Chapter VIII

CHAPTER VIII

FUZZY TOPOLOGICAL ORDERED VECTOR SPACES

Definition: 8.1

Let E be a vector space over $\mathbb{K}$, where $\mathbb{K}$ is the space of either the real or the complex numbers. If μ, ρ are fuzzy sets in E, then the fuzzy sets $\mu \oplus \rho$ and $\mu \Theta \rho$ are defined by

$$\mu \oplus \rho\ (x) = \sup\{\mu(x_1) \wedge \rho(x_2);\ x = x_1 + x_2\},$$

$$\mu \Theta \rho\ (x) = \sup\{\mu(x_1) \wedge \rho(x_2);\ x = x_1 - x_2\}.$$

Also, for $t \in \mathbb{K}$ and μ a fuzzy set in E, the fuzzy set $t\mu$ is defined as follows. If $t \neq 0$ then $t\mu(x) = 0$ if $x \neq 0$ and $t\mu(0) = \sup_{y \in E} \mu(y)$. For $x \in E$ and μ a fuzzy set $x \oplus \mu$ is defined by $x \oplus \mu(y) = \mu(y-x)$.

Definition: 8.2

A fuzzy set μ in E is called

(1) **convex** if $t\mu \oplus (1-t)\, \mu \leq \mu$ for all $0 \leq t \leq 1$;

(2) **balanced** if $t\mu \leq$ for $|t| \leq 1$;

(3) **absolutely convex** if it is convex and balanced;

(4) **absorbing** if $\sup\{\mu(tx) : t > 0\} = 1$, for all $x \in E$.

Definition: 8.3

A fuzzy set μ, in a vector space E, **absorbs** a fuzzy set ρ if $\mu(0) > 0$ and for any $\theta < \mu(0)$ there exists $t > 0$ such that $\theta \wedge (t\, \rho) \leq \mu$.

Definition: 8.4

An absolutely convex absorbing fuzzy set ρ in E, is called a **fuzzy semi-norm**. If in addition, inf { $\rho(tx)$ } = 0 for $x \neq 0$, then ρ is called a **fuzzy norm**.

Definition: 8.5

A **fuzzy linear topology** on a vector space E over $\mathbb{K}$ is a fuzzy topology (containing all constant fuzzy sets) such that the two mappings

$+ : E \times E \rightarrow E, (x, y) \rightarrow x+y,$

$\bullet : \mathbb{K} \times E \rightarrow E, (t, x) \rightarrow tx,$

are continuous where $\mathbb{K}$ is equipped with the fuzzy topology generated by the usual topology of $\mathbb{K}$ and $\mathbb{K} \times E$, $E \times E$ have the corresponding product fuzzy topologies.

A Vector space E, with a fuzzy linear topology, is called a **fuzzy topological vector space**.

Definition: 8.6

A fuzzy set μ, in a fuzzy topological vector space E, is called **bounded** if it is absorbed by every neighbourhood of zero.

Notation: 8.7

We denote by $\mathbb{R}$, the field of real numbers.

Definition: 8.8

A vector space E over $\mathbb{K}$ {$\mathbb{K} = \mathbb{R}$) is said to be an **ordered vector space** if an order relation has been given on it such that the following conditions hold:

1) if $x, y \in E$ and $x \leq y$, then $x+z \leq y+z$ for any $z \in E$;

2) if $x, y \in E$ and $x \leq y$, then $tx \leq ty$,for any $t \in \mathbb{R}$, $t \geq 0$

From condition (i) it follows that if $x \leq y$ and $z \leq w$, then $x+z \leq y+z$.

