5})>$, $\quad B_1 = <(\frac{ab}{0.2},\frac{ab}{0.4}>$,

$A_2 = <(\frac{c}{0.1},\frac{d}{0.3}),(\frac{c}{0.4},\frac{d}{0.6})>$, $\quad B_1 = <(\frac{cd}{0.1},\frac{cd}{0.3}>$. Then we have

$(\mu_{B_1}^- \circ \mu_{B_2}^-)(a,c)(a,d) = \min(\mu_{A_1}^-(a),\mu_{B_2}^-(cd)) = \min(0.2,0.1) = 0.1$

$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(a,c)(a,d) = \min(\mu_{A_1}^+(a),\mu_{B_2}^+(cd)) = \min(0.5,0.3) = 0.3$

$(\mu_{B_1}^- \circ \mu_{B_2}^-)(b,c)(b,d) = \min(\mu_{A_1}^-(b),\mu_{B_2}^-(cd)) = \min(0.3,0.1) = 0.1.$

$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(b,c)(b,d) = \min(\mu_{A_1}^+(b),\mu_{B_2}^+(cd)) = \min(0.5,0.3) = 0.3$

$(\mu_{B_1}^- \circ \mu_{B_2}^-)(a,c)(b,c) = \min(\mu_{B_1}^-(ab),\mu_{A_2}^-(c)) = \min(0.2,0.1) = 0.1$

$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(a,c)(b,c) = \min(\mu_{B_1}^+(ab),\mu_{A_2}^+(c))) = \min(0.4,0.4) = 0.4$

$(\mu_{B_1}^- \circ \mu_{B_2}^-)((a,c)(b,d) = \min(\mu_{A_2}^-(c),\mu_{A_2}^-(d),\mu_{B_1}^-(ab)) = \min(0.1,0.3,0.2) = 0.1$

$(\mu_{B_1}^+ \circ \mu_{B_2}^+)((a,c)(b,d) = \min(\mu_{A_2}^+(c),\mu_{A_2}^+(d),\mu_{B_1}^+(ab)) = \min(0.4,0.6,0.4) = 0.4$

$(\mu_{B_1}^- \circ \mu_{B_2}^-)((b,c)(a,d) = \min(\mu_{A_2}^-(c),\mu_{A_2}^-(d),\mu_{B_1}^-(ab)) = \min(0.1,0.3,0.2) = 0.1$

$(\mu_{B_1}^+ \circ \mu_{B_2}^+)((b,c)(a,d) = \min(\mu_{A_2}^+(c),\mu_{A_2}^+(d),\mu_{B_1}^+(ab)) = \min(0.4,0.6,0.4) = 0.4$

Now we see that $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ is an interval-valued fuzzy graph of $G_1^*[G_2^*]$

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(a,c)(a,d) \le \min(\mu_{A_1}^- \circ \mu_{A_2}^-(a,c), \mu_{A_1}^- \circ \mu_{A_2}^-(a,d))$$

$$0.1 \le \min(\min(\mu_{A_1}^-(a), \mu_{A_2}^-(c)), \min(\mu_{A_1}^-(a), \mu_{A_2}^-(d)))$$

$$0.1 \le \min(\min(0.2,0.1),\min(0.2,0.3))$$

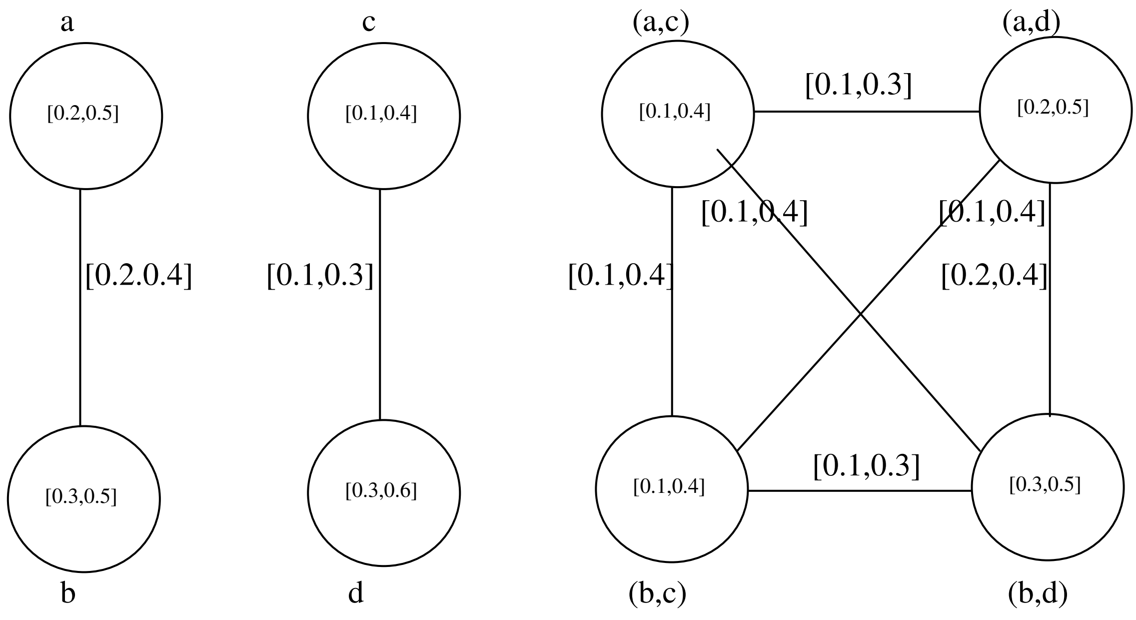
$$0.1 \le \min(0.1,0.2) = 0.1$$

$$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(a,c)(a,d) \le \min(\mu_{A_1}^+ \circ \mu_{A_2}^+(a,c), \mu_{A_1}^+ \circ \mu_{A_2}^+(a,d))$$

$$0.3 \le \min(\min(\mu_{A_1}^+(a), \mu_{A_2}^+(c)), \min(\mu_{A_1}^+(a), \mu_{A_2}^+(d)))$$

$$0.3 \le \min(\min(0.5,0.4),\min(0.5,0.6))$$

$$0.3 \le \min(0.4,0.5) = 0.4$$



G1 G2 G1[G2]

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(b,c)(b,d) \le \min(\mu_{A_1}^- \circ \mu_{A_2}^-(b,c), \mu_{A_1}^- \circ \mu_{A_2}^-(b,d))$$

$$0.1 \le \min(\min(\mu_{A_1}^-(b), \mu_{A_2}^-(c)), \min(\mu_{A_1}^-(b), \mu_{A_2}^-(d)))$$

$$0.1 \le \min(\min(0.3,0.1),\min(0.3,0.3))$$

$$0.1 \le \min(0.1,0.3) = 0.1$$

$$(\mu_{B_1}^{+} \circ \mu_{B_2}^{+})(b,c)(b,d) \le \min(\mu_{A_1}^{+} \circ \mu_{A_2}^{+}(b,c), \mu_{A_1}^{+} \circ \mu_{A_2}^{+}(b,d))$$

$$0.3 \le \min(\min(\mu_{A_1}^{+}(b), \mu_{A_2}^{+}(c)), \min(\mu_{A_1}^{+}(b), \mu_{A_2}^{+}(d)))$$

$$0.3 \le \min(\min(0.5,0.4),\min(0.5,0.6))$$

$$0.3 \le \min(0.4,0.5) = 0.4$$

$$(\mu_{B_1}^{-} \circ \mu_{B_2}^{-})(a,c)(b,c) \le \min(\mu_{A_1}^{-} \circ \mu_{A_2}^{-}(a,c), \mu_{A_1}^{-} \circ \mu_{A_2}^{-}(b,c))$$

$$0.1 \le \min(\min(\mu_{A_1}^{-}(a), \mu_{A_2}^{-}(c)), \min(\mu_{A_1}^{-}(b), \mu_{A_2}^{-}(c)))$$

$$0.1 \le \min(\min(0.2,0.1),\min(0.3,0.1))$$

$$0.1 \le \min(0.1,0.1) = 0.1$$

$$(\mu_{B_1}^{+} \circ \mu_{B_2}^{+})(a,c)(b,c) \le \min(\mu_{A_1}^{+} \circ \mu_{A_2}^{+}(a,c), \mu_{A_1}^{+} \circ \mu_{A_2}^{+}(b,c))$$

$$0.4 \le \min(\min(\mu_{A_1}^{+}(a), \mu_{A_2}^{+}(c)), \min(\mu_{A_1}^{+}(b), \mu_{A_2}^{+}(c)))$$

$$0.4 \le \min(\min(0.5,0.4),\min(0.5,0.4))$$

$$0.4 \le \min(0.4,0.4) = 0.4$$

$$(\mu_{B_1}^{-} \circ \mu_{B_2}^{-})(a,d)(b,d) \le \min(\mu_{A_1}^{-} \circ \mu_{A_2}^{-}(a,d), \mu_{A_1}^{-} \circ \mu_{A_2}^{-}(b,d))$$

$$0.2 \le \min(\min(\mu_{A_1}^{-}(a), \mu_{A_2}^{-}(d)), \min(\mu_{A_1}^{-}(b), \mu_{A_2}^{-}(d)))$$

$$0.2 \le \min(\min(0.2,0.3),\min(0.3,0.3))$$

$$0.2 \le \min(0.2,0.3) = 0.2$$

$$(\mu_{B_1}^{+} \circ \mu_{B_2}^{+})(a,d)(b,d) \le \min(\mu_{A_1}^{+} \circ \mu_{A_2}^{+}(a,d), \mu_{A_1}^{+} \circ \mu_{A_2}^{+}(b,d))$$

$$0.4 \le \min(\min(\mu_{A_1}^{+}(a), \mu_{A_2}^{+}(d)), \min(\mu_{A_1}^{+}(b), \mu_{A_2}^{+}(d)))$$

$$0.4 \le \min(\min(0.5,0.6), \min(0.5,0.6)$$

$$0.4 \le \min(0.5,0.5) = 0.5$$

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(a,c)(b,d) \le \min(\mu_{A_1}^- \circ \mu_{A_2}^-(a,c), \mu_{A_1}^- \circ \mu_{A_2}^-(b,d))$$

$$0.1 \le \min(\min(\mu_{A_1}^-(a), \mu_{A_2}^-(c)), \min(\mu_{A_1}^-(b), \mu_{A_2}^-(d)))$$

$$0.1 \le \min(\min(0.2,0.1), \min(0.3,0.3))$$

$$0.1 \le \min(0.1,0.3) = 0.1$$

$$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(a,c)(b,d) \le \min(\mu_{A_1}^+ \circ \mu_{A_2}^+(a,c), \mu_{A_1}^+ \circ \mu_{A_2}^+(b,d))$$

$$0.4 \le \min(\min(\mu_{A_1}^+(a), \mu_{A_2}^+(c)), \min(\mu_{A_1}^+(b), \mu_{A_2}^+(d)))$$

$$0.4 \le \min(\min(0.5,0.4), \min(0.5,0.6))$$

$$0.4 \le \min(0.4,0.5) = 0.4$$

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(b,c)(a,d) \le \min(\mu_{A_1}^- \circ \mu_{A_2}^-(b,c), \mu_{A_1}^- \circ \mu_{A_2}^-(a,d))$$

$$0.1 \le \min(\min(\mu_{A_1}^-(b), \mu_{A_2}^-(c)), \min(\mu_{A_1}^-(a), \mu_{A_2}^-(d)))$$

$$0.1 \le \min(\min(0.3,0.1), \min(0.2,0.3))$$

$$0.1 \le \min(0.1,0.2) = 0.2$$

$$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(b,c)(a,d) \le \min(\mu_{A_1}^+ \circ \mu_{A_2}^+(b,c), \mu_{A_1}^+ \circ \mu_{A_2}^+(a,d))$$

$$0.4 \le \min(\min(\mu_{A_1}^+(b), \mu_{A_2}^+(c)), \min(\mu_{A_1}^+(a), \mu_{A_2}^+(d)))$$

$$0.4 \le \min(\min(0.5,0.4), \min(0.5,0.6))$$

$$0.4 \le \min(0.4,0.5) = 0.4$$

Thus we have $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ is an interval-valued fuzzy graph of $G_1^*[G_2^*]$.

Proposition 2.1.8:

The composition $G_1[G_2]$ of two interval-valued fuzzy graphs G_1 and G_2 of G_1^* and G_2^* is an interval-valued fuzzy graph of $G_1^*[G_2^*]$.

Proof:

The conditions for $B_1 \circ B_2$ are verified here.

In the case $x \in V_1$ and $x_2 y_2 \in E_2$, according to (ii) we obtain

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(x, x_2)(x, y_2) = \min(\mu_{A_1}^-(x), \mu_{B_2}^-(x_2 y_2))$$

$$\leq \min(\mu_{A_1}^-(x), \min(\mu_{A_2}^-(x_2), \mu_{A_2}^-(y_2)))$$

$$= \min(\min(\mu_{A_1}^-(x), \mu_{A_2}^-(x_2)) \min(\mu_{A_1}^-(x), \mu_{A_2}^-(y_2)))$$

$$= \min((\mu_{A_1}^- \circ \mu_{A_2}^-)(x, x_2), (\mu_{A_1}^- \circ \mu_{A_2}^-)(x, y_2)$$

$$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(x, x_2)(x, y_2) = \min(\mu_{A_1}^+(x), \mu_{B_2}^+(x_2 y_2))$$

$$\leq \min(\mu_{A_1}^+(x), \min(\mu_{A_2}^+(x_2), \mu_{A_2}^+(y_2)))$$

$$= \min(\min(\mu_{A_1}^+(x), \mu_{A_2}^+(x_2)) \min(\mu_{A_1}^+(x), \mu_{A_2}^-(y_2)))$$

$$= \min((\mu_{A_1}^+ \circ \mu_{A_2}^+)(x, x_2), (\mu_{A_1}^+ \circ \mu_{A_2}^+)(x, y_2)$$

In the case $z \in V_2$ and $x_1 y_1 \in E_1$ according to (iii), we obtain

$$(\mu_{B_1}^- \circ \mu_{B_2}^-)(x_1, z)(y_1, z) = \min(\mu_{B_1}^-(x_1 y_1), \mu_{A_2}^-(z))$$

$$\leq \min(\min(\mu_{A_1}^-(x_1), \mu_{A_1}^-(y_1)), \mu_{A_2}^-(z))$$

$$= \min(\min(\mu_{A_1}^-(x_1), \mu_{A_2}^-(z)), \min(\mu_{A_1}^-(y_1), \mu_{A_2}^-(z)))$$

$$= \min(((\mu_{A_1}^- \circ \mu_{A_2}^-)(x_1, z), (\mu_{A_1}^- \circ \mu_{A_2}^-)(y_1, z)).$$

$$(\mu_{B_1}^+ \circ \mu_{B_2}^+)(x_1, z)(y_1, z) = \min(\mu_{B_1}^+(x_1 y_1), \mu_{A_2}^+(z))$$

$$\leq \min(\min(\mu^+_{A_1}(x_1), \mu^+_{A_1}(y_1)), \mu^+_{A_2}(z))$$

$$= \min(\min(\mu^+_{A_1}(x_1), \mu^+_{A_2}(z)), \min(\mu^+_{A_1}(y_1), \mu^+_{A_2}(z)))$$

$$= \min((\mu^+_{A_1} \circ \mu^+_{A_2})(x_1, z), (\mu^+_{A_1} \circ \mu^+_{A_2})(y_1, z)).$$

In the case $(x_1, x_2)(y_1, y_2) \in E^\circ - E$ we have $x_1 y_1 \in E_1$ and $x_2 \neq y_2$ which according to (iv) implies

$$(\mu^-_{B_1} \circ \mu^-_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2), \mu^-_{B_1}(x_1 y_1))$$

$$\leq \min(\mu^-_{A_2}(x_2), \mu^-_{A_2}(y_2), \min(\mu^-_{A_1}(x_1), \mu^-_{A_1}(y_1)))$$

$$=\min(\min((\mu^-_{A_1}(x_1), \mu^-_{A_2}(x_2)), \min(\mu^-_{A_1}(y_1), \mu^-_{A_2}(y_2)))$$

$$= \min((\mu^-_{A_1} \circ \mu^-_{A_2})(x_1, x_2), (\mu^-_{A_1} \circ \mu^-_{A_2})(y_1, y_2))$$

$$(\mu^+_{B_1} \circ \mu^+_{B_2})((x_1, x_2)(y_1, y_2)) = \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2), \mu^+_{B_1}(x_1 y_1)$$

$$\leq \min(\mu^+_{A_2}(x_2), \mu^+_{A_2}(y_2), \min(\mu^+_{A_1}(x_1), \mu^+_{A_1}(y_1)))$$

$$=\min(\min((\mu^+_{A_1}(x_1), \mu^+_{A_2}(x_2)), \min(\mu^+_{A_1}(y_1), \mu^+_{A_2}(y_2)))$$

$$= \min((\mu^+_{A_1} \circ \mu^+_{A_2})(x_1, x_2), (\mu^+_{A_1} \circ \mu^+_{A_2})(y_1, y_2))$$

This completes the proof.

Definition 2.1.9:

The **Union** $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ of two interval-valued fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

$$\text{(A)} \begin{cases} (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \mu^-_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \mu^-_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ (\mu^-_{A_1} \cup \mu^-_{A_2})(x) = \max(\mu^-_{A_1}(x), \mu^-_{A_2}(x)) & \text{if } x \in V_1 \cap V_2 \end{cases}$$

$$
\text{(B)}\begin{cases}
(\mu^{+}_{A_1} \cup \mu^{+}_{A_2})(x) = \mu^{+}_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\
(\mu^{+}_{A_1} \cup \mu^{+}_{A_2})(x) = \mu^{+}_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\
(\mu^{+}_{A_1} \cup \mu^{+}_{A_2})(x) = \max(\mu^{+}_{A_1}(x), \mu^{+}_{A_2}(x)) & \text{if } x \in V_1 \cap V_2
\end{cases}
$$

$$
\text{(C)}\begin{cases}
(\mu^{-}_{B_1} \cup \mu^{-}_{B_2})(xy) = \mu^{-}_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\
(\mu^{-}_{B_1} \cup \mu^{-}_{B_2})(xy) = \mu^{-}_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\
(\mu^{-}_{B_1} \cup \mu^{-}_{B_2})(xy) = \max(\mu^{-}_{B_1}(xy), \mu^{-}_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2
\end{cases}
$$

$$
\text{(D)}\begin{cases}
(\mu^{+}_{B_1} \cup \mu^{+}_{B_2})(xy) = \mu^{+}_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\
(\mu^{+}_{B_1} \cup \mu^{+}_{B_2})(xy) = \mu^{+}_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\
(\mu^{+}_{B_1} \cup \mu^{+}_{B_2})(xy) = \max(\mu^{+}_{B_1}(xy), \mu^{+}_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2
\end{cases}
$$

Example 2.1.10:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a,b,c,d,e\}$, $V_2 = \{a,b,c,d,f\}$, $E_1 = \{ab,bc,be,ce,ad,ed\}$, $E_2 = \{ab,bc,cf,bf,bd\}$ .Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ defined by

$$A_1 = <(\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.3}, \frac{d}{0.3}, , \frac{e}{0.2}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.6}, \frac{d}{0.7}, \frac{e}{0.6})>,$$

$$B_1 = <(\frac{ab}{0.1}, \frac{bc}{0.2}, \frac{ce}{0.1}, \frac{be}{0.2}, \frac{ad}{0.1}, \frac{de}{0.1}), (\frac{ab}{0.3}, \frac{bc}{0.4}, \frac{ce}{0.5}, \frac{be}{0.5}, \frac{ad}{0.3}, \frac{de}{0.6})>,$$

$$A_1 = <(\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.3}, \frac{d}{0.2}, \frac{f}{0.4}), (\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.6}, \frac{d}{0.6}, \frac{f}{0.6})>$$

$$B_2 = <(\frac{ab}{0.1}, \frac{bc}{0.2}, \frac{cf}{0.1}, \frac{bf}{0.1}, \frac{bd}{0.2}), (\frac{ab}{0.2}, \frac{bc}{0.4}, \frac{cf}{0.5}, \frac{bf}{0.2}, \frac{bd}{0.5})>$$

Then, according to the above definition:

$(\mu^{-}_{A_1} \cup \mu^{-}_{A_2})(a) = \max(\mu^{-}_{A_1}(a), \mu^{-}_{A_2}(a)) = \max(0.2,0.2) = 0.2$

$(\mu_{A_1}^- \cup \mu_{A_2}^-)(b) = \max(\mu_{A_1}^-(b), \mu_{A_2}^-(b)) = \max(0.4, 0.2) = 0.4$

$(\mu_{A_1}^- \cup \mu_{A_2}^-)(c) = \max(\mu_{A_1}^-(c), \mu_{A_2}^-(c)) = \max(0.3, 0.3) = 0.3$

$(\mu_{A_1}^- \cup \mu_{A_2}^-)(d) = \max(\mu_{A_1}^-(d), \mu_{A_2}^-(d)) = \max(0.3, 0.2) = 0.3$

