CHAPTER - IV

CHAPTER IV

FUZZY SOFT SEPARATION AXIOMS AND FUZZY SOFT COMPACTNESS

SECTION 4.1

FUZZY SOFT SEPARATION AXIOMS

Definition 4.1.1

A fuzzy soft topological space (U, E, τ) is said to be a **fuzzy soft T_o-Space** if for every pair of disjoint fuzzy soft points $e(F_A)$, $e(G_B)$, there exist a fuzzy soft open set containing one but not the other.

Example 4.1.2

A discrete fuzzy soft topological space is a fuzzy soft T_o-Space since every fuzzy soft point $e(F_A)$ is a fuzzy soft open set in the discrete space.

Theorem 4.1.3

A fuzzy soft subspace of a fuzzy soft T_o-Space is fuzzy soft T_o.

Proof

Let (Y, E, T_Y) be a fuzzy soft subspace of a fuzzy soft T_o-Space (U, E, τ) and let $e(F_{1_A})$, $e(F_{2_A})$ be two distinct fuzzy soft points of Y_E. Then these fuzzy soft points are also in F_A implies there exists a fuzzy soft open set H_A containing one fuzzy soft point but not the other implies $G_A \widetilde{\cap} H_A$, where $H_A \in \tau$ is a fuzzy soft open set in T_Y containing one fuzzy soft point but not the other.

Definition 4.1.4

A fuzzy soft topological space (U, E, τ) is said to be a **fuzzy soft T_1-Space** if for distinct pair of fuzzy soft points $e(F_A)$, $e(G_A)$ of U_E there exists a fuzzy soft open sets S_A and H_A such that

$$e(G_A) \widetilde{\in} S_A \text{ and } e(G_A) \widetilde{\notin} H_A;$$

$$e(G_A) \widetilde{\in} H_A \text{ and } e(K_A) \widetilde{\notin} S_A.$$

Theorem 4.1.5

If every fuzzy soft point of a fuzzy soft topological space (U, E, τ) is fuzzy soft closed then (U, E, τ) is fuzzy soft T_1.

Proof

Let $e(H_A)=\{\, e_j = \{H^i_{\alpha_i} : i = 1,2 \dots n\}\}$,

$e(K_A) = \{\, e_m = \{H^i_{\beta_i} : i = 1,2 \dots n\}\}$, where e_j, e_m are distinct parameter be distinct parameters be distinct fuzzy soft point of F_A.

i. $\alpha_i, \beta_i \leq 0.5$

Then we can always find some γ_i and δ_i such that $\alpha_i \leq \gamma_i, \beta_i \leq \delta_i \Rightarrow \alpha_i \leq 1 - \gamma_i$, $\beta_i \leq 1 - \delta_i \Rightarrow$ the fuzzy soft sets

$e(L_A)=\{\, e_j = \{H^i_{\alpha_i} : i = 1,2 \dots n\}\}$,

$e(T_A) = \{\, e_m = \{H^i_{\beta_i} : i = 1,2 \dots n\}\}$ are such that their complements are disjoint fuzzy soft open sets containing $e(H_A)$ and $e(H_A)$ respectively.

ii. $\alpha_i, \beta_i > 0.5$.

Then we can always find some γ_i and δ_i such that $\gamma_i \leq \alpha_i, \delta_i \leq \beta_i \Rightarrow \alpha_i \leq 1 - \gamma_i$, $\beta_i \leq 1 - \delta_i \Rightarrow$the fuzzy soft sets.

$e(L_A)=\{\, e_j = \{H^i_{\alpha_i} : i = 1,2 \dots n\}\}$,

$e(T_A) = \{\, e_m = \{H^i_{\beta_i} : i = 1,2 \dots n\}\}$ are such that their complements are disjoint fuzzy soft open sets containing $e(H_A)$ and $e(H_A)$ respectively.

Theorem 4.1.6

A fuzzy soft subspace of a fuzzy soft T_1 -Space is fuzzy soft T_1.

Definition 4.1.7

A fuzzy soft topological space (U, E, τ) is to said to be a **fuzzy soft T_2–Space** if and only if for distinct fuzzy soft points $e(G_A)$, $e(K_A)$ of U_E, there exists a disjoint fuzzy soft open sets H_A and $S_A$such that $e(G_A) \tilde{\in} H_A$ and $e(K_A) \tilde{\in} S_A$.

Theorem 4.1.8

If every fuzzy soft point of a fuzzy soft topological space (U, E, τ) is fuzzy soft closed then (U, E, τ) is fuzzy soft T_2.

Theorem 4.1.9

A fuzzy soft subspace of a fuzzy soft T_2-Space is fuzzy soft T_2.

Theorem 4.1.10

A fuzzy soft topological space (U, E, τ) is fuzzy soft T_2 if and only if for distinct fuzzy soft points $e(G_A)$, $e(K_A)$ of F_A, there exist a fuzzy soft open set S_A containing $e(G_A)$ but not $e(K_A)$ such that $e(K_A) \tilde{\in} Cl(S_A)$.