Definition: 8.9

A fuzzy set μ, on an ordered vector space E is called

1) **positive increasing** (resp. decreasing) if $0 \leq x \leq y$ implies $\mu(0) \leq \mu(x) \leq \mu(y)$. (resp. $\mu(0) \geq \mu(x) \geq \mu(y)$);

2) **positive order-convex** (resp. order-concave) if $0 \leq x \leq y$ implies $\mu(x) \geq \min\{\mu(0), \mu(y)\}$ (resp. $\mu(x) \leq \max\{\mu(0), \mu(y)$);

3) **absolutely increasing** (resp. **decreasing**) if $\mu(-x) \leq \mu(x)$ (resp. $\mu(-x) \geq \mu(x)$);

4) **Absolutely order-convex** (resp. **order-concave**) if, for $x \geq 0$, $\mu(0) \geq \min\{\mu(-x), \mu(x)\}$ (resp. $\mu(0) \leq \max\{\mu(-x), \mu(x)\}$).

Definition: 8.10

A fuzzy set μ, on an ordered vector space E, is called **co-convex** if it is both convex and order-convex.

Lemma: 8.11

Let μ_1 be an absolutely decreasing fuzzy set and μ_2 be an absolutely increasing fuzzy set in an ordered vector space E. Then the fuzzy set $\mu = \mu_1 \wedge \mu_2$ (resp. $\mu = \mu_1 \vee \mu_2$) is absolutely order-convex (resp. order-concave)

Proof

Let $x \geq 0$. Since μ_1 is absolutely decreasing and μ_2 is absolutely increasing, then $\mu_1(-x) \geq \mu(0) \geq \mu(x)$ and $\mu_2(-x) \leq \mu(0) \leq \mu(x)$; hence it follows that $\mu(0) = \max\{ \mu_1(0), \mu_2(0) \} \leq \max\{ \mu_1(x), \mu_2(-x) \}$.

But $\mu_1 \leq \mu$ and $\mu_2 \leq \mu$, so that $\mu(0) \leq \max\{\mu(x), \mu(-x)\}$. i.e. μ is absolutely order-concave. By duality the order-convex case can be proved.

Theorem: 8.12

Let $\{\mu_\alpha : \alpha \in J\}$ be an indexed family of fuzzy sets, in an ordered vector space E, and let $\mu = \sup_{\alpha \in J}\{\mu\alpha\}$. Then if each μ_α is absolutely increasing (resp. decreasing), then μ is absolutely increasing (resp. decreasing). Also, if each μ_α is absolutely order-convex, the same is true for the fuzzy set μ.

Theorem: 8.13

Let μ and ρ be increasing (resp. decreasing) fuzzy sets on an ordered vector space E. Then $\mu \oplus \rho$ and $\mu \ \Theta \ \rho$ are increasing (resp. decreasing).

Proof

Let μ and ρ be increasing fuzzy sets on an ordered vector space E. Then, for $x_1 \leq y_1$ and $x_2 \leq y_2$, $\mu(x_1) \leq \mu(y_1)$ and $\rho(x_2) \leq \rho(y_2)$ and it follows that

$$\begin{aligned} \mu \oplus \rho(x) &= \sup\{\mu(x_1), \rho(x_2) : x = x_1 + x_2\} \\ &\leq \sup\{\mu(y_1), \rho(y_2) : y = y_1 + y_2\} \\ &\leq \mu \oplus \rho(y). \end{aligned}$$

Then $\mu \oplus \rho$ is increasing. By duality the decreasing case can be proved and so $\mu \ \Theta \ \rho$.

Lemma: 8.14

If μ is an increasing (resp. decreasing) fuzzy set on an ordered vector space E, then $x \oplus \rho$ is increasing (resp. decreasing).

Proof

Let $y \leq z$, which implies $y-x \leq z-x$ for $x, y \in z \in E$. Then for an increasing fuzzy set μ in E. We have $\mu(y-x) \leq \mu(z-x)$, which implies $x \oplus \mu(y) \leq x \oplus \mu(z)$. The decreasing case can be similarly proved.

Theorem: 8.15

Let μ, ρ be a pair of order-convex fuzzy sets on an ordered vector space E. Then $\mu \oplus \rho$ and $\mu \Theta \rho$ are order-convex fuzzy sets.