$(\mu_{A_1}^- \cup \mu_{A_2}^-)(e) = \mu_{A_1}^-(e) = 0.2,$ $(\mu_{A_1}^- \cup \mu_{A_2}^-)(f) = \mu_{A_2}^-(f) = 0.4$

$(\mu_{A_1}^+ \cup \mu_{A_2}^+)(a) = \max(\mu_{A_1}^+(a), \mu_{A_2}^+(a)) = \max(0.4, 0.4) = 0.4$

$(\mu_{A_1}^+ \cup \mu_{A_2}^+)(b) = \max(\mu_{A_1}^+(b), \mu_{A_2}^+(b)) = \max(0.5, 0.5) = 0.5$

$(\mu_{A_1}^+ \cup \mu_{A_2}^+)(c) = \max(\mu_{A_1}^+(c), \mu_{A_2}^+(c)) = \max(0.6, 0.6) = 0.6$

$(\mu_{A_1}^+ \cup \mu_{A_2}^+)(d) = \max(\mu_{A_1}^+(d), \mu_{A_2}^+(d)) = \max(0.7, 0.6) = 0.7$

$(\mu_{A_1}^+ \cup \mu_{A_2}^+)(e) = \mu_{A_1}^+(e) = 0.6$ $(\mu_{A_1}^+ \cup \mu_{A_2}^+)(f) = \mu_{A_2}^+(f) = 0.6$

$(\mu_{B_1}^- \cup \mu_{B_2}^-)(ab) = \max(\mu_{B_1}^-(ab), \mu_{B_2}^-(ab)) = \max(0.1, 0.1) = 0.1$

$(\mu_{B_1}^- \cup \mu_{B_2}^-)(bc) = \max(\mu_{B_1}^-(bc), \mu_{B_2}^-(bc)) = \max(0.2, 0.2) = 0.2$

$(\mu_{B_1}^- \cup \mu_{B_2}^-)(ce) = \mu_{B_1}^-(ce) = 0.1$ $(\mu_{B_1}^- \cup \mu_{B_2}^-)(be) = \mu_{B_1}^-(be) = 0.2$

$(\mu_{B_1}^- \cup \mu_{B_2}^-)(ad) = \mu_{B_1}^-(ad) = 0.1$ $(\mu_{B_1}^- \cup \mu_{B_2}^-)(de) = \mu_{B_1}^-(de) = 0.1$

$(\mu_{B_1}^- \cup \mu_{B_2}^-)(bd) = \mu_{B_2}^-(bd) = 0.2$ $(\mu_{B_1}^- \cup \mu_{B_2}^-)(bf) = \mu_{B_2}^-(bd) = 0.1$

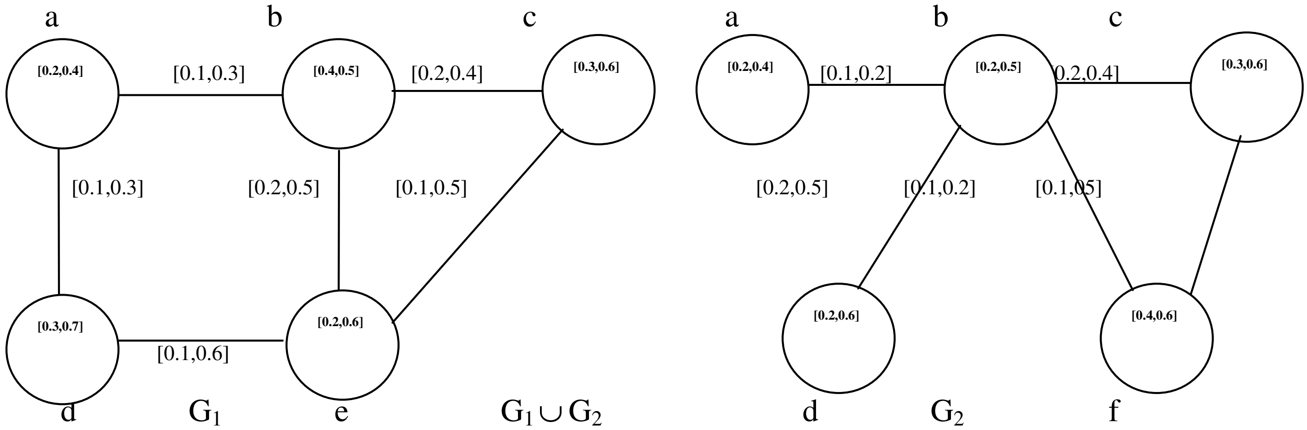
$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(ab) = \max(\mu_{B_1}^+(ab), \mu_{B_2}^+(ab)) = \max(0.3, 0.2) = 0.3$

$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(bc) = \max(\mu_{B_1}^+(bc), \mu_{B_2}^+(bc)) = \max(0.4, 0.4) = 0.4$

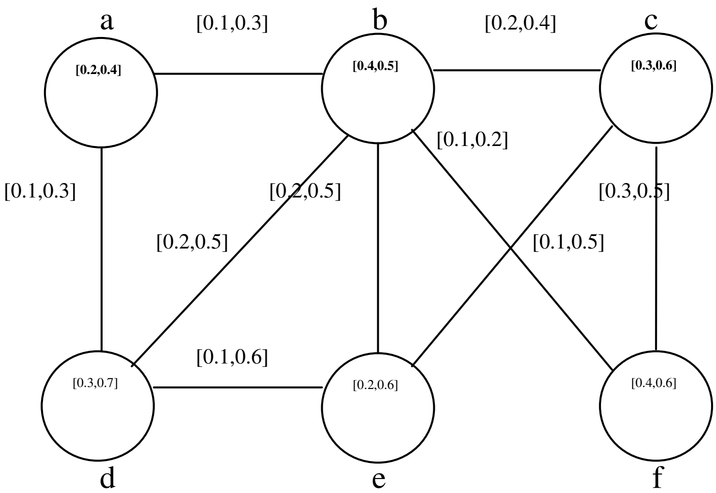
$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(ce) = \mu_{B_1}^+(ce) = 0.5$ $(\mu_{B_1}^+ \cup \mu_{B_2}^+)(be) = \mu_{B_1}^+(be) = 0.5$

$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(ad) = \mu_{B_1}^+(ad) = 0.3$ $(\mu_{B_1}^+ \cup \mu_{B_2}^+)(de) = \mu_{B_1}^+(de) = 0.6$

$(\mu^{+}_{B_1} \cup \mu^{+}_{B_2})(bd) = \ \mu^{+}_{B_2}(bd) = 0.5 \qquad (\mu^{+}_{B_1} \cup \mu^{+}_{B_2})(bf) = \ \mu^{+}_{B_2}(bd) \ = 0.2$



$G_1 \cup G_2$



Now we see that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* \cup G_2^*$

$$(\mu^{-}_{B_1} \cup \mu^{-}_{B_2})(ab) \leq \min(\mu^{-}_{A_1} \cup \mu^{-}_{A_2}(a), \mu^{-}_{A_1} \cup \mu^{-}_{A_2}(b))$$

$$0.1 \leq \min(\max(\mu^{-}_{A_1}(a), \mu^{-}_{A_2}(a)), \max(\mu^{-}_{A_1}(b), \mu^{-}_{A_2}(b)))$$

$$0.1 \leq \min(\max(0.2,0.2), \max(0.4,0.2))$$

$$0.1 \leq \min(0.2,0.4) = 0.2$$

$$(\mu^{-}_{B_1} \cup \mu^{-}_{B_2})(bc) \leq \min(\mu^{-}_{A_1} \cup \mu^{-}_{A_2}(b), \mu^{-}_{A_1} \cup \mu^{-}_{A_2}(c))$$

$$0.2 \leq \min(\max(\mu^{-}_{A_1}(b), \mu^{-}_{A_2}(b)), \max(\mu^{-}_{A_1}(c), \mu^{-}_{A_2}(c)))$$

$$0.2 \le \min(\max(0.4,0.2),\max(0.3,0.3))$$

$$0.2 \le \min(0.4,0.3) = 0.3$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(ce) \le \min(\mu_{A_1}^- \cup \mu_{A_2}^-(c), \mu_{A_1}^- \cup \mu_{A_2}^-(e))$$

$$0.1 \le \min(\max(\mu_{A_1}^-(c), \mu_{A_2}^-(c)), \mu_{A_1}^-(e))$$

$$0.1 \le \min(\max(0.3,0.3),0.2)$$

$$0.1 \le \min(0.3,0.2) = 0.2$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(be) \le \min(\mu_{A_1}^- \cup \mu_{A_2}^-(b), \mu_{A_1}^- \cup \mu_{A_2}^-(e))$$

$$0.2 \le \min(\max(\mu_{A_1}^-(b), \mu_{A_2}^-(b)), \mu_{A_1}^-(e))$$

$$0.2 \le \min(\max(0.4,0.2),0.2)$$

$$0.2 \le \min(0.4,0.2) = 0.2$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(ad) \le \min(\mu_{A_1}^- \cup \mu_{A_2}^-(d), \mu_{A_1}^- \cup \mu_{A_2}^-(d))$$

$$0.1 \le \min(\max(\mu_{A_1}^-(a), \mu_{A_2}^-(a)), \max(\mu_{A_1}^-(d), \mu_{A_2}^-(d)))$$

$$0.1 \le \min(\max(0.2,0.2),\max(0.3,0.2))$$

$$0.1 \le \min(0.2,0.3) = 0.2$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(de) \le \min(\mu_{A_1}^- \cup \mu_{A_2}^-(d), \mu_{A_1}^- \cup \mu_{A_2}^-(e))$$

$$0.1 \le \min(\max(\mu_{A_1}^-(d), \mu_{A_2}^-(d)), \mu_{A_1}^-(e))$$

$$0.1 \le \min(\max(0.3,0.2),0.2)$$

$$0.1 \le \min(0.3,0.2) = 0.2$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(bd) \le \min(\mu_{A_1}^- \cup \mu_{A_2}^-(b), \mu_{A_1}^- \cup \mu_{A_2}^-(d))$$

$$0.2 \le \min(\max(\mu_{A_1}^-(b), \mu_{A_2}^-(b)), \max(\mu_{A_1}^-(d), \mu_{A_2}^-(d)))$$

$$0.2 \leq \min(\max(0.4, 0.2), \max(0.3, 0.2))$$

$$0.2 \leq \min(0.4, 0.3) = 0.3$$

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(bf) \leq \min(\mu_{A_1}^- \cup \mu_{A_2}^-(b), \mu_{A_1}^- \cup \mu_{A_2}^-(f))$$

$$0.1 \leq \min(\max(\mu_{A_1}^-(b), \mu_{A_2}^-(b)), \mu_{A_1}^-(f))$$

$$0.1 \leq \min(\max(0.4, 0.2), 0.4)$$

$$0.1 \leq \min(0.4, 0.4) = 0.4$$

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(ab) \leq \min(\mu_{A_1}^+ \cup \mu_{A_2}^+(a), \mu_{A_1}^+ \cup \mu_{A_2}^+(b))$$

$$0.3 \leq \min(\max(\mu_{A_1}^+(a), \mu_{A_2}^+(a)), \max(\mu_{A_1}^+(b), \mu_{A_2}^+(b)))$$

$$0.3 \leq \min(\max(0.4, 0.4), \max(0.5, 0.5))$$

$$0.3 \leq \min(0.4, 0.5) = 0.4$$

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(bc) \leq \min(\mu_{A_1}^+ \cup \mu_{A_2}^+(b), \mu_{A_1}^+ \cup \mu_{A_2}^+(c))$$

$$0.4 \leq \min(\max(\mu_{A_1}^+(b), \mu_{A_2}^+(b)), \max(\mu_{A_1}^+(c), \mu_{A_2}^+(c)))$$

$$0.4 \leq \min(\max(0.5, 0.5), \max(0.6, 0.6))$$

$$0.4 \leq \min(0.5, 0.6) = 0.5$$

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(ce) \leq \min(\mu_{A_1}^+ \cup \mu_{A_2}^+(c), \mu_{A_1}^+ \cup \mu_{A_2}^+(e))$$

$$0.5 \leq \min(\max(\mu_{A_1}^+(c), \mu_{A_2}^+(c)), \mu_{A_1}^+(e))$$

$$0.5 \leq \min(\max(0.6, 0.6), 0.6)$$

$$0.5 \leq \min(0.6, 0.6) = 0.6$$

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(be) \leq \min(\mu_{A_1}^+ \cup \mu_{A_2}^+(b), \mu_{A_1}^+ \cup \mu_{A_2}^+(e))$$

$$0.5 \leq \min(\max(\mu_{A_1}^+(b), \mu_{A_2}^+(b)), \mu_{A_1}^+(e))$$

$$0.5 \le \min(\max(0.5,0.5),0.6)$$

$$0.5 \le \min(0.5,0.6) = 0.5$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(ad) \le \min(\mu^+_{A_1} \cup \mu^+_{A_2}(d), \mu^+_{A_1} \cup \mu^+_{A_2}(d))$$

$$0.3 \le \min(\max(\mu^-_{A_1}(a), \mu^-_{A_2}(a)), \max(\mu^-_{A_1}(d), \mu^-_{A_2}(d)))$$

$$0.3 \le \min(\max(0.4,0.4),\max(0.7,0.6))$$

$$0.3 \le \min(0.4,0.7) = 0.4$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(de) \le \min(\mu^+_{A_1} \cup \mu^+_{A_2}(d), \mu^+_{A_1} \cup \mu^+_{A_2}(e))$$

$$0.1 \le \min(\max(\mu^+_{A_1}(d), \mu^+_{A_2}(d)), \mu^+_{A_1}(e))$$

$$0.1 \le \min(\max(0.7,0.6),0.6)$$

$$0.1 \le \min(0.7,0.6) = 0.6$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(bd) \le \min(\mu^+_{A_1} \cup \mu^+_{A_2}(b), \mu^+_{A_1} \cup \mu^+_{A_2}(d))$$

$$0.5 \le \min(\max(\mu^+_{A_1}(b), \mu^+_{A_2}(b)), \max(\mu^+_{A_1}(d), \mu^+_{A_2}(d)))$$

$$0.5 \le \min(\max(0.5,0.5),\max(0.7,0.6))$$

$$0.5 \le \min(0.5,0.7) = 0.5$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(bf) \le \min(\mu^+_{A_1} \cup \mu^+_{A_2}(b), \mu^+_{A_1} \cup \mu^+_{A_2}(f))$$

$$0.2 \le \min(\max(\mu^+_{A_1}(b), \mu^+_{A_2}(b)), \mu^+_{A_1}(f))$$

$$0.2 \le \min(\max(0.5,0.5),0.6)$$

$$0.2 \le \min(0.5,0.6) = 0.5$$

Thus clearly we see that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* \cup G_2^*$

Proposition 2.1.11:

The Union of two interval-valued fuzzy graphs is an interval-valued fuzzy graph.

Proof:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy graphs of G_1^* and G_2^* respectively. We prove that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* \cup G_2^*$. Since all conditions for $A_1 \cup A_2$ are automatically satisfied, we verify only conditions for $B_1 \cup B_2$

At first we consider the case when $xy \in E_1 \cap E_2$. Then

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(xy) = \max((\mu_{B_1}^-)(xy), (\mu_{B_1}^-)(xy))$$

$$\leq \max(\min(\mu_{A_1}^-(x), \mu_{A_1}^-(y)), \min(\mu_{A_2}^-(x), \mu_{A_2}^-(y)))$$

$$= \min(\max(\mu_{A_1}^-(x), \mu_{A_2}^-(x)), \max(\mu_{A_1}^-(y), \mu_{A_2}^-(y)))$$

$$= \min((\mu_{A_1}^- \cup \mu_{A_2}^-)(x), (\mu_{A_1}^- \cup \mu_{A_2}^-)(y))$$

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(xy) = \max((\mu_{B_1}^+)(xy), (\mu_{B_1}^+)(xy))$$

$$\leq \max(\min(\mu_{A_1}^+(x), \mu_{A_1}^+(y)), \min(\mu_{A_2}^+(x), \mu_{A_2}^+(y)))$$

$$= \min(\max(\mu_{A_1}^+(x), \mu_{A_2}^+(x)), \max(\mu_{A_1}^+(y), \mu_{A_2}^+(y)))$$

$$= \min((\mu_{A_1}^+ \cup \mu_{A_2}^+)(x), (\mu_{A_1}^+ \cup \mu_{A_2}^+)(y))$$

If $xy \in E_1$ and $xy \notin E_2$, then

$$(\mu_{B_1}^- \cup \mu_{B_2}^-)(xy) \leq \min((\mu_{A_1}^- \cup \mu_{A_2}^-)(x), (\mu_{A_1}^- \cup \mu_{A_2}^-)(y)),$$

$$(\mu_{B_1}^+ \cup \mu_{B_2}^+)(xy) \leq \min((\mu_{A_1}^+ \cup \mu_{A_2}^+)(x), (\mu_{A_1}^+ \cup \mu_{A_2}^+)(y))$$

If $xy \in E_2$ and $xy \notin E_1$, then

$$(\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \leq \min((\mu^-_{A_1} \cup \mu^-_{A_2})(x), (\mu^-_{A_1} \cup \mu^-_{A_2})(y)),$$

$$(\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \leq \min((\mu^+_{A_1} \cup \mu^+_{A_2})(x), (\mu^+_{A_1} \cup \mu^+_{A_2})(y))$$

This completes the proof.

Definition 2.1.12:

The **join** $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ of two interval-valued fuzzy graphs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

$$\text{(A)} \begin{cases} (\mu^-_{A_1} + \mu^-_{A_2})(x) = (\mu^-_{A_1} \cup \mu^-_{A_2})(x) \\ (\mu^+_{A_1} + \mu^+_{A_2})(x) = (\mu^+_{A_1} \cup \mu^+_{A_2})(x) \quad \text{if } x \in V_1 \bigcup V_2 \end{cases}$$

$$\text{(B)} \begin{cases} (\mu^-_{B_1} + \mu^-_{B_2})(xy) = (\mu^-_{B_1} \cup \mu^-_{B_2})(xy) \\ (\mu^+_{B_1} + \mu^+_{B_2})(xy) = (\mu^+_{B_1} \cup \mu^+_{B_2})(xy) \quad \text{if } xy \in E_1 \bigcap E_2 \end{cases}$$

$$\text{(C)} \begin{cases} (\mu^-_{B_1} + \mu^-_{B_2})(xy) = \min(\mu^-_{A_1}(x), \mu^-_{A_2}(y)) \\ (\mu^+_{B_1} + \mu^+_{B_2})(xy) = \min(\mu^+_{A_1}(x), \mu^+_{A_2}(y)) \end{cases}$$

if $xy \in E'$, where E' is the set of all edges joining the nodes of V_1 and V_2

Proposition 2.1.13:

The join of interval-valued fuzzy graphs is an interval-valued fuzzy graph.

Proof:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy graphs of G_1^* and G_2^* respectively. We prove that $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ is an interval-valued fuzzy graph of the graph $G_1^* + G_2^*$ .When $xy \in E'$, we have

$$(\mu_{B_1}^- + \mu_{B_2}^-)(xy) = \min(\mu_{A_1}^-(x), \mu_{A_2}^-(y))$$

$$\leq \min((\mu_{A_1}^- \cup \mu_{A_2}^-)(x), (\mu_{A_1}^- \cup \mu_{A_2}^-)(y))$$

$$= \min((\mu_{A_1}^- + \mu_{A_2}^-)(x), (\mu_{A_1}^- + \mu_{A_2}^-)(y))$$

$$(\mu_{B_1}^+ + \mu_{B_2}^+)(xy) = \min(\mu_{A_1}^+(x), \mu_{A_2}^+(y))$$

$$\leq \min((\mu_{A_1}^+ \cup \mu_{A_2}^+)(x), (\mu_{A_1}^+ \cup \mu_{A_2}^+)(y))$$

$$= \min((\mu_{A_1}^+ + \mu_{A_2}^+)(x), (\mu_{A_1}^+ + \mu_{A_2}^+)(y))$$

This completes the proof.

Proposition 2.1.14:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs with $V_1 \cap V_2 = \phi$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets of V_1, V_2, E_1 and E_2 respectively. Then $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of $G_1^* \cup G_2^*$ if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*,respectively.

Proof:

Suppose that $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an interval-valued fuzzy graph of $G_1^* \cup G_2^*$.Let $xy \in E_1$.Then $xy \notin E_2$ and $x, y \in V_1 - V_2$.Thus

$$\mu_{B_1}^-(xy) = (\mu_{B_1}^- \cup \mu_{B_2}^-)(xy)$$

$$\leq \min((\mu_{A_1}^- \cup \mu_{A_2}^-)(x), (\mu_{A_1}^- \cup \mu_{A_2}^-)(y)$$

$$= \min(\mu_{A_1}^-(x), \mu_{A_1}^-(y))$$

$$\mu_{B_1}^+(xy) = (\mu_{B_1}^+ \cup \mu_{B_2}^+)(xy)$$

$$\leq \min((\mu_{A_1}^+ \cup \mu_{A_2}^+)(x), (\mu_{A_1}^+ \cup \mu_{A_2}^+)(y)$$

$$= \min(\mu_{A_1}^+(x), \mu_{A_1}^+(y)).$$

This shows that $G_1 = (A_1, B_1)$ is an interval-valued fuzzy graph. Similarly, we can show that $G_2 = (A_2, B_2)$ is an interval-valued fuzzy graph.