Proof

$$\mu(y_1) \geq \min\{\mu(x_1), \mu(z_1) : x_1 \leq y_1 \leq z_1\} \text{ and}$$

$$\rho(y_2) \geq \min\{\rho(x_2), \rho(z_2) : x_2 \leq y_2 \leq z_2\}.$$

Since $x_1+x_2 \leq y_1+y_2 \leq z_1+z_2$, then

$$\begin{aligned} \mu \oplus \rho\ (y) = &\ \sup\{\mu(y_1) \wedge \rho(y_2) : y = y_1 + y_2\} \\ \geq &\ \sup\{\min\{\min\{\mu(x_1), \mu(z_1)\}, \min\{\rho(x_2), \rho(z_2)\} : \\ &\ x_1+x_2 \leq y_1+y_2 \leq z_1+z_2\}\} \\ \geq &\ \sup\{\min\{\min\{\mu(x_1), \rho(x_2)\}, \min\{\mu(z_1), \rho(z_2)\} : \\ &\ x_1+x_2 \leq y_1+y_2 \leq z_1+z_2\}\} \\ \geq &\ \min\{\sup\{\mu(x_1) \wedge \rho(x_2)\}, \sup\{\mu(z_1) \wedge \rho(z_2)\} : \\ &\ x_1+x_2 \leq y_1+y_2 \leq z_1+z_2\} \\ \geq &\ \min\{\mu\oplus\rho(x), \mu\oplus\rho(z) : x \leq y \leq z\} \end{aligned}$$

Then $\mu \oplus \rho$ is an order-convex fuzzy set in E and the same is true for $\mu\Theta\rho$.

Lemma: 8.16

If μ be an order-convex fuzzy set on an ordered vector space E. Then fuzzy set $X \oplus \mu$ is order-convex.

Proof

Let μ be an order-convex fuzzy set in E. Then, for $x_1 \leq x_2 \leq x_3$.

$$\mu(x_2) \geq \min\{\mu(x_1), \mu(x_3)\}$$

But $x_1 - x \leq x_2 - x \leq x_3 - x$, so that

$$\mu(x_2 - x) \geq \min\{\mu(x_1 - x), \mu(x_3 - x) : x_1 - x \leq x_2 - x \leq x_3 - x\}.$$

Hence

$$X \oplus \mu(x_2) \geq \min\{x \oplus \mu(x_1), x \oplus \mu(x_3) : x_1 \leq x_2 \leq x_3\}.$$

i.e. $x \oplus \mu$ is order-convex.

Theorem: 8.16

Let μ be a fuzzy set on an ordered vector space E. If μ is increasing (resp. decreasing, order-convex) then so is the fuzzy set $t\mu$ for $t > 0$, $t \in \mathbb{R}$.

Remark: 8.17

It is clear that the constant fuzzy sets are increasing, decreasing and order convex. Also if μ is increasing then $1-\mu$ is decreasing.

Theorem: 8.18

The mappings d, i : $I^E \to I^E$ satisfy the closure axioms; namely, we have, for $\mu, \rho \in I^E$,

(C1) $d(0) = 0$,

(C2) $\mu \leq d(\mu)$,

(C3) $d(d(\mu)) = d(\mu)$,

(C4) $d(\mu V \rho) = d(\mu) V d(\rho)$.

Proof

(C1) and (C2) follow trivially from the definitions.