The converse statement is given by Proposition 2.1.11.

Proposition 2.1.15:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs with $V_1 \bigcap V_2 = \phi$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets of V_1, V_2, E_1 and E_2 respectively. Then $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ is an interval-valued fuzzy graph of $G_1^* \cup G_2^*$ if and only if $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*, respectively.

Proof: The proof follows from Proposition 2.1.13 and Proposition 2.1.14.

SECTION 2.2

ISOMORPHISMS OF INTERVAL-VALUED FUZZY GRAPHS

Here we characterize various types of (weak) isomorphisms of interval-valued fuzzy graphs.

Definition 2.2.1:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs.

A **homomorphism** $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ such that

(a) $\mu^-_{A_1}(x_1) \le \mu^-_{A_2}(f(x_1))$; $\mu^+_{A_1}(x_1) \le \mu^+_{A_2}(f(x_1))$

(b) $\mu^-_{B_1}(x_1 y_1) \le \mu^-_{B_2}(f(x_1) f(y_1))$; $\mu^+_{B_1}(x_1 y_1) \le \mu^+_{B_2}(f(x_1) f(y_1))$

for all $x_1 \in V_1$, $x_1 y_1 \in E_1$.

Definition 2.2.2:

A bijective homomorphism with the property

(c) $\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1))$, $\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1))$ is called a **weak isomorphism**.

Remark 2.2.3:

A weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.

Definition 2.2.4:

A bijective homomorphism preserving the weights of the arcs but not necessarily the weights of the nodes, i.e., a bijective homomorphism f : $G_1 \to G_2$ such that

(d) $\mu^-_{B_1}(x_1y_1) = \mu^-_{B_2}(f(x_1)f(y_1))$ $\quad\quad$ $\mu^+_{B_1}(x_1y_1) = \mu^+_{B_2}(f(x_1)f(y_1))$

for all $x_1y_1 \in V$ is called a **weak co- isomorphism**.

Definition 2.2.5:

A bijective mapping $f : G_1 \to G_2$ satisfying (c) and (d) is called an **isomorphism.**

Example 2.2.2:

Consider graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 = \{a_1, b_1\}$, $V_2 = \{a_2, b_2\}$ **,** $E_1 = \{a_1, b_1\}$ and $E_2 = \{a_2, b_2\}$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets defined by

$A_1 =< (\frac{a_1}{0.2}, \frac{b_1}{0.3}), (\frac{a_1}{0.5}, \frac{b_1}{0.6}) >$ $\quad\quad$ $B_1 =< (\frac{a_1b_1}{0.1}, \frac{a_1b_1}{0.3}) >$

$A_2 =< (\frac{a_2}{0.3}, \frac{b_2}{0.2}), (\frac{a_2}{0.6}, \frac{b_2}{0.5}) >$ $\quad\quad$ $B_2 =< (\frac{a_2b_2}{0.1}, \frac{a_2b_2}{0.4}) >$

Then $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*,respectively

The map $f : V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$

To check weak isomorphism

$\mu^-_{A_1}(a_1) = 0.2$ $\quad\quad$ $\mu^-_{A_2}(f(a_1)) = \mu^-_{A_2}(b_2) = 0.2$

$\mu^-_{A_1}(b_1) = 0.2$ $\quad\quad$ $\mu^-_{A_2}(f(b_1)) = \mu^-_{A_2}(a_2) = 0.3$

$\mu^+_{A_1}(a_1) = 0.5$ $\quad\quad$ $\mu^+_{A_2}(f(a_1)) = \mu^+_{A_2}(b_2) = 0.5$

$\mu^+_{A_1}(b_1) = 0.6$ $\quad\quad$ $\mu^+_{A_2}(f(b_1)) = \mu^+_{A_2}(a_2) = 0.6$

Thus the map $f: V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a weak isomorphism

Example 2.2.3:

Consider graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 = \{a_1, b_1\}$, $V_2 = \{a_2, b_2\}$ $E_1 = \{a_1, b_1\}$ and $E_2 = \{a_2, b_2\}$. Let A_1, A_2, B_1 and B_2 be interval-valued fuzzy subsets defined by

$$A_1 =< (\frac{a_1}{0.2}, \frac{b_1}{0.3}), (\frac{a_1}{0.4}, \frac{b_1}{0.5}) > \qquad B_1 =< (\frac{a_1 b_1}{0.1}, \frac{a_1 b_1}{0.3}) >$$

$$A_2 =< (\frac{a_2}{0.4}, \frac{b_2}{0.3}), (\frac{a_2}{0.5}, \frac{b_2}{0.6}) > \qquad B_2 =< (\frac{a_2 b_2}{0.1}, \frac{a_2 b_2}{0.3}) >$$

Then $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs of G_1^* and G_2^*,respectively

The map $f: V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$

To check weak isomorphism

$$\mu_{A_1}^-(a_1) = 0.2 \qquad \mu_{A_2}^-(f(a_1)) = \mu_{A_2}^-(b_2) = 0.3$$

$$\mu_{A_1}^-(a_1) \neq \mu_{A_2}^-(f(a_1))$$

To check weak co-isomorphism

$$\mu_{B_1}^-(a_1 b_1) = 0.1 \qquad \mu_{B_2}^-(f(a_1) f(b_1)) = \mu_{B_2}^-(b_2 a_2) = 0.1$$

$$\mu_{B_1}^+(a_1 b_1) = 0.3 \qquad \mu_{B_2}^+(f(a_1) f(b_1)) = \mu_{B_2}^+(b_2 a_2) = 0.3$$

Thus the map $f: V_1 \to V_2$ defined by $f(a_1) = b_2$ and $f(b_1) = a_2$ is a weak co-isomorphism.

Proposition 2.2.4: An isomorphism between interval-valued fuzzy graph is an equivalence relation.

Chapter III

CHAPTER III

COMPLETE INTERVAL-VALUED FUZZY GRAPHS

SECTION 3.1

INTERVAL-VALUED FUZZY COMPLETE GRAPHS

Definition 3.1.1:

An interval-valued fuzzy graph $G=(A,B)$ is called **complete** if

$\mu_B^-(xy)=\min(\mu_A^-(x),\mu_A^-(y))$ and $\mu_B^-(xy)=\min(\mu_A^-(x),\mu_A^-(y))$ for all $xy\in E$

Example 3.1.2:

Consider the graph $G^*=(V,E)$ such that $V=\{x,y,z\}$, $E=\{xy,yz,zx\}$ .If A and B are interval-valued fuzzy subsets defined by

$$A=<(\frac{x}{0.2},\frac{y}{0.3},\frac{z}{0.4}),(\frac{x}{0.4},\frac{y}{0.5},\frac{z}{0.5})>$$

$$B=<(\frac{xy}{0.2},\frac{yz}{0.3},\frac{zx}{0.2}),(\frac{xy}{0.4},\frac{yz}{0.5},\frac{zx}{0.4})>$$

Then it is easy to check whether the graph is complete.

$\mu_B^-(xy)=\min(\mu_A^-(x),\mu_A^-(y))$ $\quad$ $\mu_B^+(xy)=\min(\mu_A^+(x),\mu_A^+(y))$

$0.2=\min(0.2,0.3)=0.2$ $\quad$ $0.4=\min(0.4,0.5)=0.4$

$\mu_B^-(yz)=\min(\mu_A^-(y),\mu_A^-(z))$ $\mu_B^+(yz)=\min(\mu_A^+(y),\mu_A^+(z))$

$0.3=\min(0.3,0.4)=0.3$ $\quad$ $0.5=\min(0.5,0.5)=0.5$

$\mu_B^-(zx)=\min(\mu_A^-(z),\mu_A^-(x))$ $\quad$ $\mu_B^+(zx)=\min(\mu_A^+(z),\mu_A^+(x))$

$0.2=\min(0.4,0.2)=0.2$ $\quad$ $0.4=\min(0.5,0.4)=0.4$

Thus $G=(A,B)$ is an interval-valued fuzzy complete graph of G^*

Proposition 3.1.3:

If $G=(A,B)$ be an interval-valued fuzzy complete graph, then also G[G] is an interval-valued fuzzy complete graph.

Definition 3.1.4:

The **Complement of an interval-valued fuzzy graph** $G=(A,B)$ of $G^*=(V,E)$ is an interval-valued fuzzy complete graph $\overline{G}=(\overline{A},\overline{B})$ on $\overline{G^*}=(V,\overline{E})$, where $\overline{A}=A=[\mu_A^-,\mu_A^+]$ and $\overline{B}=[\overline{\mu_B^-},\overline{\mu_B^+}]$ is defined by

$$\overline{\mu_B^-}(\mathrm{xy})=\begin{cases}0 & \text{if } \mu_B^-(xy)>0\\ \min(\mu_A^-(x),\mu_A^-(y)) & \text{if } \mu_B^-(xy)=0\end{cases}$$

$$\overline{\mu_B^+}(\mathrm{xy})=\begin{cases}0 & \text{if } \mu_B^+(xy)>0\\ \min(\mu_A^+(x),\mu_A^+(y)) & \text{if } \mu_B^+(xy)=0.\end{cases}$$

Definition 3.1.5:

An interval-valued fuzzy graph $G=(A,B)$ is called **self-complementary** if $\overline{\overline{G}}=G$.

Example 3.1.6:

Consider a graph $G^*=(V,E)$ such that $V=\{a,b,c\}$, $E=\{ab,bc\}$.Then an interval-valued fuzzy graph $G=(A,B)$

$A=<(\frac{a}{0.1},\frac{b}{0.2},\frac{c}{0.3}),(\frac{a}{0.3},\frac{b}{0.4},\frac{c}{0.5})>$ $\quad B=<(\frac{ab}{0.1},\frac{bc}{0.2}),(\frac{ab}{0.3},\frac{bc}{0.4})>$ is self complementary.

$\overline{\mu_B^-}$ (ab) = 0 $\quad$ $\overline{\mu_B^+}$ (ab) = 0 $\quad$ $\overline{\mu_B^-}$ (bc) = 0 $\quad$ $\overline{\mu_B^+}$ (bc) = 0

$\overline{\mu_B^-}$ (ca) = min ($\mu_A^-(c), \mu_A^-(a)$)

= min (0.3,0.1) = 0.1

$\overline{\mu_B^+}$ (ca) = min ($\mu_A^+(c), \mu_A^+(a)$)

= min(0.5,0.3) = 0.3

$\overline{B} =< (\frac{ca}{0.1}, \frac{ca}{0.3}) >$

$\mu_B^-(ab)$ =0 $\mu_B^-(bc)$ =0 $\mu_B^-(ca)$ =0.1 $\mu_B^+(ab)$ =0 $\mu_B^+(bc)$ =0

$\mu_B^+(ca)$ =0.3

$\overline{\mu_B^-}(ab)$ = min($\mu_A^-(a), \mu_A^-(b)$) = min (0.1,0.2) = 0.1

$\overline{\mu_B^-}(bc)$ = min($\mu_A^-(b), \mu_A^-(c)$) = min (0.2,0.3) = 0.2 $\overline{\mu_B^-}(ca) = 0$

$\overline{\mu_B^+}(bc)$ = min($\mu_A^+(b), \mu_A^+(c)$) = min (0.4,0.5) = 0.4 $\overline{\mu_B^+}(ca) = 0$

$\overline{\overline{B}} =< (\frac{ab}{0.1}, \frac{bc}{0.2}), (\frac{ab}{0.3}, \frac{bc}{0.4}) >$ = B.

Thus the interval-valued fuzzy graph $G = (A, B)$ is self complementary.

Proposition 3.1.7:

In a self complementary interval-valued fuzzy complete graph $G = (A, B)$ we have a) $\sum_{x \neq y} \mu_B^-(xy) = \sum_{x \neq y} \min(\mu_A^-(x), \mu_A^-(y))$ b) $\sum_{x \neq y} \mu_B^+(xy) = \sum_{x \neq y} \min(\mu_A^+(x), \mu_A^+(y))$

Proof:

Let $G = (A, B)$ be a self complementary interval-valued fuzzy complete graph. Then there exists an automorphism $f : V \to V$ such that

$\mu_A^-(f(x)) = \mu_A^-(x)$ $\mu_A^+(f(x)) = \mu_A^+(x)$

$\overline{\mu_B^+}(f(x)f(y)) = \mu_B^+(xy)$ for all $x, y \in V$

Hence for $x, y \in V$ we obtain,

$$\mu_B^-(xy) = \overline{\mu_B^-}(f(x)f(y)) = \min(\mu_A^-(f(x)), \mu_A^-(f(y))) = \min(\mu_A^-(x), \mu_A^-(y))$$

Similarly,

$$\mu_B^+(xy) = \overline{\mu_B^+}(f(x)f(y)) = \min(\mu_A^+(f(x)), \mu_A^+(f(y))) = \min(\mu_A^+(x), \mu_A^+(y))$$

Proposition 3.1.8: Let $G = (A, B)$ be an interval-valued fuzzy complete graph. If $\mu_B^-(xy) = \min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) = \min(\mu_A^+(x), \mu_A^+(y))$ for all $x, y \in V$, then G is self complementary.

Proof: Let $G = (A, B)$ be an interval-valued fuzzy complete graph such that $\mu_B^-(xy) = \min(\mu_A^-(x), \mu_A^-(y))$ and $\mu_B^+(xy) = \min(\mu_A^+(x), \mu_A^+(y))$ for all $x, y \in V$. Then $G = \overline{G}$ under the identity map $I : V \to V$. So $\overline{\overline{G}} = G$.

Hence G is self complementary.

Proposition 3.1.9:

Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be interval-valued fuzzy complete graphs. Then $G_1 \cong G_2$ if and only if $\overline{G_1} \cong \overline{G_2}$.

Proof: Assume that G_1 and G_2 are isomorphic, there exists a bijective map $f : V_1 \to V_2$ satisfying $\mu_{A_1}^-(x) = \mu_{A_2}^-(f(x))$, $\mu_{A_1}^+(x) = \mu_{A_2}^+(f(x))$ for all $x \in V_1$ $\mu_{B_1}^-(xy) = \mu_{B_2}^-(f(x)f(y))$, $\mu_{B_1}^+(xy) = \mu_{B_2}^+(f(x)f(y))$ for all $xy \in E_1$

By definition of complement, we have

$$\overline{\mu_{B_1}^-}(xy) = \min(\mu_{A_1}^-(x), \mu_{A_1}^-(y)) = \min(\mu_{A_2}^-(f(x)), \mu_{A_2}^-(f(y))) = \overline{\mu_{B_2}^-}(f(x)f(y))$$

$\overline{\mu_{B_1}^+}(xy) = \min(\mu_{A_1}^+(x), \mu_{A_1}^+(y)) = \min(\mu_{A_2}^+(f(x)), \mu_{A_2}^+(f(y))) = \overline{\mu_{B_2}^+}(f(x)f(y))$ for all $xy \in E_1$. Hence $\overline{G_1} \cong \overline{G_2}$.

SECTION 3.2

VARIOUS PRODUCTS ON INTERVAL-VALUED FUZZY COMPLETE GRAPHS

Definition 3.2.1:

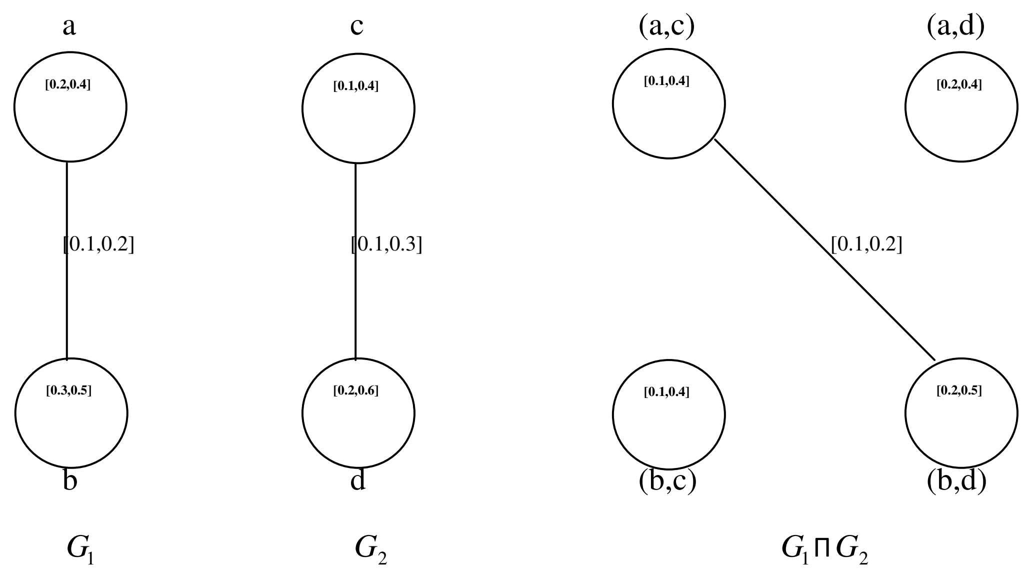
The direct product of two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ with crisp graph $G_1^* = (V_1, E_1)$ and $G_2 = (A_2, B_2)$ with crisp graph $G_2^* = (V_2, E_2)$, where we assume that $V_1 \bigcap V_2 = \phi$, is defined to be the interval-valued fuzzy graph G_1 ⊓ G_2 : (σ_1 ⊓ σ_2 , μ_1 ⊓ μ_2) with crisp graph $G^* : (V_1 \times V_2, E)$ where $E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$

$$\begin{cases} (\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u, v) = \mu_{A_1}^-(u) \wedge \mu_{A_2}^-(v), \quad \text{for all } (u, v) \in V_1 \times V_2 \\ (\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(u, v) = \mu_{A_1}^+(u) \wedge \mu_{A_2}^+(v), \end{cases}$$

$$\begin{cases} (\mu_{B_1}^- \sqcap \mu_{B_2}^-)((u_1, v_1)(u_2, v_2)) = \mu_{B_1}^-(u_1 u_2) \wedge \mu_{B_2}^-(v_1 v_2), \\ (\mu_{B_1}^+ \sqcap \mu_{B_2}^+)((u_1, v_1)(u_2, v_2)) = \mu_{B_1}^+(u_1 u_2) \wedge \mu_{B_2}^+(v_1 v_2). \end{cases}$$

Example 3.2.2:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{ab\}$, $E_2 = \{cd\}$ .Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ and $G_1 \sqcap G_2$ as follows.



Here $V_1 \times V_2 = \{(a,c)(a,d)(b,c)(b,d)\}$ $E = \{(a,c)(b,d) : ab \in E_1, cd \in E_2\}$

$(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(a,c) = \mu_{A_1}^-(a) \wedge \mu_{A_2}^-(c) = (0.2 \wedge 0.1) = 0.1$

$(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(a,d) = \mu_{A_1}^-(a) \wedge \mu_{A_2}^-(d) = (0.2 \wedge 0.2) = 0.2$

$(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(b,c) = \mu_{A_1}^-(b) \wedge \mu_{A_2}^-(c) = (0.3 \wedge 0.1) = 0.1$

$(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(b,d) = \mu_{A_1}^-(b) \wedge \mu_{A_2}^-(d) = (0.3 \wedge 0.2) = 0.2$

$(\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(a,c) = \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(c) = (0.4 \wedge 0.4) = 0.4$

$(\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(a,d) = \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(d) = (0.4 \wedge 0.6) = 0.4$

$(\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(b,c) = \mu_{A_1}^+(b) \wedge \mu_{A_2}^+(c) = (0.5 \wedge 0.4) = 0.4$

$(\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(b,d) = \mu_{A_1}^+(b) \wedge \mu_{A_2}^+(d) = (0.5 \wedge 0.6) = 0.5$

$(\mu_{B_1}^- \sqcap \mu_{B_2}^-)((a,c)(b,d)) = \mu_{B_1}^-(ab) \wedge \mu_{B_2}^-(cd) = (0.1 \wedge 0.1) = 0.1$

$(\mu_{B_1}^+ \sqcap \mu_{B_2}^+)((a,c)(b,d)) = \mu_{B_1}^+(ab) \wedge \mu_{B_2}^+(cd) = (0.2 \wedge 0.3) = 0.2.$

Now we check the condition for interval-valued fuzzy graph.

$(\mu_{B_1}^- \sqcap \mu_{B_2}^-)((a,c)(b,d)) \le \mu_{A_1}^- \sqcap \mu_{A_2}^-(a,c) \wedge \mu_{A_1}^- \sqcap \mu_{A_2}^-(b,d)$

$$0.1 \le (\mu_{A_1}^-(a) \wedge \mu_{A_2}^-(c)) \wedge (\mu_{A_1}^-(b) \wedge \mu_{A_2}^-(d))$$

$$0.1 \le (0.2 \wedge 0.1) \wedge (0.3 \wedge 0.2)$$

$$0.1 \le (0.1 \wedge 0.2) = 0.1$$

$(\mu_{B_1}^+ \sqcap \mu_{B_2}^+)((a,c)(b,d)) \le \mu_{A_1}^+ \sqcap \mu_{A_2}^+(a,c) \wedge \mu_{A_1}^+ \sqcap \mu_{A_2}^+(b,d)$

$$0.1 \le (\mu_{A_1}^+(a) \wedge \mu_{A_2}^+(c)) \wedge (\mu_{A_1}^+(b) \wedge \mu_{A_2}^+(d))$$

$$0.1 \le (0.2 \wedge 0.1) \wedge (0.3 \wedge 0.2)$$

$$0.1 \le (0.1 \wedge 0.2) = 0.1$$

Thus we see that $G_1 \sqcap G_2$ is an interval-valued fuzzy graph.

Theorem 3.2.3:

If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete interval-valued fuzzy graphs, then $G_1 \sqcap G_2$ is complete.

Proof:

If $(u_1, v_1)(u_2, v_2) \in E$, then since G_1 and G_2 are complete , we have

$$(\mu_{B_1}^- \sqcap \mu_{B_2}^-)((u_1, v_1)(u_2, v_2)) = \mu_{B_1}^-(u_1 u_2) \wedge \mu_{B_2}^-(v_1 v_2)$$

$$= \mu_{A_1}^-(u_1) \wedge \mu_{A_2}^-(u_2) \wedge \mu_{A_2}^-(v_1) \wedge \mu_{A_2}^-(v_2)$$

$$= (\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u_1, v_1) \wedge (\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u_2, v_2)$$

$$(\mu_{B_1}^+ \sqcap \mu_{B_2}^+)((u_1, v_1)(u_2, v_2)) = \mu_{B_1}^+(u_1 u_2) \wedge \mu_{B_2}^+(v_1 v_2)$$

$$= \mu_{A_1}^+(u_1) \wedge \mu_{A_2}^+(u_2) \wedge \mu_{A_2}^+(v_1) \wedge \mu_{A_2}^+(v_2)$$

$$= (\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(u_1, v_1) \wedge (\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(u_2, v_2)$$

To define semi-strong product and strong product of interval-valued fuzzy complete graph we need the following results.

Definition 3.2.4:

An interval-valued fuzzy graph $G = (A, B)$ of a given graph $G^* = (V, E)$ is called an **interval-valued strong fuzzy graph** if $\mu_B^-(xy) = \mu_A^-(x) \wedge \mu_A^-(y)$ and $\mu_B^+(xy) = \mu_A^+(x) \wedge \mu_A^+(y)$ for all $xy \in E$.

Definition 3.2.5:

The **Complement of a strong interval-valued fuzzy graph** G is $\overline{G} = (\overline{A}, \overline{B})$ where $\overline{A} = [\overline{\mu_A^-}(x), \overline{\mu_A^-}(x)]$ is an interval-valued fuzzy set on $\overline{V}$ and $\overline{B} = [\overline{\mu_B^-}, \overline{\mu_B^+}]$ is an interval-valued fuzzy set on $\overline{E} \subseteq \overline{V} \times \overline{V}$ such that

1) $\overline{V} = V$,
2) $\overline{\mu_A^-}(x) = \mu_A^-(x)$ and $\overline{\mu_A^+}(x) = \mu_A^+(x)$ for all $x \in V$,

3) $\overline{\mu_B^-}(xy) = \begin{cases} 0 & \text{if } \mu_B^-(xy) > 0, \\ \mu_A^-(x) \wedge \mu_A^-(y) & \text{if } \mu_B^-(xy) = 0. \end{cases}$

4) $\overline{\mu_B^+}(xy) = \begin{cases} 0 & \text{if } \mu_B^+(xy) > 0, \\ \mu_A^+(x) \wedge \mu_A^+(y) & \text{if } \mu_B^+(xy) = 0. \end{cases}$

Definition 3.2.6:

Let $G = (A, B)$ be an interval-valued fuzzy graph where $A = [\mu_A^-, \mu_A^+]$ and $B = [\mu_B^-, \mu_B^+]$ be two interval-valued fuzzy sets on a non-empty finite set V and $E \subseteq V \times V$ respectively. **The total degree of a vertex $u \in V$ is denoted by** $td(u)$ and defined as $td(u) = [td^+(u), td^-(u)]$ where

$$td^+(u) = \sum_{uv\in E} \mu_B^+(uv) + \mu_A^+(u), \qquad td^-(u) = \sum_{uv\in E} \mu_B^-(uv) + \mu_A^-(u).$$

Remark 3.2.7:

If the total degrees of all vertices of an interval-valued fuzzy graph are equal, then the graph is said to be totally regular interval-valued fuzzy graph.

Definition 3.2.8:

The semi-strong product of two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ with crisp graph $G_1^* = (V_1, E_1)$ and $G_2 = (A_2, B_2)$ with crisp graph $G_2^* = (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \phi$, is defined to be the interval-valued fuzzy graph $G_1 \bullet G_2 : (A_1 \bullet A_2, B_1 \bullet B_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where
$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$

(i) $\begin{cases} (\mu_{A_1}^- \bullet \mu_{A_2}^-)(u, v) = \mu_{A_1}^-(u) \wedge \mu_{A_2}^-(v), & \text{for all } (u, v) \in V_1 \times V_2 \\ (\mu_{A_1}^+ \bullet \mu_{A_2}^+)(u, v) = \mu_{A_1}^+(u) \wedge \mu_{A_2}^+(v) \end{cases}$

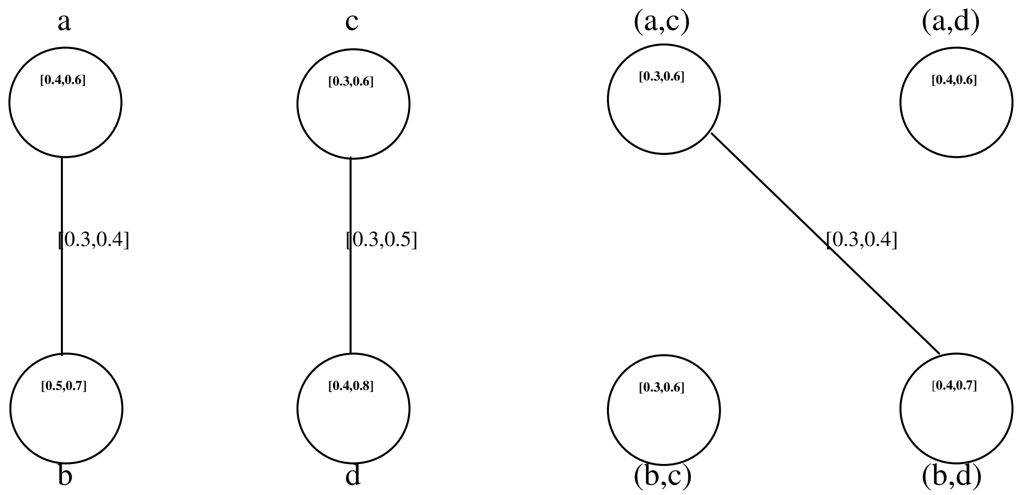
(ii) $\begin{cases} (\mu_{B_1}^- \bullet \mu_{B_2}^-)((u, v_1), (u, v_2)) = \mu_{A_1}^-(u) \wedge \mu_{B_2}^-(v_1 v_2), \\ (\mu_{B_1}^+ \bullet \mu_{B_2}^+)((u, v_1), (u, v_2)) = \mu_{A_1}^+(u) \wedge \mu_{B_2}^+(v_1 v_2) \end{cases}$

$$\text{(iii)} \begin{cases} (\mu_{B_1}^- \bullet \mu_{B_2}^-)((u_1, v_1),(u_2, v_2)) = \mu_{B_1}^-(u_1u_2) \wedge \mu_{B_2}^-(v_1v_2), \\ (\mu_{B_1}^+ \bullet \mu_{B_2}^+)((u_1, v_1),(u_2, v_2)) = \mu_{B_1}^+(u_1u_2) \wedge \mu_{B_2}^+(v_1v_2). \end{cases}$$

Example 3.2.9:

In this example we consider two interval-valued fuzzy graphs

$G_1 = (A_1, B_1)$, $G_2 = (A_2, B_2)$ and $G_1 \bullet G_2$ as follows



G_1 $\qquad$ G_2 $\qquad$ $G_1 \bullet G_2$

Here $V_1 \times V_2 = \{(a,c)(a,d)(b,c)(b,d)\}$ $\qquad$ $E = \{(a,c)(b,d) : ab \in E_1, cd \in E_2\}$

$(\mu_{A_1}^- \bullet \mu_{A_2}^-)(a,c) = \mu_{A_1}^-(a) \wedge \mu_{A_2}^-(c) = (0.4 \wedge 0.3) = 0.3$

$(\mu_{A_1}^- \bullet \mu_{A_2}^-)(a,d) = \mu_{A_1}^-(a) \wedge \mu_{A_2}^-(d) = (0.4 \wedge 0.4) = 0.4$

$(\mu_{A_1}^- \bullet \mu_{A_2}^-)(b,c) = \mu_{A_1}^-(b) \wedge \mu_{A_2}^-(c) = (0.5 \wedge 0.3) = 0.3$

$(\mu_{A_1}^- \bullet \mu_{A_2}^-)(b,d) = \mu_{A_1}^-(b) \wedge \mu_{A_2}^-(d) = (0.5 \wedge 0.4) = 0.4$

$(\mu_{A_1}^+ \bullet \mu_{A_2}^+)(a,c) = \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(c) = (0.6 \wedge 0.6) = 0.6$

$(\mu_{A_1}^+ \bullet \mu_{A_2}^+)(a,d) = \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(d) = (0.6 \wedge 0.8) = 0.6$

$(\mu_{A_1}^+ \bullet \mu_{A_2}^+)(b,c) = \mu_{A_1}^+(b) \wedge \mu_{A_2}^+(c) = (0.7 \wedge 0.6) = 0.6$

$(\mu_{A_1}^+ \bullet \mu_{A_2}^+)(b,d) = \mu_{A_1}^+(b) \wedge \mu_{A_2}^+(d) = (0.7 \wedge 0.8) = 0.7$

$$(\mu^-_{B_1} \bullet \mu^-_{B_2})((a,c)(a,d)) = \mu^-_{A_1}(a) \wedge \mu^-_{B_2}(cd) = (0.4 \wedge 0.3) = 0.3$$
$$(\mu^-_{B_1} \bullet \mu^-_{B_2})((b,c)(b,d)) = \mu^-_{A_1}(b) \wedge \mu^-_{B_2}(cd) = (0.5 \wedge 0.3) = 0.3$$

$$(\mu^+_{B_1} \bullet \mu^+_{B_2})((a,c)(a,d)) = \mu^+_{A_1}(a) \wedge \mu^+_{B_2}(cd) = (0.6 \wedge 0.5) = 0.5$$
$$(\mu^+_{B_1} \bullet \mu^+_{B_2})((b,c)(b,d)) = \mu^+_{A_1}(b) \wedge \mu^+_{B_2}(cd) = (0.7 \wedge 0.5) = 0.5$$

$$(\mu^-_{B_1} \bullet \mu^-_{B_2})((a,c)(b,d)) = \mu^-_{B_1}(ab) \wedge \mu^-_{B_2}(cd) = (0.3 \wedge 0.3) = 0.3$$
$$(\mu^+_{B_1} \bullet \mu^+_{B_2})((a,c)(b,d)) = \mu^+_{B_1}(ab) \wedge \mu^+_{B_2}(cd) = (0.4 \wedge 0.5) = 0.4$$

Now we shall verify the condition for the interval-valued fuzzy graph.

$$(\mu^-_{B_1} \bullet \mu^-_{B_2})((a,c)(a,d)) \le \mu^-_{A_1} \bullet \mu^-_{A_2}(a,c) \wedge \mu^-_{A_1} \bullet \mu^-_{A_2}(a,d)$$

$$0.3 \le (\mu^-_{A_1}(a) \wedge \mu^-_{A_2}(c)) \wedge (\mu^-_{A_1}(a) \wedge \mu^-_{A_2}(d))$$

$$0.3 \le (0.4 \wedge 0.3) \wedge (0.4 \wedge 0.4)$$

$$0.3 \le (0.3 \wedge 0.4) = 0.3$$

$$(\mu^-_{B_1} \bullet \mu^-_{B_2})((b,c)(b,d)) \le \mu^-_{A_1} \bullet \mu^-_{A_2}(b,c) \wedge \mu^-_{A_1} \bullet \mu^-_{A_2}(b,d)$$

$$0.3 \le (\mu^-_{A_1}(b) \wedge \mu^-_{A_2}(c)) \wedge (\mu^-_{A_1}(b) \wedge \mu^-_{A_2}(d))$$

$$0.3 \le (0.5 \wedge 0.3) \wedge (0.5 \wedge 0.4)$$

$$0.3 \le (0.3 \wedge 0.4) = 0.3$$

$$(\mu^-_{B_1} \bullet \mu^-_{B_2})((a,c)(b,d)) \le \mu^-_{A_1} \bullet \mu^-_{A_2}(a,c) \wedge \mu^-_{A_1} \bullet \mu^-_{A_2}(b,d)$$

$$0.3 \le (\mu^-_{A_1}(a) \wedge \mu^-_{A_2}(c)) \wedge (\mu^-_{A_1}(b) \wedge \mu^-_{A_2}(d))$$

$$0.3 \le (0.4 \wedge 0.3) \wedge (0.5 \wedge 0.4)$$

$$0.3 \le (0.3 \wedge 0.4) = 0.3$$

$$(\mu^+_{B_1} \bullet \mu^+_{B_2})((a,c)(a,d)) \le \mu^+_{A_1} \bullet \mu^+_{A_2}(a,c) \wedge \mu^+_{A_1} \bullet \mu^+_{A_2}(a,d)$$

$$0.5 \le (\mu^+_{A_1}(a) \wedge \mu^+_{A_2}(c)) \wedge (\mu^+_{A_1}(a) \wedge \mu^+_{A_2}(d))$$

$$0.5 \le (0.6 \wedge 0.6) \wedge (0.6 \wedge 0.8)$$

$$0.3 \le (0.6 \wedge 0.6) = 0.6$$

$$(\mu^{+}_{B_1} \bullet \mu^{+}_{B_2})((b,c)(b,d)) \le \mu^{+}_{A_1} \bullet \mu^{+}_{A_2}(b,c) \wedge \mu^{+}_{A_1} \bullet \mu^{+}_{A_2}(b,d)$$

$$0.5 \le (\mu^{+}_{A_1}(b) \wedge \mu^{+}_{A_2}(c)) \wedge (\mu^{+}_{A_1}(b) \wedge \mu^{+}_{A_2}(d))$$

$$0.5 \le (0.7 \wedge 0.6) \wedge (0.7 \wedge 0.8)$$

$$0.5 \le (0.6 \wedge 0.7) = 0.6$$

$$(\mu^{+}_{B_1} \bullet \mu^{+}_{B_2})((a,c)(b,d)) \le \mu^{+}_{A_1} \bullet \mu^{+}_{A_2}(a,c) \wedge \mu^{+}_{A_1} \bullet \mu^{+}_{A_2}(b,d)$$

$$0.4 \le (\mu^{+}_{A_1}(a) \wedge \mu^{+}_{A_2}(c)) \wedge (\mu^{+}_{A_1}(b) \wedge \mu^{+}_{A_2}(d))$$

$$0.4 \le (0.6 \wedge 0.6) \wedge (0.7 \wedge 0.8)$$

$$0.4 \le (0.6 \wedge 0.7) = 0.6.$$

Thus we see that $G_1 \bullet G_2$ is an interval-valued fuzzy graph.

Theorem 3.2.10:

If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete interval-valued fuzzy graphs, then $G_1 \bullet G_2$ is complete.

Proof:

If $(u,v_1)(u,v_2) \in E$, then

$$(\mu^{-}_{B_1} \bullet \mu^{-}_{B_2})((u,v_1)(u,v_2)) = \mu^{-}_{A_1}(u) \wedge \mu^{-}_{B_2}(v_1 v_2)$$

$$= \mu^{-}_{A_1}(u) \wedge \mu^{-}_{A_2}(v_1) \wedge \mu^{-}_{A_2}(v_2) \text{ (since } G_2 \text{ is complete)}$$

$$= (\mu^{-}_{A_1} \bullet \mu^{-}_{A_2})(u,v_1) \wedge (\mu^{-}_{A_1} \bullet \mu^{-}_{A_2})(u,v_2)$$

If $((u_1,v_1)(u_2,v_2)) \in E$, then since G_1 and G_2 are complete , we have

$$(\mu^{-}_{B_1} \bullet \mu^{-}_{B_2})((u_1,v_1)(u_2,v_2)) = \mu^{-}_{B_1}(u_1 u_2) \wedge \mu^{-}_{B_2}(v_1 v_2)$$

$$= \mu^{-}_{A_1}(u_1) \wedge \mu^{-}_{A_2}(u_2) \wedge \mu^{-}_{A_2}(v_1) \wedge \mu^{-}_{A_2}(v_2)$$

$$= (\mu^{-}_{A_1} \bullet \mu^{-}_{A_2})(u_1,v_1) \wedge (\mu^{-}_{A_1} \bullet \mu^{-}_{A_2})(u_2,v_2).$$

Similarly, if $(u,v_1)(u,v_2)\in E$, then

$$(\mu_{B_1}^{+}\bullet\mu_{B_2}^{+})((u,v_1)(u,v_2))=\mu_{A_1}^{+}(u)\wedge\mu_{B_2}^{+}(v_1v_2)$$

$$=\mu_{A_1}^{+}(u)\wedge\mu_{A_2}^{+}(v_1)\wedge\mu_{A_2}^{+}(v_2)$$

$$=(\mu_{A_1}^{+}\bullet\mu_{A_2}^{+})(u,v_1)\wedge(\mu_{A_1}^{+}\bullet\mu_{A_2}^{+})(u,v_2)$$

If $((u_1,v_1)(u_2,v_2))\in E$, we have

$$(\mu_{B_1}^{+}\bullet\mu_{B_2}^{+})((u_1,v_1)(u_2,v_2))=\mu_{B_1}^{+}(u_1u_2)\wedge\mu_{B_2}^{+}(v_1v_2)$$

$$=\mu_{A_1}^{+}(u_1)\wedge\mu_{A_2}^{+}(u_2)\wedge\mu_{A_2}^{+}(v_1)\wedge\mu_{A_2}^{+}(v_2)$$

$$=(\mu_{A_1}^{+}\bullet\mu_{A_2}^{+})(u_1,v_1)\wedge(\mu_{A_1}^{+}\bullet\mu_{A_2}^{+})(u_2,v_2).$$

Definition 3.2.11:

The strong product of two interval-valued fuzzy graphs $G_1=(A_1,B_1)$ with crisp graph $G_1^*=(V_1,E_1)$ and $G_2=(A_2,B_2)$ with crisp graph $G_2^*=(V_2,E_2)$, where we assume that $V_1\cap V_2=\phi$, is defined to be the interval-valued fuzzy graph $G_1\otimes G_2:(A_1\otimes A_2,B_1\otimes B_2)$ with crisp graph $G^*:(V_1\times V_2,E)$ where

$$E=\{(u,v_1)(u,v_2):u\in V_1,(v_1,v_2)\in E_2\}\cup\{(u_1,w)(u_2,w):w\in V_2,(u_1,u_2)\in E_1\}$$
$$\cup\{(u_1,v_1)(u_2,v_2):(u_1,u_2)\in E_1,(v_1,v_2)\in E_2\}$$

(i) $$\begin{cases}(\mu_{A_1}^{-}\otimes\mu_{A_2}^{-})(u,v)=\mu_{A_1}^{-}(u)\wedge\mu_{A_2}^{-}(v), & \text{for all } (u,v)\in V_1\times V_2\\ (\mu_{A_1}^{+}\otimes\mu_{A_2}^{+})(u,v)=\mu_{A_1}^{+}(u)\wedge\mu_{A_2}^{+}(v)\end{cases}$$

(ii) $$\begin{cases}(\mu_{B_1}^{-}\otimes\mu_{B_2}^{-})((u,v_1),(u,v_2))=\mu_{A_1}^{-}(u)\wedge\mu_{B_2}^{-}(v_1v_2),\\ (\mu_{B_1}^{+}\otimes\mu_{B_2}^{+})((u,v_1),(u,v_2))=\mu_{A_1}^{+}(u)\wedge\mu_{B_2}^{+}(v_1v_2)\end{cases}$$

(iii) $$\begin{cases}(\mu_{B_1}^{-}\otimes\mu_{B_2}^{-})((u_1,w),(u_2,w))=\mu_{A_2}^{-}(w)\wedge\mu_{B_1}^{-}(u_1u_2),\\ (\mu_{B_1}^{+}\otimes\mu_{B_2}^{+})((u_1,w),(u_2,w))=\mu_{A_2}^{+}(w)\wedge\mu_{B_1}^{+}(u_1u_2)\end{cases}$$