(C3) follows from

$$\begin{aligned} d(d(\mu))(x) &= \sup\{d(\mu)(y) : y \geq x\} \\ &= \sup\{\sup\{\mu(z) : z \geq y\} : y \geq x\} \\ &= \sup\{\mu(z) : z \geq x\} \\ &= d(\mu)(x) \end{aligned}$$

(C4): Trivially $d(\mu V \rho) \leq d(\mu) V d(\rho)$. For the converse

$$\begin{aligned} d(\mu) V d(\rho)(x) &= \{\sup\{\mu(y) : y \geq x\} V \{\sup\{\rho(y) : y \geq x\}\} \\ &\leq \sup\{\mu V \rho(y) : y \geq x\} \\ &= d(\mu V \rho)(x). \end{aligned}$$

Theorem: 8.19

Let μ, ρ be fuzzy sets on an ordered vector space E. Then, for each $x \in E$, we have

(1) $i(\mu \oplus \rho) = i(\mu) \oplus i(\rho)$;

(2) $d(\mu \oplus \rho) = d(\mu) \oplus d(\rho)$;

(3) $c(\mu \oplus \rho) = c(\mu) \oplus c(\rho)$

And the same is true for $\mu \oplus \rho$.

Proof

For (i),

$i(\mu \oplus \rho)(x) = \sup\{\mu \oplus \rho(y) : y \leq x\}$

$$= \sup\{\sup\{\mu(y_1) \wedge \rho(y_2) : y = y_1 + y_2\} : y \leq x\}$$

$$= \sup\{\min\{\sup\{\mu(y_1) : y_1 \leq x_1\}, \sup\{\rho(y_2) : y_2 \leq x_2\}\}\}$$

$$= \sup\{i(\mu)\,(x_1) \wedge i(\rho)\,(x_2) : y_1 \leq x_1,\ y_2 \leq x_2,\ y_1+y_2 \leq x_1+x_2 = x\}$$

$$= i(\mu) \oplus i(\rho)(x).$$

(2) Can be, analogously, proved.

As to (3),

$$c(\mu\oplus\rho)(y) = \sup\{\min\{\mu\oplus\rho(x), \mu\oplus\rho(z)\} : x \leq y \leq z\}$$

$$= \sup\{\min\{\sup\{\mu(x_1) \wedge \rho(x_2) : x = x_1 + x_2\}, \sup\{\mu(z_1) \wedge \rho(z_2) : z = z_1 + z_2\}\}\}.$$

$$= \sup\{\sup\{\mu(x_1) \wedge \rho(x_2) \wedge \mu(z_1) \wedge \rho(z_2) : x = x_1 + x_2,\ z = z_1 + z_2\} : x \leq y \leq z\}$$

$$= \sup\{\sup\{\mu(x_1) \wedge \mu(z_1)\} \wedge \{\rho(x_2) \wedge \rho(z_2)\} : x_1 + x_2 \leq y_1 + y_2 \leq z_1 + z_2\}\}$$

$$= \sup\{\sup\{\mu(x_1) \wedge \mu(z_1) : x_1 \leq y_1 \leq z_1\} \wedge \sup\{\rho(x_2) \wedge \rho(z_2) : x_2 \leq y_2 \leq z_2\}$$

$$= \sup\{c(\mu)\,(y_1) \wedge c(\rho)\,(y_2)\} : y = y_1 + y_2\}$$

$$= c(\mu) \oplus c(\rho)\,(y)$$

Theorem: 8.20

Let μ be a fuzzy sets on an ordered vector space E. Then for each x, $y \in E$, we have

(1) $i(x \oplus \rho)(y) = x \oplus i(\mu)\,(y);$

(2) $d(x \oplus \rho)(y) = x \oplus d(\mu)\,(y);$

(3) $c(x \oplus \rho)(y) = x \oplus c(\mu)\,(y);$

Proof

$$
\begin{aligned}
i(x \oplus \mu)(y) &= \sup\{x \oplus \mu(z) : z \leq y\} \\
&= \sup\{\mu(z-x) : z-x \leq y-x\} \\
&= i(\mu)(y-x) \\
&= x \oplus i(\mu)(y).
\end{aligned}
$$

(2) is immediate analogously.