(iv) $$\begin{cases}(\mu_{B_1}^{-}\otimes\mu_{B_2}^{-})((u_1,v_1),(u_2,v_2))=\mu_{B_1}^{-}(u_1u_2)\wedge\mu_{B_2}^{-}(v_1v_2),\\ (\mu_{B_1}^{+}\otimes\mu_{B_2}^{+})((u_1,v_1),(u_2,v_2))=\mu_{B_1}^{+}(u_1u_2)\wedge\mu_{B_2}^{+}(v_1v_2).\end{cases}$$

Example 3.2.12:

Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be graphs such that $V_1 = \{a,b\}$, $V_2 = \{c,d\}$, $E_1 = \{ab\}$, $E_2 = \{cd\}$. Consider two interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ where $A_1 = <(\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})>$, $B_1 = <\frac{ab}{0.2}, \frac{ab}{0.3}>$, $A_2 = <(\frac{c}{0.2}, \frac{d}{0.3}), (\frac{c}{0.5}, \frac{d}{0.7})>$, $B_2 = <\frac{cd}{0.2}, \frac{cd}{0.4}>$

Here $V_1 \times V_2 = \{(a,c)(a,d)(b,c)(b,d)\}$
$E = \{(a,c)(a,d), (b,c)(b,d), (a,c)(b,c), (a,d)(b,d), (a,c)(b,d)\}$. Then

$(\mu_{A_1}^- \otimes \mu_{A_2}^-)(a,c) = \mu_{A_1}^-(a) \wedge \mu_{A_2}^-(c) = (0.3 \wedge 0.2) = 0.2$

$(\mu_{A_1}^- \otimes \mu_{A_2}^-)(a,d) = \mu_{A_1}^-(a) \wedge \mu_{A_2}^-(d) = (0.3 \wedge 0.3) = 0.3$

$(\mu_{A_1}^- \otimes \mu_{A_2}^-)(b,c) = \mu_{A_1}^-(b) \wedge \mu_{A_2}^-(c) = (0.4 \wedge 0.2) = 0.2$

$(\mu_{A_1}^- \otimes \mu_{A_2}^-)(b,d) = \mu_{A_1}^-(b) \wedge \mu_{A_2}^-(d) = (0.4 \wedge 0.3) = 0.3$

$(\mu_{A_1}^+ \otimes \mu_{A_2}^+)(a,c) = \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(c) = (0.5 \wedge 0.5) = 0.5$

$(\mu_{A_1}^+ \otimes \mu_{A_2}^+)(a,d) = \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(d) = (0.5 \wedge 0.7) = 0.5$

$(\mu_{A_1}^+ \otimes \mu_{A_2}^+)(b,c) = \mu_{A_1}^+(b) \wedge \mu_{A_2}^+(c) = (0.6 \wedge 0.5) = 0.5$

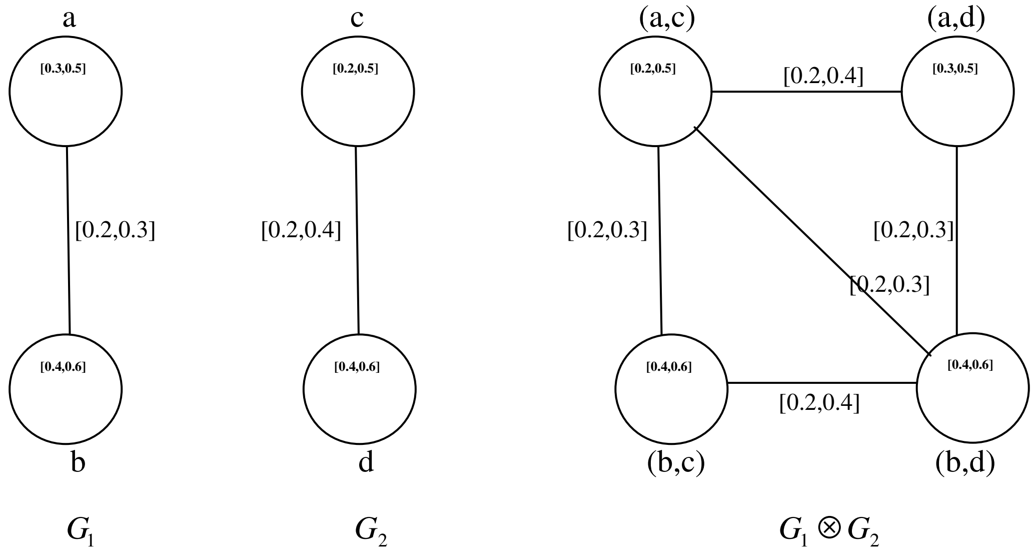
$(\mu_{A_1}^+ \otimes \mu_{A_2}^+)(b,d) = \mu_{A_1}^+(b) \wedge \mu_{A_2}^+(d) = (0.6 \wedge 0.7) = 0.6$

$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((a,c)(a,d)) = \mu_{A_1}^-(a) \wedge \mu_{B_2}^-(cd) = (0.3 \wedge 0.2) = 0.2$
$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((b,c)(b,d)) = \mu_{A_1}^-(b) \wedge \mu_{B_2}^-(cd) = (0.4 \wedge 0.2) = 0.2$
$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((a,c)(b,c)) = \mu_{A_2}^-(c) \wedge \mu_{B_1}^-(ab) = (0.2 \wedge 0.2) = 0.2$
$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((a,d)(b,d)) = \mu_{A_2}^-(d) \wedge \mu_{B_1}^-(ab) = (0.3 \wedge 0.2) = 0.2$
$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((a,c)(b,d)) = \mu_{B_1}^-(ab) \wedge \mu_{B_2}^-(cd) = (0.2 \wedge 0.2) = 0.2$

$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((a,c)(a,d)) = \mu_{A_1}^+(a) \wedge \mu_{B_2}^+(cd) = (0.5 \wedge 0.4) = 0.4$
$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((b,c)(b,d)) = \mu_{A_1}^+(b) \wedge \mu_{B_2}^+(cd) = (0.6 \wedge 0.4) = 0.4$
$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((a,c)(b,c)) = \mu_{A_2}^+(c) \wedge \mu_{B_1}^+(ab) = (0.5 \wedge 0.3) = 0.3$

$$(\mu_{B_1}^{+}\otimes\mu_{B_2}^{+})((a,d)(b,d))=\mu_{A_2}^{+}(d)\wedge\mu_{B_1}^{+}(ab)=(0.7\wedge 0.3)=0.3$$

$$(\mu_{B_1}^{+}\otimes\mu_{B_2}^{+})((a,c)(b,d))=\mu_{B_1}^{+}(ab)\wedge\mu_{B_2}^{+}(cd)=(0.3\wedge 0.4)=0.3$$



We shall verify the condition for interval-valued fuzzy graph

$$(\mu_{B_1}^{-}\otimes\mu_{B_2}^{-})((a,c)(a,d))\le\mu_{A_1}^{-}\otimes\mu_{A_2}^{-}(a,c)\wedge\mu_{A_1}^{-}\otimes\mu_{A_2}^{-}(a,d)$$

$$0.2\le(\mu_{A_1}^{-}(a)\wedge\mu_{A_2}^{-}(c))\wedge(\mu_{A_1}^{-}(a)\wedge\mu_{A_2}^{-}(d))$$

$$0.2\le(0.3\wedge 0.2)\wedge(0.3\wedge 0.3)$$

$$0.2\le(0.2\wedge 0.3)=0.2$$

$$(\mu_{B_1}^{-}\otimes\mu_{B_2}^{-})((b,c)(b,d))\le\mu_{A_1}^{-}\otimes\mu_{A_2}^{-}(b,c)\wedge\mu_{A_1}^{-}\otimes\mu_{A_2}^{-}(b,d)$$

$$0.2\le(\mu_{A_1}^{-}(b)\wedge\mu_{A_2}^{-}(c))\wedge(\mu_{A_1}^{-}(b)\wedge\mu_{A_2}^{-}(d))$$

$$0.2\le(0.4\wedge 0.2)\wedge(0.4\wedge 0.3)$$

$$0.2\le(0.2\wedge 0.3)=0.2$$

$$(\mu_{B_1}^{-}\otimes\mu_{B_2}^{-})((a,c)(b,c))\le\mu_{A_1}^{-}\otimes\mu_{A_2}^{-}(a,c)\wedge\mu_{A_1}^{-}\otimes\mu_{A_2}^{-}(b,c)$$

$$0.2\le(\mu_{A_1}^{-}(a)\wedge\mu_{A_2}^{-}(c))\wedge(\mu_{A_1}^{-}(b)\wedge\mu_{A_2}^{-}(c))$$

$$0.2\le(0.3\wedge 0.2)\wedge(0.4\wedge 0.2)$$

$$0.2\le(0.2\wedge 0.2)=0.2$$

$$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((a,d)(b,d)) \le \mu_{A_1}^- \otimes \mu_{A_2}^-(a,d) \wedge \mu_{A_1}^- \otimes \mu_{A_2}^-(b,d)$$

$$0.2 \le (\mu_{A_1}^-(a) \wedge \mu_{A_2}^-(d)) \wedge (\mu_{A_1}^-(b) \wedge \mu_{A_2}^-(d))$$

$$0.2 \le (0.3 \wedge 0.3) \wedge (0.4 \wedge 0.3) = 0..3$$

$$(\mu_{B_1}^- \otimes \mu_{B_2}^-)((a,c)(b,d)) \le \mu_{A_1}^- \otimes \mu_{A_2}^-(a,c) \wedge \mu_{A_1}^- \otimes \mu_{A_2}^-(b,d)$$

$$0.2 \le (\mu_{A_1}^-(a) \wedge \mu_{A_2}^-(c)) \wedge (\mu_{A_1}^-(b) \wedge \mu_{A_2}^-(d))$$

$$0.2 \le (0.3 \wedge 0.2) \wedge (0.4 \wedge 0.3)$$

$$0.2 \le (0.2 \wedge 0.3) = 0.2$$

$$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((a,c)(a,d)) \le \mu_{A_1}^+ \otimes \mu_{A_2}^+(a,c) \wedge \mu_{A_1}^+ \otimes \mu_{A_2}^+(a,d)$$

$$0.4 \le (\mu_{A_1}^+(a) \wedge \mu_{A_2}^+(c)) \wedge (\mu_{A_1}^+(a) \wedge \mu_{A_2}^+(d))$$

$$0.4 \le (0.5 \wedge 0.5) \wedge (0.5 \wedge 0.7)$$

$$0.4 \le (0.5 \wedge 0.5) = 0.5$$

$$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((b,c)(b,d)) \le \mu_{A_1}^+ \otimes \mu_{A_2}^+(b,c) \wedge \mu_{A_1}^+ \otimes \mu_{A_2}^+(b,d)$$

$$0.4 \le (\mu_{A_1}^+(b) \wedge \mu_{A_2}^+(c)) \wedge (\mu_{A_1}^+(b) \wedge \mu_{A_2}^+(d))$$

$$0.4 \le (0.6 \wedge 0.5) \wedge (0.6 \wedge 0.7)$$

$$0.4 \le (0.5 \wedge 0.6) = 0.5$$

$$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((a,c)(b,c)) \le \mu_{A_1}^+ \otimes \mu_{A_2}^+(a,c) \wedge \mu_{A_1}^+ \otimes \mu_{A_2}^+(b,c)$$

$$0.3 \le (\mu_{A_1}^+(a) \wedge \mu_{A_2}^+(c)) \wedge (\mu_{A_1}^+(b) \wedge \mu_{A_2}^+(c))$$

$$0.3 \le (0.5 \wedge 0.5) \wedge (0.6 \wedge 0.5)$$

$$0.3 \le (0.5 \wedge 0.5) = 0.5$$

$$(\mu_{B_1}^+ \otimes \mu_{B_2}^+)((a,d)(b,d)) \le \mu_{A_1}^+ \otimes \mu_{A_2}^+(a,d) \wedge \mu_{A_1}^+ \otimes \mu_{A_2}^+(b,d)$$

$$0.3 \le (\mu_{A_1}^+(a) \wedge \mu_{A_2}^+(d)) \wedge (\mu_{A_1}^+(b) \wedge \mu_{A_2}^+(d))$$

$$0.3 \le (0.5 \wedge 0.7) \wedge (0.6 \wedge 0.7)$$

$$0.3 \le (0.5 \wedge 0.6) = 0.5$$

$$(\mu^+_{B_1} \otimes \mu^+_{B_2})((a,c)(b,d)) \le \mu^+_{A_1} \otimes \mu^+_{A_2}(a,c) \wedge \mu^+_{A_1} \otimes \mu^+_{A_2}(b,d)$$

$$0.3 \le (\mu^+_{A_1}(a) \wedge \mu^+_{A_2}(c)) \wedge (\mu^+_{A_1}(b) \wedge \mu^+_{A_2}(d))$$

$$0.3 \le (0.5 \wedge 0.5) \wedge (0.6 \wedge 0.7)$$

$$0.3 \le (0.5 \wedge 0.6) = 0.5$$

Thus we have $G_1 \otimes G_2$ is an interval-valued fuzzy graph.

Theorem 3.2.13:

If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are complete interval-valued fuzzy graphs, then $G_1 \otimes G_2$ is complete.

Proof:

If $(u,v_1)(u,v_2) \in E$, then

$$(\mu^-_{B_1} \otimes \mu^-_{B_2})((u,v_1)(u,v_2)) = \mu^-_{A_1}(u) \wedge \mu^-_{B_2}(v_1 v_2)$$

$$= \mu^-_{A_1}(u) \wedge \mu^-_{A_2}(v_1) \wedge \mu^-_{A_2}(v_2) \text{ (since } G_2 \text{ is complete)}$$

$$= (\mu^-_{A_1} \otimes \mu^-_{A_2})(u,v_1) \wedge (\mu^-_{A_1} \otimes \mu^-_{A_2})(u,v_2)$$

$$(\mu^+_{B_1} \otimes \mu^+_{B_2})((u,v_1)(u,v_2)) = \mu^+_{A_1}(u) \wedge \mu^+_{B_2}(v_1 v_2)$$

$$= \mu^+_{A_1}(u) \wedge \mu^+_{A_2}(v_1) \wedge \mu^+_{A_2}(v_2) \text{ (since } G_2 \text{ is complete)}$$

$$= (\mu^+_{A_1} \otimes \mu^+_{A_2})(u,v_1) \wedge (\mu^+_{A_1} \otimes \mu^+_{A_2})(u,v_2)$$

If $((u_1,w)(u_2,w)) \in E$, then

$$(\mu^-_{B_1} \otimes \mu^-_{B_2})((u_1,w)(u_2,w)) = \mu^-_{A_2}(w) \wedge \mu^-_{B_1}(u_1 u_2)$$

$$= \mu^-_{A_2}(w) \wedge \mu^-_{A_1}(u_1) \wedge \mu^-_{A_1}(u_2) \text{ (since } G_1 \text{ is complete)}$$

$$= (\mu^-_{A_1} \otimes \mu^-_{A_2})(u_1,w) \wedge (\mu^-_{A_1} \otimes \mu^-_{A_2})(u_2,w)$$

Similarly, we have

$$(\mu_{B_1}^{+} \otimes \mu_{B_2}^{+})((u_1, w)(u_2, w)) = \mu_{A_2}^{+}(w) \wedge \mu_{B_1}^{+}(u_1u_2)$$

$$= \mu_{A_2}^{+}(w) \wedge \mu_{A_1}^{+}(u_1) \wedge \mu_{A_1}^{+}(u_2) \text{ (since } G_1 \text{ is complete)}$$

$$= (\mu_{A_1}^{+} \otimes \mu_{A_2}^{+})(u_1, w) \wedge (\mu_{A_1}^{+} \otimes \mu_{A_2}^{+})(u_2, w)$$

If $((u_1, v_1)(u_2, v_2)) \in E$, then since G_1 and G_2 are complete, we have

$$(\mu_{B_1}^{-} \otimes \mu_{B_2}^{-})((u_1, v_1)(u_2, v_2)) = \mu_{B_1}^{-}(u_1u_2) \wedge \mu_{B_2}^{-}(v_1v_2)$$

$$= \mu_{A_1}^{-}(u_1) \wedge \mu_{A_2}^{-}(u_2) \wedge \mu_{A_2}^{-}(v_1) \wedge \mu_{A_2}^{-}(v_2)$$

$$= (\mu_{A_1}^{-} \otimes \mu_{A_2}^{-})(u_1, v_1) \wedge (\mu_{A_1}^{-} \otimes \mu_{A_2}^{-})(u_2, v_2).$$

$$(\mu_{B_1}^{+} \otimes \mu_{B_2}^{+})((u_1, v_1)(u_2, v_2)) = \mu_{B_1}^{+}(u_1u_2) \wedge \mu_{B_2}^{+}(v_1v_2)$$

$$= \mu_{A_1}^{+}(u_1) \wedge \mu_{A_2}^{+}(u_2) \wedge \mu_{A_2}^{+}(v_1) \wedge \mu_{A_2}^{+}(v_2)$$

$$= (\mu_{A_1}^{+} \otimes \mu_{A_2}^{+})(u_1, v_1) \wedge (\mu_{A_1}^{+} \otimes \mu_{A_2}^{+})(u_2, v_2).$$

Hence $G_1 \otimes G_2$ is complete.

Theorem 3.2.14:

If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs such that $G_1 \sqcap G_2$ is complete, then atleast G_1 or G_2 must be complete.

Proof:

Suppose that G_1 and G_2 are not complete. Then there exists atleast one $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$ such that

$$\begin{cases} \mu_{B_1}^{-}(u_1v_1) < \mu_{A_1}^{-}(u_1) \wedge \mu_{A_1}^{-}(v_1) \\ \mu_{B_1}^{+}(u_1v_1) < \mu_{A_1}^{+}(u_1) \wedge \mu_{A_1}^{+}(v_1) \end{cases} \text{ and } \begin{cases} \mu_{B_2}^{-}(u_2v_2) < \mu_{A_2}^{-}(u_2) \wedge \mu_{A_2}^{-}(v_2) \\ \mu_{B_2}^{+}(u_2v_2) < \mu_{A_2}^{+}(u_2) \wedge \mu_{A_2}^{+}(v_2) \end{cases}$$

Now, $(\mu_{B_1}^- \sqcap \mu_{B_2}^-)((u_1,v_1)(u_2,v_2)) = \mu_{B_1}^-(u_1u_2) \wedge \mu_{B_2}^-(v_1v_2)$

$$< \mu_{A_1}^-(u_1) \wedge \mu_{A_1}^-(u_2) \wedge \mu_{A_2}^-(v_1) \wedge \mu_{A_2}^-(v_2)$$

(since G_1 and G_2 are not complete).

Similarly, $(\mu_{B_1}^+ \sqcap \mu_{B_2}^+)((u_1,v_1)(u_2,v_2)) < \mu_{A_1}^+(u_1) \wedge \mu_{A_1}^+(u_2) \wedge \mu_{A_2}^+(v_1) \wedge \mu_{A_2}^+(v_2)$

But, $(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u_1,v_1) = \mu_{A_1}^-(u_1) \wedge \mu_{A_2}^-(v_1)$

$(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u_2,v_2) = \mu_{A_1}^-(u_2) \wedge \mu_{A_2}^-(v_2)$

Thus,

$$(\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u_1,v_1) \wedge (\mu_{A_1}^- \sqcap \mu_{A_2}^-)(u_2,v_2) = \mu_{A_1}^-(u_1) \wedge \mu_{A_1}^-(u_2) \wedge \mu_{A_2}^-(v_1) \wedge \mu_{A_2}^-(v_2)$$

$$> (\mu_{B_1}^- \sqcap \mu_{B_2}^-)((u_1,v_1)(u_2,v_2))$$

Similarly, we have

$$(\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(u_1,v_1) \wedge (\mu_{A_1}^+ \sqcap \mu_{A_2}^+)(u_2,v_2) > (\mu_{B_1}^+ \sqcap \mu_{B_2}^+)((u_1,v_1)(u_2,v_2))$$

Hence, $G_1 \sqcap G_2$ is not complete, a contradiction.

Theorem 3.2.15:

If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are interval-valued fuzzy graphs such that $G_1 \bullet G_2$ or $G_1 \otimes G_2$ is complete, then atleast G_1 or G_2 must be complete.

Proof:

The proof is similar to the proof of the preceding theorem.

Chapter IV

CHAPTER IV

TYPES OF INTERVAL-VALUED FUZZY GRAPHS

SECTION 4.1

BALANCED INTERVAL-VALUED FUZZY GRAPHS

Definition 4.1.1:

Let G be an interval-valued fuzzy graph. The **open neighbourhood** degree of a vertex x in G is defined by $\deg(x) = [\deg_{\mu^-}(x), \deg_{\mu^+}(x)]$, where $\deg_{\mu^-}(x) = \sum_{y \in N(x)} \mu_A^-(y)$ and $\deg_{\mu^+}(x) = \sum_{y \in N(x)} \mu_A^+(y)$. Notice that $\mu_B^-(xy) > 0, \mu_B^+(xy) > 0$ for $xy \in E$, and $\mu_B^-(xy) = \mu_B^+(xy) = 0$ for $xy \notin E$

Remark 4.1.2:

If all the vertices have the same open neighbourhood degree n, then G is called an n-regular interval-valued fuzzy graph.

Definition 4.1.3:

Let G be an interval-valued fuzzy graph. The **closed neighbourhood** degree of a vertex x in G is defined by $\deg(x) = [\deg_{\mu^-}(x), \deg_{\mu^+}(x)]$, where

$$\deg_{\mu^-}[x] = \deg_{\mu^-}(x) + \mu_A^-(x), \qquad \deg_{\mu^+}[x] = \deg_{\mu^+}(x) + \mu_A^+(x)$$

Remark 4.1.4:

If all the vertices have the same closed neighbourhood degree m ,then G is called a m -totally regular interval-valued fuzzy graph.

Example 4.1.5:

Consider a graph G^* such that $V = \{a, b, c, d\}$, $E = \{ab, bc, cd, ad\}$. Let A be an interval-valued fuzzy subset of V, and let B be an interval-valued fuzzy subset of $E \subseteq V \times V$ defined by

$$\begin{bmatrix} & a & b & c & d \\ \hline \mu_A^- & 0.3 & 0.3 & 0.3 & 0.3 \\ \mu_A^+ & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \qquad \begin{bmatrix} & ab & bc & cd & da \\ \hline \mu_B^- & 0.1 & 0.1 & 0.1 & 0.1 \\ \mu_B^+ & 0.2 & 0.4 & 0.2 & 0.4 \end{bmatrix}$$

$\deg(a) = [\deg_{\mu^-}(a), \deg_{\mu^+}(a)]$

$\deg_{\mu^-}(a) = \mu_A^-(b) + \mu_A^-(d) = 0.3 + 0.3 = 0.6$

$\deg_{\mu^+}(a) = \mu_A^+(b) + \mu_A^+(d) = 0.5 + 0.5 = 1$

$\deg(a) = [0.6, 1]$

$\deg(b) = [\deg_{\mu^-}(b), \deg_{\mu^+}(b)]$

$\deg_{\mu^-}(b) = \mu_A^-(a) + \mu_A^-(c) = 0.3 + 0.3 = 0.6$

$\deg_{\mu^+}(b) = \mu_A^+(a) + \mu_A^+(c) = 0.5 + 0.5 = 1$

$\deg(b) = [0.6, 1]$

$\deg(c) = [\deg_{\mu^-}(c), \deg_{\mu^+}(c)]$

$\deg_{\mu^-}(c) = \mu_A^-(b) + \mu_A^-(d) = 0.3 + 0.3 = 0.6$

$\deg_{\mu^+}(c) = \mu_A^+(b) + \mu_A^+(d) = 0.5 + 0.5 = 1$

$\deg(c) = [0.6, 1]$

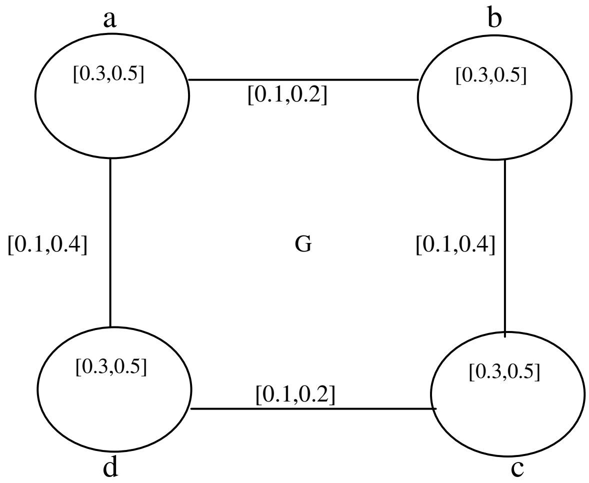
$\deg(d) = [\deg_{\mu^-}(d), \deg_{\mu^+}(d)]$

$\deg_{\mu^-}(d) = \mu_A^-(a) + \mu_A^-(c) = 0.3 + 0.3 = 0.6$

$\deg_{\mu^+}(d) = \mu_A^+(a) + \mu_A^+(c) = 0.5 + 0.5 = 1$

$\deg(d) = [0.6, 1]$

All the vertices have the same neighbourhood degree. Hence G^* is regular



$$\deg[a] = [\deg_{\mu^-}[a], \deg_{\mu^+}[a]]$$
$$\deg_{\mu^-}[a] = \deg_{\mu^-}(a) + \mu_A^-(a) = 0.6 + 0.3 = 0.9$$
$$\deg_{\mu^+}[a] = \deg_{\mu^+}(a) + \mu_A^+(a) = 1 + 0.5 = 1.5$$
$$\deg[a] = [0.9, 1.5]$$

$$\deg[b] = [\deg_{\mu^-}[b], \deg_{\mu^+}[b]]$$
$$\deg_{\mu^-}[b] = \deg_{\mu^-}(b) + \mu_A^-(b) = 0.6 + 0.3 = 0.9$$
$$\deg_{\mu^+}[b] = \deg_{\mu^+}(b) + \mu_A^+(b) = 1 + 0.5 = 1.5$$
$$\deg[b] = [0.9, 1.5]$$

$$\deg[c] = [\deg_{\mu^-}[c], \deg_{\mu^+}[c]]$$
$$\deg_{\mu^-}[c] = \deg_{\mu^-}(c) + \mu_A^-(c) = 0.6 + 0.3 = 0.9$$
$$\deg_{\mu^+}[c] = \deg_{\mu^+}(c) + \mu_A^+(c) = 1 + 0.5 = 1.5$$
$$\deg[c] = [0.9, 1.5]$$

$$\deg[d] = [\deg_{\mu^-}[d], \deg_{\mu^+}[d]]$$
$$\deg_{\mu^-}[d] = \deg_{\mu^-}(d) + \mu_A^-(d) = 0.6 + 0.3 = 0.9$$
$$\deg_{\mu^+}[d] = \deg_{\mu^+}(d) + \mu_A^+(d) = 1 + 0.5 = 1.5$$
$$\deg[d] = [0.9, 1.5]$$

Since the vertices have the same closed neighbourhood degree, then it is totally regular. Thus the interval-valued fuzzy graph G is both totally regular and regular.

Example 4.1.6:

Consider a graph G^* such that $V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_1v_3\}$. Let A be an interval-valued fuzzy subset of V and let B be an interval-valued fuzzy subset of E defined by $\mu_A^-(v_1) = 0.4,\quad \mu_A^-(v_2) = 0.7,$

$\mu_A^-(v_3) = 0.6,$

$\mu_A^+(v_1) = 0.4,\quad \mu_A^+(v_2) = 0.8,\quad \mu_A^+(v_3) = 0.7,$

$\mu_B^-(v_1v_2) = 0.2,\quad \mu_B^-(v_1v_3) = 0.2,\quad \mu_B^+(v_1v_2) = 0.3,\quad \mu_B^+(v_1v_3) = 0.4.$

$$\deg(v_1) = [\deg_{\mu^-}(v_1), \deg_{\mu^+}(v_1)]$$
$$\deg_{\mu^-}(v_1) = \mu_A^-(v_2) + \mu_A^-(v_3) = (0.7 + 0.6) = 1.3$$
$$\deg_{\mu^+}(v_1) = \mu_A^+(v_2) + \mu_A^+(v_3) = 0.8 + 0.7 = 1.5$$
$$\deg(v_1) = [1.3, 1.5]$$

$$\deg(v_2) = [\deg_{\mu^-}(v_2), \deg_{\mu^+}(v_2)]$$
$$\deg_{\mu^-}(v_2) = \mu_A^-(v_1) = 0.4$$
$$\deg_{\mu^+}(v_2) = \mu_A^+(v_1) = 0.4$$
$$\deg(v_2) = 0.4$$

$$\deg(v_3) = [\deg_{\mu^-}(v_3), \deg_{\mu^+}(v_3)]$$
$$\deg_{\mu^-}(v_3) = \mu_A^-(v_1) = 0.4$$
$$\deg_{\mu^+}(v_3) = \mu_A^+(v_1) = 0.4$$
$$\deg(v_3) = 0.4$$

All the vertices does not have the same neighbourhood degree. Hence G^* is not regular

$$\deg[v_1] = [\deg_{\mu^-}[v_1], \deg_{\mu^+}[v_1]]$$
$$\deg_{\mu^-}[v_1] = \deg_{\mu^-}(v_1) + \mu_A^-(v_1) = 1.3 + 0.4 = 1.7$$
$$\deg_{\mu^+}[v_1] = \deg_{\mu^+}(v_1) + \mu_A^+(v_1) = 1.5 + 0.4 = 1.9$$
$$\deg[v_1] = [1.7, 1.9]$$

$$\deg[v_2] = [\deg_{\mu^-}[v_2], \deg_{\mu^+}[v_2]]$$
$$\deg_{\mu^-}[v_2] = \deg_{\mu^-}(v_2) + \mu_A^-(v_2) = 0.4 + 0.7 = 1.1$$
$$\deg_{\mu^+}[v_2] = \deg_{\mu^+}(v_2) + \mu_A^+(v_2) = 0.4 + 0.8 = 1.2$$
$$\deg[v_2] = [1.1, 1.2]$$

$\deg[v_3] = [\deg_{\mu^-}[v_3], \deg_{\mu^+}[v_3]]$

$\deg_{\mu^-}[v_3] = \deg_{\mu^-}(v_3) + \mu_A^-(v_3) = 0.4 + 0.6 = 1$

$\deg_{\mu^+}[v_3] = \deg_{\mu^+}(v_3) + \mu_A^+(v_3) = 0.4 + 0.7 = 1.1$

$\deg[v_2] = [1, 1.1]$

Since the vertices does not have the same closed neighbourhood degree, then it is not totally regular.

Thus the interval-valued fuzzy graph G is neither totally regular nor regular.

Definition 4.1.7:

We define the **order** $O(G)$ **and size** $S(G)$ of an interval-valued fuzzy graph $G = (A, B)$ by $O(G) = \sum_{x \in V} \frac{1 + \mu_A^+(x) - \mu_A^-(x)}{2}$, $S(G) = \sum_{xy \in E} \frac{1 + \mu_B^+(xy) - \mu_B^-(xy)}{2}$.

Theorem 4.1.8:

Every complete interval-valued fuzzy graph is totally regular.

Theorem 4.1.9:

Let $G = (A, B)$ be an interval-valued fuzzy graph of a graph G^*. Then, $A = [\mu_A^-, \mu_A^+]$ is a constant function if and only if the following statements are equivalent:

a) G is a regular interval-valued fuzzy graph

b) G is a totally regular interval-valued fuzzy graph

Proof:

Suppose that $A=[\mu_A^-,\mu_A^+]$ is a constant function. Let $\mu_A^-(x)=c_1$ and $\mu_A^+(x)=c_2$ for all $x\in V$.

$(a)\Rightarrow(b)$: Assume that G is an n-regular interval-valued fuzzy graph. Then, $\deg_{\mu^-}(x)=n_1$ and $\deg_{\mu^+}(x)=n_2$ for all $x\in V$. So,

$$\deg_{\mu^-}[x]=\deg_{\mu^-}(x)+\mu_A^-(x),\ \deg_{\mu^+}[x]=\deg_{\mu^+}(x)+\mu_A^+(x)\ \ \forall x\in V.$$

Thus $\deg_{\mu^-}[x]=n_1+c_1$, $\deg_{\mu^+}[x]=n_2+c_2$, $\forall x\in V$.

Hence, G is a totally regular interval-valued fuzzy graph.

$(b)\Rightarrow(a)$: Suppose that G is a totally regular interval-valued fuzzy graph. Then $\deg_{\mu^-}[x]=k_1$, $\deg_{\mu^+}[x]=k_2$, $\forall x=V$

or $\deg_{\mu^-}(x)+\mu_A^-(x)=k_1$, $\deg_{\mu^+}(x)+\mu_A^+(x)=k_2$, $\forall x=V$

or $\deg_{\mu^-}(x)+c_1=k_1$, $\deg_{\mu^+}(x)+c_2=k_2$, $\forall x=V$

or $\deg_{\mu^-}(x)=k_1-c_1$, $\deg_{\mu^+}(x)=k_2-c_2$, $\forall x=V$

Thus, G is a regular interval-valued fuzzy graph. Hence, (a) and (b) are equivalent.

The converse part is obvious.

Theorem 4.1.10:

Let G be an interval-valued fuzzy graph where a crisp graph G^* is an odd cycle. Then, G is a regular interval-valued fuzzy graph if and only if B is a constant function.

Proof:

If $B=[\mu_B^-,\mu_B^+]$ is a constant function, say $\mu_B^-=c_1$ and $\mu_B^+=c_2$ for all $xy\in E$; then, $\deg_{\mu^-}(x)=2c_1$ and $\deg_{\mu^+}(x)=2c_2$ for every $x\in V$. Hence G is a regular interval valued fuzzy graph.

Conversely, suppose that G is a (k_1,k_2) regular interval-valued fuzzy graph. Let $e_1,e_2,e_3,......e_{2n+1}$ be the edges of G in that order. Let $\mu_B^-(e_1)=c_1$, $\mu_B^-(e_2)=k_1-c_1$, $\mu_B^-(e_3)=k_1-(k_1-c_1)=c_1$, $\mu_B^-(e_4)=k_1-c_1$, and so on. Therefore,

$$\mu_B^-(e_i)=\begin{cases} c_1, & \text{if } i \text{ is odd} \\ k_1-c_1, & \text{if } i \text{ is even.} \end{cases}$$

Thus, $\mu_B^-(e_1)=\mu_B^-(e_{2n+1})=c_1$. So, if e_1 and e_{2n+1} incident at a vertex v_1, then $\deg_{\mu^-}(v_1)=k_1$, $\deg_{\mu^-}(e_1)+\deg_{\mu^-}(e_{2n+1})=k_1$, $c_1+c_1=k_1$, $2c_1=k_{1,}$ and $c_1=k_1/2$. This shows that μ_B^- is a regular function.

Similarly, let $\mu_B^+(e_1)=c_2$, $\mu_B^+(e_2)=k_2-c_2$, $\mu_B^+(e_3)=k_2-(k_2-c_2)=c_2$, $\mu_B^+(e_4)=k_2-c_2$, and so on. Therefore,

$$\mu_B^+(e_i)=\begin{cases} c_2, & \text{if } i \text{ is odd} \\ k_2-c_2, & \text{if } i \text{ is even.} \end{cases}$$

Thus, $\mu_B^+(e_2)=\mu_B^+(e_{2n+1})=c_2$. So, if e_2 and e_{2n} incident at a vertex v_2, then $\deg_{\mu^+}(v_2)=k_2$, $\deg_{\mu^+}(e_2)+\deg_{\mu^+}(e_{2n+1})=k_2$, $c_2+c_2=k_2$, $2c_2=k_2$ and $c_2=k_2/2$. This shows that μ_B^+ is a constant function. Hence, $B=[\mu_B^-,\mu_B^+]$ is a constant function.

Theorem 4.1.11:

Let G be an interval-valued fuzzy graph where a crisp graph G^* is an even cycle. Then, G is a regular interval-valued fuzzy graph if and only if either $B = [\mu_B^-, \mu_B^+]$ is a constant function alternate edges have the same membership values.

Definition 4.1.12:

The **density** of an interval-valued fuzzy graph G is $D(G) = (D^-(G), D^+(G)),$

where $D^-(G) = (2\sum_{x,y\in V}(\mu_B^-(xy)))/(\sum_{x,y\in V}(\mu_A^-(x) \wedge \mu_A^-(y)))$ for $x, y \in V$ and

$$D^+(G) = (2\sum_{x,y\in V}(\mu_B^+(xy)))/(\sum_{x,y\in V}(\mu_A^+(x) \wedge \mu_A^+(y))) \text{ for } x, y \in V.$$

Definition 4.1.13:

An interval-valued fuzzy graph G is **balanced** if $D(H) \le D(G)$; that is, $D^-(H) \le D^-(G)$, $D^+(H) \le D^+(G)$ for all subgroups H of G.

Definition 4.1.14:

An interval-valued fuzzy graph G is **strictly balanced** if for every $x, y \in V$, $D(H) = D(G)$ for all nonempty sub graphs.

Example 4.1.15:

Consider a graph G^* such that $V = \{a, b, c, d\}$, $E = \{ab, bc, cd, ad\}$. Let A be an interval-valued fuzzy subset of V, and let B be an interval-valued fuzzy subset of $E \subseteq V \times V$ defined by

$$\left[\begin{array}{c|cccc} & a & b & c & d \\ \hline \mu_A^- & 0.3 & 0.3 & 0.3 & 0.3 \\ \mu_A^+ & 0.5 & 0.5 & 0.5 & 0.5 \end{array}\right] \qquad \left[\begin{array}{c|cccc} & ab & bc & cd & da \\ \hline \mu_B^- & 0.1 & 0.1 & 0.1 & 0.1 \\ \mu_B^+ & 0.2 & 0.4 & 0.2 & 0.4 \end{array}\right]$$

$$D(G) = (D^-(G), D^+(G))$$

$$D^-(G) = (2(\mu_B^-(ab) + \mu_B^-(bc) + \mu_B^-(cd) + \mu_B^-(da))) / ((\mu_A^-(a) \wedge \mu_A^-(b)) + (\mu_A^-(b) \wedge \mu_A^-(c)) + (\mu_A^-(c) \wedge \mu_A^-(d)) + (\mu_A^-(d) \wedge \mu_A^-(a))$$

$$= (2(0.1 + 0.1 + 0.1 + 0.1)) / ((0.3 \wedge 0.3) + (0.3 \wedge 0.3) + (0.3 \wedge 0.3) + (0.3 \wedge 0.3))$$
$$= 2(0.4) / (0.3 + 0.3 + 0.3 + 0.3) = (0.8) / (1.2) = 0.67$$

$$D^+(G) = (2(\mu_B^+(ab) + \mu_B^+(bc) + \mu_B^+(cd) + \mu_B^+(da))) / ((\mu_A^+(a) \wedge \mu_A^+(b)) + (\mu_A^+(b) \wedge \mu_A^+(c)) + (\mu_A^+(c) \wedge \mu_A^+(d)) + (\mu_A^+(d) \wedge \mu_A^+(a))$$
$$= (2(0.2 + 0.4 + 0.2 + 0.4)) / ((0.5 \wedge 0.5) + (0.5 \wedge 0.5) + (0.5 \wedge 0.5) + (0.5 \wedge 0.5))$$
$$= 2(1.2) / (0.5 + 0.5 + 0.5 + 0.5) = (2.4) / 2 = 1.2$$

$$D(G) = (D^-(G), D^+(G)) = (0.67, 1.2)$$

$H_1 = \{a, b, c\}$ $\qquad$ $D(H_1) = (D^-(H_1), D^+(H_1))$

$$D^-(H_1) = (2(\mu_B^-(ab) + \mu_B^-(bc))) / ((\mu_A^-(a) \wedge \mu_A^-(b)) + (\mu_A^-(b) \wedge \mu_A^-(c))$$

$$= 2(0.1+0.1)/(0.3+0.3)=2(0.2)/0.6=0.4/0.6=0.67$$

$$D^+(H_1) = (2(\mu_B^+(ab) + \mu_B^+(bc))) / ((\mu_A^+(a) \wedge \mu_A^+(b)) + (\mu_A^+(b) \wedge \mu_A^+(c))$$

$$= 2(0.2+0.4)/(0.5+0.5)=2(0.6)/1.0=1.2$$

$$D(H_1) = (D^-(H_1), D^+(H_1)) = (0.67, 1.2)$$

$H_2 = \{a, c\}$ $\qquad$ $D(H_2) = (D^-(H_2), D^+(H_2))$

$D^-(H_2) = 0, \quad D^+(H_2) = 0 \qquad D(H_2) = (0,0).$

$H_3 = \{a, d\}$ $\qquad$ $D(H_3) = (D^-(H_3), D^+(H_3))$

$$D^-(H_3) = (2(\mu_B^-(ad)))/\mu_A^-(a) \wedge \mu_A^-(d) = (2\times 0.1)/0.3 = 0.2/0.3 = 0.67$$
$$D^+(H_3) = (2(\mu_B^+(ad)))/\mu_A^+(a) \wedge \mu_A^+(d) = (2\times 0.4)/0.5 = 0.8/0.5 = 1.6$$
$$D(H_3) = (0.67, 1.6)$$

$H_4 = \{b, c\}$ $\qquad$ $D(H_4) = (D^-(H_4), D^+(H_4))$

$$D^-(H_4) = (2(\mu_B^-(bc)))/\mu_A^-(b) \wedge \mu_A^-(c) = (2\times 0.1)/0.3 = 0.2/0.3 = 0.67$$
$$D^+(H_4) = (2(\mu_B^+(bc)))/\mu_A^+(b) \wedge \mu_A^+(c) = (2\times 0.4)/0.5 = 0.8/0.5 = 1.6$$
$$D(H_4) = (0.67, 1.6)$$

Here we see that

$$D(G) = (0.67, 1.2)$$
$$D(H_1) = (0.67, 1.2), D(H_2) = (0, 0), D(H_3) = (0.67, 1.6), D(H_4) = (0.67, 1.6)$$

Thus we see that regular interval-valued fuzzy graph is not balanced.

Remark 4.1.16:

Every regular interval-valued fuzzy graph may not be balanced.