As to (3),

$$
\begin{aligned}
c(x\oplus\mu)(y) &= \sup\{\min(x\oplus\mu)(y_1), (x\oplus\mu)(y_2)\} : y_1 \leq y \leq y_2\} \\
&= \sup\{\min\{\mu(y_1-x), \mu(y_2-x)\}: y_1 - x \leq y-x \leq y_2-x\} \\
&= c(\mu)(y-x) \\
&= x \oplus c(\mu)(y).
\end{aligned}
$$

Theorem: 8.21

Two ordered vector spaces E_1 and E_2 are said to be isomorphic if there exists a one-to-one mapping f of E_1 onto E_2 such that the following conditions satisfied:

(1) for any $x,y \in E$ and numbers $t_1, t_2 \in \mathbb{R}$ we have $f(t_1x + t_2y) = t_1 f(x) + t_2 f(y)$;

(2) If $x, y \in E_1$, then $x \leq y$ iff $f(x) \leq f(y)$.

Theorem: 8.22

Let E_1 and E_2 be an isomorphic pair of ordered vector spaces. Then:

(1) The inverse image of each increasing, decreasing and order-convex fuzzy set in E_2, is respectively, increasing, decreasing and order-convex.

(2) The image of each increasing, decreasing and order-convex fuzzy set in E_1, is respectively, increasing, decreasing and order-convex.

Definition: 8.23

A non-empty subset C of a vector space E is called a **cone** if $tC \subseteq C$ for all scalar $t > 0$, ($t \in \mathbb{R}$). Clearly, a cone C in E determines a transitive and reflexive relation '$\leq$' by

$$x \leq y \text{ if } y - x \in C.$$

Remark: 8.24

If E is an ordered vector space with '$\leq$' as an order relation compatible with the vector structure of E and if we define

$$C' = \{x \in E : x \geq 0\}.$$

Then C' is a cone in E and $\leq$ is exactly the vector ordering of E induced by C'.

Lemma: 8.25

Let μ be a fuzzy set on an ordered vector space E and C is a cone in E. Then the order-convex hull of μ is

$$c(\mu) = \min\{(C \oplus \mu), (-C \oplus \mu)\}$$

Proof

Let $\rho(y) = \min\{(C \oplus \mu)(y), (-C \oplus \mu)(y)\}$

Then

$$\rho(y) = \min\{ \sup_{x \in C} \{x \oplus \mu(y)\}, \sup_{x \in C} \{-x \oplus \mu(y)\}\}$$

$$= \sup\{\min\{\mu(y-x), \mu(y+x)\} : y-x \leq y \leq y+x\}$$

$$= c(\mu)(y).$$

Definition: 8.26

Any ordered vector space E which, at the same time, is a fuzzy topological vector space (with a fuzzy linear topology) is called a **fuzzy topological ordered vector space** if the following condition is satisfied: Given $\{x_\alpha\}$, $\{y_\alpha\}$ be two convergent nets in the fuzzy topological vector space E with $x_\alpha \to x$ and $y_\alpha \to y$ such that $x_\alpha \le y_\alpha$ for all $\alpha \in J$, then it is true that $x \le y$.

Theorem: 8.27

Let E be an ordered vector space which, at the same time, is a fuzzy topological vector space. In order that E be a fuzzy topological ordered vector space it is sufficient that a neighbourhood system consisting of order-convex neighbourhood of zero should exist.

Notation: 8.28

In an ordered vector space E, in which a fuzzy linear topology τ is also given, we shall say that a fuzzy set, which is bounded with respect to the fuzzy linear topology τ, is τ-bounded. We shall denote by $x = \tau\text{-limit}\{x_\alpha\}$ or $x_\alpha \xrightarrow{\tau} x$ the convergence with respect to the fuzzy linear topology τ. The pharses 'τ-closed fuzzy set' 'τ-continuous function', etc., have an obvious meaning.