Example 4.1.17:

Consider a graph G^* such that $V = \{x, y, z\}, E = \{xy, yz, zx\}$. Let A be an interval-valued fuzzy subset of V and let B be an interval-valued fuzzy subset of E defined by

$\mu_A^-(x) = 0.3,\quad \mu_A^-(y) = 0.4,\quad \mu_A^-(z) = 0.5,\quad \mu_A^+(x) = 0.5,$

$\mu_A^+(y) = 0.7,\quad \mu_A^+(z) = 0.6,\quad \mu_B^-(xy) = 0.3,\quad \mu_B^-(yz) = 0.4,$

$\mu_B^-(zx) = 0.3,\quad \mu_B^+(xy) = 0.5,\quad \mu_B^+(yz) = 0.6,\quad \mu_B^+(zx) = 0.5.$

$$D(G) = (D^-(G), D^+(G))$$
$$\begin{aligned} D^-(G) &= (2(\mu_B^-(xy) + \mu_B^-(yz) + \mu_B^-(zx)))/((\mu_A^-(x) \wedge \mu_A^-(y)) + (\mu_A^-(y) \wedge \mu_A^-(z)) + (\mu_A^-(z) \wedge \mu_A^-(x)) \\ &= (2(0.3 + 0.4 + 0.3))/((0.3 \wedge 0.4) + (0.4 \wedge 0.5) + (0.5 \wedge 0.3)) \\ &= (2(1.0))/(0.3 + 0.4 + 0.3) = 2/1.0 = 2 \end{aligned}$$

$$D^+(G) = (2(\mu_B^+(xy) + \mu_B^+(yz) + \mu_B^+(zx)))/((\mu_A^+(x) \wedge \mu_A^+(y)) + (\mu_A^+(y) \wedge \mu_A^+(z)) + (\mu_A^+(z) \wedge \mu_A^+(x))$$
$$= (2(0.5 + 0.6 + 0.5))/((0.5 \wedge 0.7) + (0.7 \wedge 0.6) + (0.6 \wedge 0.5))$$
$$= (2(1.6))/(0.5 + 0.6 + 0.5) = 3.2/1.6 = 2$$

$$D(G) = (D^-(G), D^+(G)) = (2,2)$$

$H_1 = \{x, y\}$ $\qquad D(H_1) = (D^-(H_1), D^+(H_1))$

$$D^-(H_1) = (2(\mu_B^-(xy)))/\mu_A^-(x) \wedge \mu_A^-(y) = (2 \times 0.3)/(0.3 \wedge 0.4) = 0.6/0.3 = 2$$
$$D^+(H_1) = (2(\mu_B^+(xy)))/\mu_A^+(x) \wedge \mu_A^+(y) = (2 \times 0.5)/(0.5 \wedge 0.7) = 1.0/0.5 = 2$$
$$D(H_1) = (2,2)$$

$H_2 = \{x, z\}$ $\qquad D(H_2) = (D^-(H_2), D^+(H_2))$

$$D^-(H_2) = (2(\mu_B^-(xz)))/\mu_A^-(x) \wedge \mu_A^-(z) = (2 \times 0.8)/(0.3 \wedge 0.5) = 0.6/0.3 = 2$$
$$D^+(H_2) = (2(\mu_B^+(xz)))/\mu_A^+(x) \wedge \mu_A^+(z) = (2 \times 0.5)/(0.5 \wedge 0.6) = 1.0/0.5 = 2$$
$$D(H_2) = (2,2)$$

$H_3 = \{y, z\}$ $\qquad D(H_3) = (D^-(H_3), D^+(H_3))$

$$D^-(H_3) = (2(\mu_B^-(yz)))/\mu_A^-(y) \wedge \mu_A^-(z) = (2 \times 0.4)/(0.4 \wedge 0.5) = 0.8/0.4 = 2$$
$$D^+(H_3) = (2(\mu_B^+(yz)))/\mu_A^+(y) \wedge \mu_A^+(z) = (2 \times 0.6)/(0.7 \wedge 0.6) = 1.2/0.6 = 2$$
$$D(H_3) = (2,2)$$

$$D(G) = D(H_1) = D(H_2) = D(H_3)$$

Thus we see that G is strictly balanced.

Proposition 4.1.18:

Any complete interval-valued fuzzy graph is balanced.

Proposition 4.1.19:

Let G be a self-complementary interval-valued fuzzy graph. Then, $D(G) = (1,1)$

Proposition 4.1.20:

Let G_1 and G_2 be two balanced interval-valued fuzzy graphs. Then, $G_1 \times G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \times G_2)$.

Theorem 4.1.21:

Let G be as strictly balanced interval-valued fuzzy graph, and let $\overline{G}$ be its complement; then $D(G)+D(\overline{G})=(2,2)$.

Proof:

Let G be a strictly balanced interval-valued fuzzy graph and $\overline{G}$ its complement.

Let H be a nonempty subgraph of $G.$ Since G is strictly balanced, $D(G)=D(H)$ for every $H \subseteq G$ and $x, y \in V.$ In $\overline{G},$

$$\overline{\mu_B^-(xy)} = \mu_A^-(x) \wedge \mu_A^-(y) - \mu_B^-(xy), \qquad \text{(A)}$$

$$\overline{\mu_B^+(xy)} = \mu_A^+(x) \wedge \mu_A^+(y) - \mu_B^+(xy), \qquad \text{(B)}$$

for every $x, y \in V.$ Dividing (A) by $\mu_A^-(xy) \wedge \mu_A^-(y),$ we get

$$\frac{\overline{\mu_B^-(xy)}}{\mu_A^-(x) \wedge \mu_A^-(y)} = 1 - \frac{\mu_B^-(xy)}{\mu_A^-(x) \wedge \mu_A^-(y)} \text{ for every } x, y \in V.$$

Dividing (B) by $\mu_A^+(xy) \wedge \mu_A^+(y),$ we get

$$\frac{\overline{\mu_B^+(xy)}}{\mu_A^+(x) \wedge \mu_A^+(y)} = 1 - \frac{\mu_B^+(xy)}{\mu_A^+(x) \wedge \mu_A^+(y)} \text{ for every } x, y \in V.$$

Then, $\displaystyle\sum_{x,y \in V} \frac{\overline{\mu_B^-(xy)}}{\mu_A^-(x) \wedge \mu_A^-(y)} = 1 - \sum_{x,y \in V} \frac{\mu_B^-(xy)}{\mu_A^-(x) \wedge \mu_A^-(y)}$ for every $x, y \in V.$

$$\sum_{x,y \in V} \frac{\overline{\mu_B^+(xy)}}{\mu_A^+(x) \wedge \mu_A^+(y)} = 1 - \sum_{x,y \in V} \frac{\mu_B^+(xy)}{\mu_A^+(x) \wedge \mu_A^+(y)} \text{ for every } x, y \in V.$$

Multiplying both sides the above equations by 2,

$$2\sum_{x,y\in V}\frac{\overline{\mu_B^-(xy)}}{\mu_A^-(x)\wedge\mu_A^-(y)} = 2-2\sum_{x,y\in V}\frac{\mu_B^-(xy)}{\mu_A^-(x)\wedge\mu_A^-(y)} \qquad \text{for every } x,y\in V.$$

$$2\sum_{x,y\in V}\frac{\overline{\mu_B^+(xy)}}{\mu_A^+(x)\wedge\mu_A^+(y)} = 2-2\sum_{x,y\in V}\frac{\mu_B^+(xy)}{\mu_A^+(x)\wedge\mu_A^+(y)} \qquad \text{for every } x,y\in V.$$

Thus, $D^-(\overline{G}) = 2 - D^-(G)$ and $D^+(\overline{G}) = 2 - D^+(G)$.

Now, $D(G) + D(\overline{G}) = (D^-(G), D^+(G)) + (D^-(\overline{G}), D^+(\overline{G}))$

$$= (D^-(G) + D^-(\overline{G})), (D^+(G) + D^+(\overline{G}))$$

$$= (2,2).$$

This completes the proof.

Corollary 4.1.22: The complement of strictly balanced interval-valued fuzzy graph is strictly balanced

Theorem 4.1.23:

Let G_1 and G_2 be isomorphic interval-valued fuzzy graphs. If G_2 is balanced, then G_1 is balanced.

SECTION 4.2

IRREGULARITY IN INTERVAL-VALUED FUZZY GRAPHS

Definition 4.2.1:

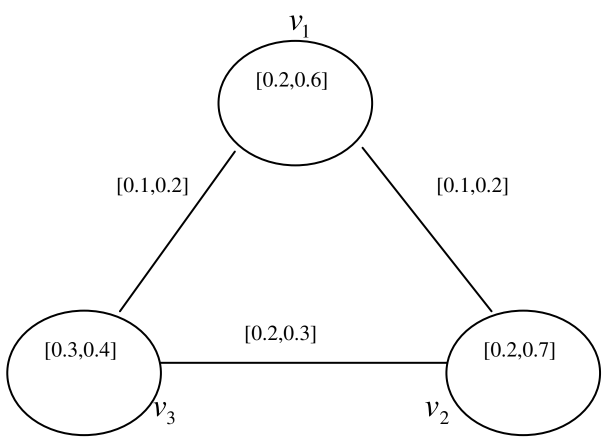
Let G be an interval-valued fuzzy graph on G^*. If there is a vertex which is adjacent to vertices with distinct neighbourhood degrees, then G is called an **irregular interval-valued fuzzy graph**. That is, $\deg(x) \neq n$ for all $x \in V$.

Example 4.2.2:

Consider a graph G^* such that $V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_1v_3\}$. Let A be an interval-valued fuzzy subset of V and let B be an interval-valued fuzzy subset of $E \subseteq V \times V$ defined by

$$\begin{bmatrix} & v_1 & v_2 & v_3 \\ \hline \mu_A^- & 0.2 & 0.2 & 0.3 \\ \mu_A^+ & 0.6 & 0.7 & 0.4 \end{bmatrix} \qquad \begin{bmatrix} & v_1 & v_2 & v_3 \\ \hline \mu_A^- & 0.2 & 0.2 & 0.3 \\ \mu_A^+ & 0.6 & 0.7 & 0.4 \end{bmatrix}$$

$$\deg(v_1) = [\deg_{\mu^-}(v_1), \deg_{\mu^+}(v_1)]$$
$$\deg_{\mu^-}(v_1) = \mu_A^-(v_2) + \mu_A^-(v_3) = 0.2 + 0.3 = 0.5$$
$$\deg_{\mu^+}(v_1) = \mu_A^+(v_2) + \mu_A^+(v_3) = 0.7 + 0.4 = 1.1$$
$$\deg(v_1) = [0.5, 1.1]$$



$\deg(v_2)=[\deg_{\mu^-}(v_2),\deg_{\mu^+}(v_2)]$

$\deg_{\mu^-}(v_2)=\mu_A^-(v_3)+\mu_A^-(v_1)=0.3+0.2=0.5$

$\deg_{\mu^+}(v_2)=\mu_A^+(v_3)+\mu_A^+(v_1)=0.4+0.6=1.0$

$\deg(v_2)=[0.5,1.0]$

$\deg(v_3)=[\deg_{\mu^-}(v_3),\deg_{\mu^+}(v_3)]$

$\deg_{\mu^-}(v_3)=\mu_A^-(v_1)+\mu_A^-(v_2)=0.2+0.2=0.5$

$\deg_{\mu^+}(v_3)=\mu_A^+(v_1)+\mu_A^+(v_2)=0.6+0.7=1.3$

$\deg(v_3)=[0.4,1.3]$

$\deg(v_1)\neq\deg(v_2)\neq\deg(v_3)$.

Thus we have G is an irregular interval-valued fuzzy graph.

Definition 4.2.3:

Let G be an interval-valued fuzzy graph. If there is a vertex which is adjacent to vertices with distinct closed neighbourhood degrees, then G is called a **totally irregular interval-valued fuzzy graph**.

Example 4.2.4: Consider an interval-valued fuzzy graph G such that $V=\{v_1,v_2,v_3,v_4,v_5\}$, $E=\{v_1v_2,v_2v_3,v_2v_4,v_3v_1,v_3v_4,v_4v_1,v_4v_5\}$.

$\deg[v_1]=[\deg_{\mu^-}[v_1],\deg_{\mu^+}[v_1]]$

$\deg_{\mu^-}[v_1]=\deg_{\mu^-}(v_1)+\mu_A^-(v_1)$

$\deg_{\mu^-}(v_1)=\mu_A^-(v_2)+\mu_A^-(v_3)+\mu_A^-(v_4)$

= 0.3+0.3+0.4=1.0

$\deg_{\mu^+}[v_1]=\deg_{\mu^+}(v_1)+\mu_A^+(v_1)$

$\deg_{\mu^+}(v_1)=\mu_A^+(v_2)+\mu_A^+(v_3)+\mu_A^+(v_4)$

= 0.5+0.7+0.6=1.8

$\deg_{\mu^-}[v_1]=1.0+0.4=1.4$

$\deg_{\mu^+}[v_1]=1.8+0.6=2.4$

$\deg[v_1]=[1.4,2.4]$

$\deg[v_2] = [\deg_{\mu^-}[v_2], \deg_{\mu^+}[v_2]]$

$\deg_{\mu^-}[v_2] = \deg_{\mu^-}(v_2) + \mu_A^-(v_2)$ $\deg_{\mu^+}[v_2] = \deg_{\mu^+}(v_2) + \mu_A^+(v_2)$

$\deg_{\mu^-}(v_2) = \mu_A^-(v_1) + \mu_A^-(v_3) + \mu_A^-(v_4)$ $\deg_{\mu^+}(v_2) = \mu_A^+(v_1) + \mu_A^+(v_3) + \mu_A^+(v_4)$

$= 0.4+0.3+0.4=1.1$ $= 0.6+0.7+0.6=1.9$

$\deg_{\mu^-}[v_2] = 1.1 + 0.3 = 1.4$ $\deg_{\mu^+}[v_2] = 1.9 + 0.5 = 2.4$

$\deg[v_2] = [1.4, 2.4]$

$\deg[v_3] = [\deg_{\mu^-}[v_3], \deg_{\mu^+}[v_3]]$

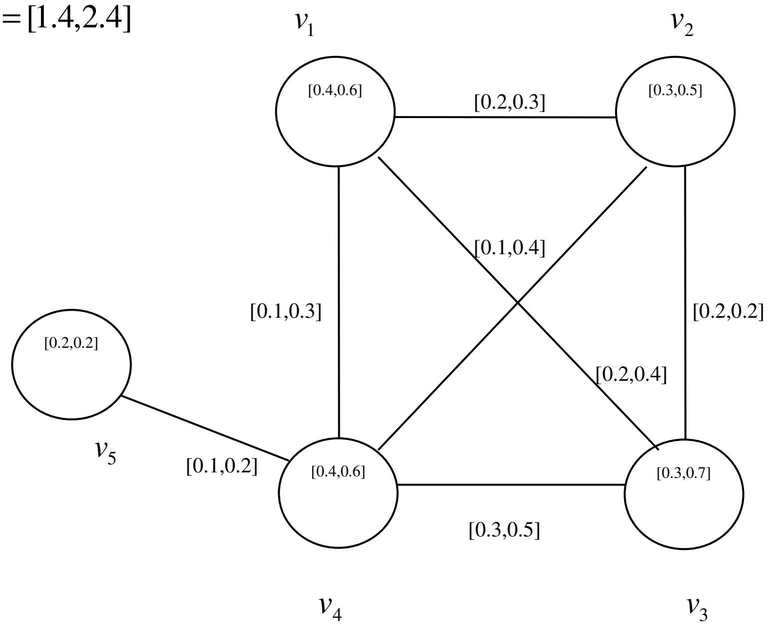
$\deg_{\mu^-}[v_3] = \deg_{\mu^-}(v_3) + \mu_A^-(v_3)$ $\deg_{\mu^+}[v_3] = \deg_{\mu^+}(v_3) + \mu_A^+(v_3)$

$\deg_{\mu^-}(v_3) = \mu_A^-(v_1) + \mu_A^-(v_2) + \mu_A^-(v_4)$ $\deg_{\mu^+}(v_3) = \mu_A^+(v_1) + \mu_A^+(v_2) + \mu_A^+(v_4)$

$= 0.4+0.3+0.4=1.1$ $= 0.6+0.5+0.6=1.7$

$\deg_{\mu^-}[v_3] = 1.1 + 0.3 = 1.4$ $\deg_{\mu^+}[v_3] = 1.7 + 0.7 = 2.4$

$\deg[v_3] = [1.4, 2.4]$



$\deg[v_4] = [\deg_{\mu^-}[v_4], \deg_{\mu^+}[v_4]]$ $\deg[v_5] = [\deg_{\mu^-}[v_5], \deg_{\mu^+}[v_5]]$

$\deg_{\mu^-}[v_4] = \deg_{\mu^-}(v_4) + \mu_A^-(v_4)$ $\deg_{\mu^-}[v_5] = \deg_{\mu^-}(v_5) + \mu_A^-(v_5)$

$\deg_{\mu^-}(v_4) = \mu_A^-(v_1) + \mu_A^-(v_2) + \mu_A^-(v_3) + \mu_A^-(v_5)$ $\deg_{\mu^-}(v_5) = \mu_A^-(v_4) = 0.5$

$= 0.4+0.3+0.3+0.2=1.2$

$\deg_{\mu^-}[v_4] = 1.2 + 0.4 = 1.6$

$\deg_{\mu^-}[v_5] = 0.4 + 0.2 = 0.6$

$\deg[v_4] = [\deg_{\mu^-}[v_4], \deg_{\mu^+}[v_4]]$

$\deg_{\mu^+}[v_4] = \deg_{\mu^+}(v_4) + \mu_A^+(v_4)$

$\deg_{\mu^+}(v_4) = \mu_A^+(v_1) + \mu_A^+(v_2) + \mu_A^+(v_3) + \mu_A^+(v_5)$

$= 0.6+0.5+0.7+0.2=2.0$

$\deg_{\mu^-}[v_4] = 2.0 + 0.6 = 2.6$

$\deg[v_4] = [1.6, 2.6]$

$\deg[v_5] = [\deg_{\mu^-}[v_5], \deg_{\mu^+}[v_5]]$

$\deg_{\mu^+}[v_5] = \deg_{\mu^+}(v_5) + \mu_A^+(v_5)$

$\deg_{\mu^+}(v_5) = \mu_A^+(v_4)$

$= 0.6$

$\deg_{\mu^-}[v_5] = 0.6 + 0.2 = 0.8$

$\deg[v_5] = [0.6, 0.8]$

Thus it is clear that G is a totally irregular interval-valued fuzzy graph.

Definition 4.2.5:

A connected interval-valued fuzzy graph G is said to be **neighbourly irregular** if every two adjacent vertices of G have distinct open neighbourhood degree.

Example 4.2.6:

Consider an interval-valued fuzzy graph G such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$.

$\deg(v_1) = [\deg_{\mu^-}(v_1), \deg_{\mu^+}(v_1)]$

$\deg_{\mu^-}(v_1) = \mu_A^-(v_2) + \mu_A^-(v_4)$

$= 0.3+0.5=0.8$

$\deg_{\mu^+}(v_1) = \mu_A^+(v_2) + \mu_A^+(v_4)$

$=0.7+0.5=1.2$

$\deg(v_1) = [0.8, 1.2]$

$\deg(v_2) = [\deg_{\mu^-}(v_2), \deg_{\mu^+}(v_2)]$

$\deg_{\mu^-}(v_2) = \mu_A^-(v_1) + \mu_A^-(v_3)$

$= 0.2+0.4=0.6$

$\deg_{\mu^+}(v_2) = \mu_A^+(v_1) + \mu_A^+(v_3)$

$= 0.6+0.4=1.0$

$\deg(v_2) = [0.6, 1.0]$

$\deg(v_3)=[\deg_{\mu^-}(v_3),\deg_{\mu^+}(v_3)]$ $\quad\quad$ $\deg(v_4)=[\deg_{\mu^-}(v_4),\deg_{\mu^+}(v_4)]$

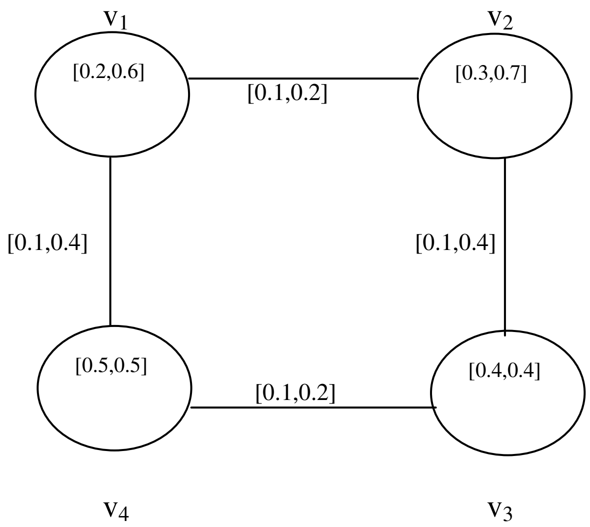
$\deg_{\mu^-}(v_3)=\mu_A^-(v_2)+\mu_A^-(v_4)$ $\quad\quad$ $\deg_{\mu^-}(v_4)=\mu_A^-(v_1)+\mu_A^-(v_3)$

$= 0.3+0.5=0.8$ $\quad\quad$ $= 0.2+0.4=0.6$

$\deg_{\mu^+}(v_3)=\mu_A^+(v_2)+\mu_A^+(v_4)$ $\quad\quad$ $\deg_{\mu^+}(v_4)=\mu_A^+(v_1)+\mu_A^+(v_3)$

$=0.7+0.5=1.2$ $\quad\quad$ $= 0.6+0.4=1.0$

$\deg(v_3)=[0.8,1.2]$ $\quad\quad$ $\deg(v_4)=[0.6,1.0]$



Thus, we have $\deg(v_1)=[0.8,1.2], \deg(v_2)=[0.6,1.0]$, $\deg(v_3)=[0.8,1.2]$, and $\deg(v_4)=[0.6,1.0]$ Hence, G is neighbourly irregular.