Theorem: 8.29

Let $\{x_\alpha\}_{\alpha\in J}$, $\{y_\alpha\}_{\alpha\in J}$ and $\{z_\alpha\}_{\alpha\in J}$ be nets in a fuzzy topological ord4ered vector E such that $x_\alpha \le y_\alpha \le z_\alpha$, for any $\alpha \in J$. If $x_\alpha \xrightarrow{\tau} 0$ and $z_\alpha \xrightarrow{\tau} 0$, then $y_\alpha \xrightarrow{\tau} 0$.

Remark: 8.30

If μ is an order-convex neighbourhood of zero, then there exists an index $\alpha_0 \in J$ such that μ is a neighbourhood of both $\{x_\alpha\}$ and $\{z_\alpha\}$ for any $\alpha \le \alpha_0$. From $x_\alpha \le y_\alpha \le z_\alpha$ it follows that μ is a neighbourhood of y_α for any $\alpha \in J$.

Definition: 8.31

Let E be an ordered vector space which, at the same time, is a fuzzy topological vector space with the fuzzy linear topology τ. Let $\mathbb{B}$ be a set of all the τ-neighbourhoods of zero. The fuzzy topology on E, which has the system $\{\lambda : \lambda = c(\mu), \mu \in \mathbb{B}\}$ as a neighbourhood system of a neighbourhood of zero, is called a **locally order-convex fuzzy topology associated to** τ, and is denoted by $c(\tau)$.

Remark: 8.32

One can show that $c(\tau) \le \tau$, $c(c(\tau)) = c(\tau)$ and that from $\tau_1 \le \tau_2$ it follows that $c(\tau_1) \le c(\tau_2)$.

Definition: 8.33

A fuzzy topological ordered vector space, in which the fuzzy linear topology admits a neighbourhood system at zero consisting of order-convex fuzzy sets is called **locally order-convex**.

Theorem: 8.34

In a fuzzy topological ordered vector space, which is locally order-convex, there exists a neighbourhood system at zero consisting of balanced convex and order-convex fuzzy sets.

Proof

If μ is any neighbourhood of zero then there exists an order-convex neighbourhood $\mu_1 \leq \mu$ of zero. There exists also an absolutely convex neighbourhood μ_2 of zero, such that $\mu_2 \leq \mu_1$ and $\mu_2(0) = \mu_1(0)$. By putting $\rho = c(\mu_2)$, we get a neighbourhood of zero which is balanced, convex and order-convex fuzzy set, as one can easily check. But $\rho \leq \mu_1$, because μ_1 is order-convex. It follows that $\rho \leq \mu$ and the theorem is proved.

Theorem: 8.35

Let E be a fuzzy topological ordered vector space , with locally order- convex fuzzy linear topology τ . Let $\{ x_\alpha\}_{\alpha\in J}$, $\{ y_\alpha\}_{\alpha\in J}$ be two nets in E with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha\in J$.If $y_\alpha \xrightarrow{\tau} 0$, then $x_\alpha \xrightarrow{\tau} 0$.

Theorem: 8.36

Let E be a fuzzy topological ordered vector space (with τ as a fuzzy linear topology). If τ is locally order-convex, then the order-convex hull of each τ-bounded fuzzy set is in E is τ-bounded.

Proof

Let μ be a τ-bounded set in E, and let λ be a neighbourhood of zero. If the fuzzy linear topology τ is locally order-convex, then there exists an order-convex neighbourhood of $\lambda_1 \leq \lambda$ of zero. Since μ is absorbed by every neighbourhood of zero, then for every $\theta < \lambda_1(0)$ there exists $t > 0$ ($t\in \mathbb{R}$) such that $\theta \wedge (t\mu) < \lambda_1$. It follows that the order-convex hull $c(\theta \wedge (t\mu)) \leq c(\lambda_1) = \lambda_1$, which implies $\theta \wedge (tc(\mu)) \leq \lambda_1$, i.e., $c(\mu)$ is absorbed by λ_1 and so by λ , then $c(\mu)$ is τ-bounded fuzzy set in E.