Definition 4.2.7:

A connected interval-valued fuzzy graph G is said to be **neighbourly totally irregular** if every two adjacent vertices of G have distinct closed neighbourhood degree.

Example 4.2.8:

Consider an interval-valued fuzzy graph G such that $V=\{v_1,v_2,v_3,v_4\}$, $E=\{v_1v_2,v_2v_3,v_3v_4,v_4v_1\}$.

$\deg[v_1]=[\deg_{\mu^-}[v_1],\deg_{\mu^+}[v_1]]$

$\deg_{\mu^-}[v_1]=\deg_{\mu^-}(v_1)+\mu_A^-(v_1)$ $\qquad$ $\deg_{\mu^+}[v_1]=\deg_{\mu^+}(v_1)+\mu_A^+(v_1)$

$\deg_{\mu^-}(v_1)=\mu_A^-(v_2)+\mu_A^-(v_4)$ $\qquad$ $\deg_{\mu^+}(v_1)=\mu_A^+(v_2)+\mu_A^+(v_4)$

$=0.4+0.4=0.8$ $\qquad$ $=0.5+0.5=1.0$

$\deg_{\mu^-}[v_1]=0.8+0.3=1.1$ $\qquad$ $\deg_{\mu^+}[v_1]=1.0+0.6=1.6$

$\deg[v_1]=[1.1,1.6]$

$\deg[v_2]=[\deg_{\mu^-}[v_2],\deg_{\mu^+}[v_2]]$

$\deg_{\mu^-}[v_2]=\deg_{\mu^-}(v_2)+\mu_A^-(v_2)$ $\qquad$ $\deg_{\mu^+}[v_2]=\deg_{\mu^+}(v_2)+\mu_A^+(v_2)$

$\deg_{\mu^-}(v_2)=\mu_A^-(v_1)+\mu_A^-(v_3)$ $\qquad$ $\deg_{\mu^+}(v_2)=\mu_A^+(v_1)+\mu_A^+(v_3)$

$=0.3+0.2=0.5$ $\qquad$ $=0.6+0.7=1.3$

$\deg_{\mu^-}[v_2]=0.5+0.4=0.9$ $\qquad$ $\deg_{\mu^+}[v_2]=1.3+0.5=1.8$

$\deg[v_2]=[0.9,1.8]$

$\deg[v_3]=[\deg_{\mu^-}[v_3],\deg_{\mu^+}[v_3]]$

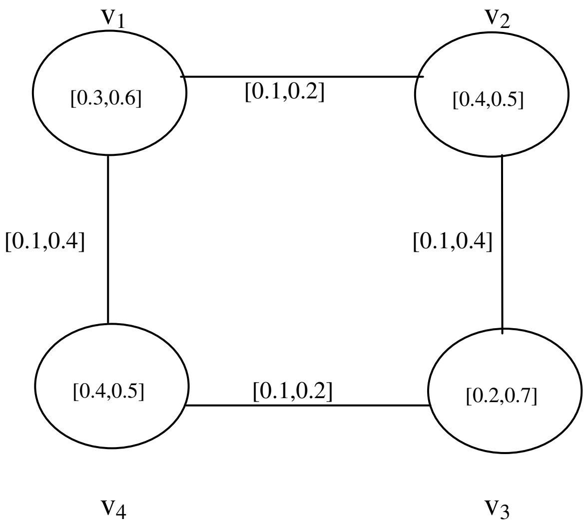
$\deg_{\mu^-}[v_3]=\deg_{\mu^-}(v_3)+\mu_A^-(v_3)$ $\qquad$ $\deg_{\mu^+}[v_3]=\deg_{\mu^+}(v_3)+\mu_A^+(v_3)$

$\deg_{\mu^-}(v_3)=\mu_A^-(v_2)+\mu_A^-(v_4)$ $\qquad$ $\deg_{\mu^+}(v_3)=\mu_A^+(v_2)+\mu_A^+(v_4)$

$=0.4+0.4=0.8$ $\qquad$ $=0.5+0.5=1.0$

$\deg_{\mu^-}[v_3]=0.8+0.2=1.0$ $\qquad$ $\deg_{\mu^+}[v_3]=1.0+0.7=1.7$

$\deg[v_3]=[1.0,1.7]$



$\deg[v_4] = [\deg_{\mu^-}[v_4], \deg_{\mu^+}[v_4]]$

$\deg_{\mu^-}[v_4] = \deg_{\mu^-}(v_4) + \mu_A^-(v_4)$ $\quad$ $\deg_{\mu^+}[v_4] = \deg_{\mu^+}(v_4) + \mu_A^+(v_4)$

$\deg_{\mu^-}(v_4) = \mu_A^-(v_1) + \mu_A^-(v_3)$ $\quad$ $\deg_{\mu^+}(v_4) = \mu_A^+(v_1) + \mu_A^+(v_3)$

$= 0.3+0.2=0.5$ $\quad$ $= 0.6+0.7=1.3$

$\deg_{\mu^-}[v_4] = 0.5 + 0.4 = 0.9$ $\quad$ $\deg_{\mu^+}[v_4] = 1.3 + 0.5 = 1.8$

$\deg[v_4] = [0.9,1.8]$

Thus, we have $\deg[v_1] = [1.1,1.6]$, $\deg[v_2] = [0.9,1.8]$, $\deg[v_3] = [1.0,1.7]$, $\deg[v_4] = [0.9,1.8]$.

Hence is neighbourly totally irregular.

Definition 4.2.9:

Let G be a connected interval-valued fuzzy graph. G is called **highly irregular** if every vertex of G is adjacent to vertices with distinct neighbourhood degrees.

Remark 4.2.10:

A neighbourly irregular interval-valued fuzzy graph may not be highly irregular.

Theorem 4.2.11:

An interval-valued fuzzy graph G is highly irregular and neighbourly irregular interval-valued fuzzy graph if and only if the neighbourhood degrees of all the vertices of G are distinct.

Proof:

Let G be an interval-valued fuzzy graph with $n-$vertices $v_1, v_2, ...v_n$. Assume that G is highly irregular and neighbourly irregular.

Claim 1: The neighbourhood degrees of all vertices of G are distinct. Let $\deg(v_i) = [k_i, l_i], i = 1,2,....., n.$ Let the adjacent vertices of v_1 be $v_2, v_3, ...v_n$ with neighbourhood degrees $[k_2, l_2], [k_3, l_3], ..., [k_n, l_n],$ respectively. Then, $k_2 \neq k_3 \neq .... \neq k_n$ and $l_2 \neq l_3 \neq .... \neq l_n$, since G is highly irregular. Also $k_1 \neq k_2 \neq k_3 \neq .... \neq k_n$ and $l_1 \neq l_2 \neq l_3 \neq .... \neq l_n$, since G is neighbourly irregular. Hence, the neighbourhood degree of all the vertices of G is distinct.

Conversely, assume that the neighbourhood degrees of all the vertices of G distinct.

Claim 2: G is highly irregular and neighbourly irregular interval-valued fuzzy graph. Let $\deg(v_i) = [k_i, l_i], i = 1,2,...n.$ Given that $k_1 \neq k_2 \neq k_3 \neq .... \neq k_n$ and $l_1 \neq l_2 \neq l_3 \neq .... \neq l_n$, which implies that every two adjacent vertices have distinct neighbourhood degrees and to every vertex, the adjacent vertices have distinct neighbourhood degrees.

Theorem 4.2.12:

An interval-valued fuzzy graph G of G^*, where G^* is a cycle with 3 vertices that is neighbourly irregular and highly irregular if and only if the lower and upper membership values of the vertices between every pair of vertices are all distinct.

Proof:

Assume that lower and upper membership values of the vertices are all distinct.

Claim 1: G is neighbourly irregular and highly irregular interval-valued fuzzy graph. Let $v_i, v_j, v_k \in V$. Given that $\mu_A^-(v_i) \neq \mu_A^-(v_j) \neq \mu_A^-(v_k)$ and $\mu_A^+(v_i) \neq \mu_A^+(v_j) \neq \mu_A^+(v_k)$, which implies that $\sum_{x\in N(x)} \mu_A^-(v_i) \neq \sum_{x\in N(x)} \mu_A^-(v_j) \neq \sum_{x\in N(x)} \mu_A^-(v_k)$ and

$\sum_{x\in N(x)} \mu_A^+(v_i) \neq \sum_{x\in N(x)} \mu_A^+(v_j) \neq \sum_{x\in N(x)} \mu_A^+(v_k)$. That is,

$\deg(v_i) \neq \deg(v_j) \neq \deg(v_k)$. Hence, G is neighbourly irregular and highly irregular.

Conversely, assume that G is neighbourly irregular and highly irregular.

Claim 2: Lower and upper membership values of the vertices are all distinct.

Let $\deg(v_i) = [k_i, l_i], i = 1,2,....n.$ Suppose that lower and upper membership value of any two vertices are the same. Let $v_1, v_2 \in V$. Let $\mu_A^-(v_1) = \mu_A^-(v_2)$ and $\mu_A^+(v_1) = \mu_A^+(v_2)$ Then , $\deg(v_1) = \deg(v_2)$, since G^* is cycle, which is a contradiction to the fact that G is neighbourly irregular and highly irregular interval-valued fuzzy graph. Hence, lower membership and upper membership value of the vertices are all distinct.

Remark 4.2.13:

A complete interval-valued fuzzy graph may not be neighbourly irregular.

Remark 4.2.14:

A neighbourly total irregular interval-valued fuzzy graph may not be neighbourly irregular.

Proposition 4.2.15:

If an interval-valued fuzzy graph G is neighbourly irregular and $[\mu_A^-, \mu_A^+]$ is a constant function, then it is neighbourly totally irregular.

Proof:

Assume that G is a neighbourly irregular interval-valued fuzzy graph. Then, the open neighbourhood degrees of every two adjacent vertices are distinct. Let $v_1, v_j \in V$ be adjacent vertices with distinct open neighbourhood degrees $[k_1, l_1]$ and $[k_2, l_2]$, where $k_1 \neq k_2, l_1 \neq l_2$. Let us assume that $(\mu_1(v_i), \nu_1(v_i)) = (\mu_1(v_j), \nu_1(v_j)) = [c_1, c_2]$, where c_1, c_2 are constant and $c_1, c_2 \in [0,1]$. Therefore, $\deg_{\mu^-}[v_i] = \deg_{\mu^-}(v_i) + \mu_1(v_i) = k_1 + c_1$, $\deg_{\mu^+}[v_i] = \deg_{\mu^+}(v_i) + \nu_1(v_i) = l_1 + c_2$, $\deg_{\mu^-}[v_j] = \deg_{\mu^-}(v_j) + \mu_1(v_j) = k_2 + c_1$, and $\deg_{\mu^+}[v_j] = \deg_{\mu^+}(v_j) + \nu_1(v_j) = l_2 + c_2$.

Claim: Consider that $\deg_{\mu^-}[v_i] \neq \deg_{\mu^-}[v_j]$ and $\deg_{\mu^+}[v_i] \neq \deg_{\mu^+}[v_j]$. Suppose that, $\deg_{\mu^-}[v_i] = \deg_{\mu^-}[v_j]$ and $\deg_{\mu^+}[v_i] = \deg_{\mu^+}[v_j]$. Consider that

$$\deg_{\mu^-}[v_i] = \deg_{\mu^-}[v_j],$$

$$k_1 + c_1 = k_2 + c_1,$$

$$k_1 - k_2 = c_1 - c_1 = 0,$$

$k_1 = k_2$, which is a contradiction to $k_1 \neq k_2$.

Therefore, $\deg_{\mu^-}[v_i] \neq \deg_{\mu^-}[v_j]$. Similarly, we consider that

$$\deg_{\mu^+}[v_i] = \deg_{\mu^+}[v_j],$$

$$l_1 + c_2 = l_2 + c_2,$$

$$l_1 - l_2 = c_2 - c_2 = 0,$$

$l_1 = l_2$, which is a contradiction to $l_1 \neq l_2$.

Therefore, $\deg_{\mu^-}[v_i] \neq \deg_{\mu^-}[v_j]$ .Hence, G is a neighbourly totally irregular interval-valued fuzzy graph.

Theorem 4.2.16:

If an interval-valued fuzzy graph G is neighbourly totally irregular and $[\mu_A^-, \mu_A^+]$ is a constant function, then it is a neighbourly irregular interval-valued fuzzy graph.

Proof:

Assume that G is a neighbourly totally irregular interval-valued fuzzy graph. Then, the closed neighbourhood degrees of every two adjacent vertices are distinct. Let $v_1, v_j \in V$ and $\deg[v_i] = [k_1, l_1], \deg[v_j] = [k_2, l_2]$ where $k_1 \neq k_2$ and $l_1 \neq l_2$. Assume that $(\mu_1(v_i), \nu_1(v_i)) = [c_1, c_2]$ and $(\mu_1(v_j), \nu_1(v_j)) = [c_1, c_2]$, where $c_1, c_2 \in [0,1]$ are constant and $\deg[v_i] \neq \deg[v_j]$.

Claim: Consider that $\deg[v_i] \neq \deg[v_j]$.

Given that $\deg[v_i] \neq \deg[v_j]$ which implies $\deg_{\mu^-}[v_i] \neq \deg_{\mu^-}[v_j]$ and $\deg_{\mu^+}[v_i] \neq \deg_{\mu^+}[v_j]$, now, we consider that

$$\deg_{\mu^-}[v_i] \neq \deg_{\mu^-}[v_j],$$

$$k_1 + c_1 \neq k_2 + c_1,$$

$$k_1 \neq k_2.$$

We now consider that

$$\deg_{\mu^+}[v_i] \neq \deg_{\mu^+}[v_j],$$

$$l_1 + c_2 = l_2 + c_2,$$

$$l_1 \neq l_2.$$

That is, the neighbourhood degrees of adjacent vertices of G are distinct. Hence, neighbourhood degree of every pair of adjacent vertices is distinct in G.

Proposition 4.2.17:

If an interval-valued fuzzy graph G is neighbourly irregular and neighbourly totally irregular, then $[\mu_A^-, \mu_A^+]$ need not be a constant function.

Remark 4.2.18:

If G is a neighbourly irregular interval-valued fuzzy graph, then interval-valued subgraph $H = (A', B')$ of G may not be neighbourly irregular.

Remark 4.2.19:

If G is a totally irregular interval-valued fuzzy graph, then interval-valued fuzzy subgraph $H = (A', B')$ of G may not be totally irregular.

SECTION 4.3

RELATIONSHIP BETWEEN INTUITIONISTIC FUZZY GRAPHS AND INTERVAL-VALUED FUZZY GRAPHS

Definition 4.3.1:(Shannon,1994)

By an **intuitionistic fuzzy graph** (IFG, for short) G of a graph G^*, we mean a pair $G = (A, B)$, where $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set on V and $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy relation on E such that

$\mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)),\ \ \mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)),$ for all $xy \in E$

Remark 4.3.2:

The class of all IFGs on G^* will be denoted by I F G (G^*).

Theorem 4.3.3:(Parvathi,2009)

If G_1 and G_2 are intuitionistic fuzzy graphs, then $G_1 \cap G_2, G_1 \cup G_2$ and $\overline{G_1}$ are intuitionistic fuzzy graphs.

Remark 4.3.4:

The class of all Interval-Valued Fuzzy Graphs(IVFG, for short) will be denoted by $I\mathcal{V}\mathcal{F}\mathcal{G}(G^*)$.

Theorem 4.3.5:

If G_1 and G_2 are intuitionistic fuzzy graphs, then $G_1 \cap G_2, G_1 \cup G_2$ and $\overline{G_1}$ are intuitionistic fuzzy graphs.

Lemma 4.3.5:

($I\mathcal{V}\mathcal{F}\mathcal{G}(G^*)$,∪,∩) and ($I\mathcal{F}\mathcal{G}(G^*)$,∪,∩) are complete lattices.

Theorem 4.3.6:

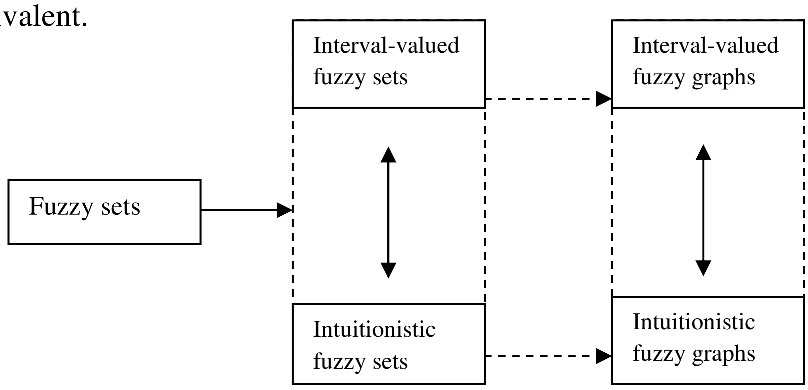
The mapping $\psi_1 : \text{IVFG}(G^*) \rightarrow \text{IFG}(G^*)$ defined by

$$\psi_1(B) = (\{(x, \mu_A^-(x), 1-\mu_A^+(x)) | x \in V\}, \{(xy, \mu_B^-(xy), 1-\mu_B^+(xy)) | xy \in E\}),$$

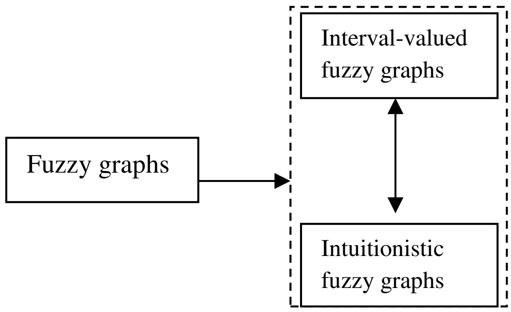
where $B = (\{(x, \mu_A^-(x), \mu_A^+(x)) | x \in V\}, \{(xy, \mu_B^-(xy), \mu_B^+(xy)) | xy \in E\})$, is an isomorphism between lattices $(\text{IVFG}(G^*), \cup, \cap)$ and $(\text{IFG}(G^*), \cup, \cap)$.

Remark 4.3.6:

From a pure mathematical point of view, the above theorem shows that the two concepts intuitionistic fuzzy graphs and interval-valued fuzzy graphs are equivalent.



Fig(a):Links between models



Fig(b):Link between IFGs & IVFGs

In the above figures, we present the relationships that exists between different models. In these figures, a double arrow between two theorems means that they are equivalent, a single arrow $X \rightarrow Y$ denotes that model Y is an extension of X. In figure (a), a dash arrow X denotes that the model Y is based on the previous model X.

Summary and Conclusion

SUMMARY AND CONCLUSION

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. In many real world problems, we get partial information about that problem. So there is vagueness in the description of the objects or in its relationships or in both. To describe this type of relation, we need to design graph model with fission of type 1 fuzzy set. This fission of fuzzy set with graph is known as fuzzy graph. Interval-valued fuzzy sets provide more adequate description of uncertainty than traditional fuzzy sets.

In this thesis Interval-valued fuzzy graph, Interval-valued fuzzy complete graphs, certain types of Interval-valued fuzzy graphs, Isomorphism on Interval-valued fuzzy graph, relationship between Intuitionistic fuzzy graphs and Interval-valued fuzzy graphs are studied in detail. Certain operations like Cartesian product, composition, union, join are defined on Interval-valued fuzzy graphs and some of their properties are investigated.

The notion of complete interval-valued fuzzy graph is brought out. Some interesting properties of self-complementary and self-weak complementary interval-valued fuzzy complete graphs are discussed. Three new operations namely direct product, semi strong product and strong product are also provided.

Types of interval-valued fuzzy graphs including the balanced interval-valued fuzzy graphs, neighbourly irregular interval-valued fuzzy graphs, neighbourly total irregular interval-valued fuzzy graphs, highly irregular interval-valued fuzzy graphs, and highly total irregular interval-valued fuzzy graphs are proposed. Some interesting properties associated with these new interval-valued fuzzy graphs are investigated. The relationship between the Intuitionistic fuzzy graph and Interval-valued fuzzy graph is also discussed in this chapter. The study can be extended to Intuitionistic fuzzy graph, Neutrosophic fuzzy graph, Bipolar fuzzy graphs further.

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