Proof

Let (U, E, τ) be fuzzy soft T_2 and $e(G_A)$, $e(K_A)$ be distinct fuzzy soft points. So there exists a distinct fuzzy soft open sets H_A and B_A such that $e(K_A) \tilde{\in} H_A$, $e(G_A) \tilde{\in} B_A \Rightarrow e(G_A) \tilde{\in} H_A^C$. So H_A^C is a fuzzy soft open set containing $e(G_A)$ but not $e(K_A)$ and $Cl(H_A^C) = H_A^C$.

Conversely, take a pair of distinct fuzzy soft points $e(G_A)$ and $e(K_A)$ of F_A, there exists a fuzzy soft open set $S_A$containing $e(G_A)$ but not $e(K_A)$ such that $e(K_A) \tilde{\notin} Cl(S_A) \Rightarrow e(K_A) \tilde{\in} (Cl(S_A))^C \Rightarrow S_A$ and $(Cl(S_A))^C$ are disjoint fuzzy soft open set containing $e(G_A)$ and $e(K_A)$ respectively.

Definition 4.1.11

A fuzzy soft topological (U, E, τ) is said to be a **fuzzy soft regular space** if for every fuzzy soft point $e(H_A)$ and fuzzy soft closed set K_A not containing $e(H_A)$, there exist a disjoint fuzzy soft open sets G_{1_A}, G_{2_A} such that $e(H_A) \tilde{\in} G_A$ and $K_A \tilde{\subseteq} G_{2_A}$.

A fuzzy soft regular T_1 –space is called a fuzzy soft T_3 space.

Remark 4.1.12

It can be shown that the property of being fuzzy soft T_3 is hereditary.

Remark 4.1.13

Every fuzzy soft T_3-Space is fuzzy soft T_2-Space, every fuzzy soft T_2-Space is fuzzy soft T_1-Space and every fuzzy soft T_1-Space is fuzzy soft T_o-Space.

Theorem 4.1.14

A fuzzy soft topological space (U, E, τ) in which every fuzzy soft point is fuzzy soft closed, is fuzzy soft regular if and only if for a fuzzy soft open set G_A containing a fuzzy soft point $e(H_A)$, there exists a fuzzy soft open set S_A containing $e(H_A)$ such that $Cl(S_A) \tilde{\subseteq} G_A$.

Proof

Take a fuzzy soft open set G_A containing $e(H_A)$ in a regular fuzzy soft topological (U, E, τ). Then G_A^C is fuzzy soft closed. By hypothesis, there exists a disjoint fuzzy soft set S_A and W_A are disjoint, so $S_A \tilde{\subseteq} W_A^C \Rightarrow Cl(S_A) \tilde{\subseteq} W_A^C \Rightarrow Cl(S_A) \tilde{\subseteq} G_A$.

Conversely, assume the hypothesis. Take a fuzzy soft closed set K_A not containing a fuzzy soft point $e(H_A) \tilde{\notin} K_A$. Then K_A^C is a fuzzy soft open set containing $e(H_A) \Rightarrow$ there exists a fuzzy soft open set S_A containing $e(H_A)$ such that $Cl(S_A) \tilde{\subseteq} K_A^C \Rightarrow K_A \tilde{\subseteq} Cl(S_A)^C \Rightarrow Cl(S_A)^C$ is a fuzzy soft open set containing K_A and $S_A \tilde{\cap} Cl(S_A)^C = \phi$.

Definition 4.1.15

A fuzzy soft topological space (U, E, τ) is said to be **fuzzy soft normal space** if for every pair of disjoint fuzzy soft closed sets H_A and K_A, there exist a disjoint fuzzy soft open sets G_{1_A}, G_{2_A} such that $H_A \tilde{\subseteq} G_{1_A}$ and $K_A \tilde{\subseteq} G_{2_A}$.

A fuzzy soft normal T_1-Space is called a fuzzy soft T_4-Space.

Remark 4.1.16

Every fuzzy soft T_4-Space is fuzzy soft T_3.

Theorem 4.1.17

A fuzzy soft topological space (U, E, τ) is fuzzy soft normal if and only if for any fuzzy soft closed set H_A and fuzzy soft open set G_A containing H_A, there exist a fuzzy soft open set S_A such that $H_A \tilde{\subseteq} S_A$ and $Cl(S_A) \tilde{\subseteq} G_A$).

Proof

Let (U, E, τ) be fuzzy soft normal space and H_A be a fuzzy soft closed set and G_A be a fuzzy soft open set containing $H_A \implies H_A$ and G_A^C are disjoint fuzzy soft closed sets $\implies$ there exists a disjoint fuzzy soft open sets ${G_1}_A$, ${G_2}_A$such that $H_A \tilde{\subseteq} {G_1}_A$and $G_A^C \tilde{\subseteq} {G_2}_A$. Now ${G_2}_A \tilde{\subseteq} {{G_2}_A}^C \implies Cl({G_1}_A) \tilde{\subseteq} Cl({G_2}_A)^C = {{G_2}_A}^C$. Also, $G_A^C \tilde{\subseteq} {G_2}_A \implies {{G_2}_A}^C \tilde{\subseteq} G_A$.

Conversely, let L_A and K_A be any disjoint pair fuzzy soft closed set $\implies L_A \tilde{\subseteq} K_A^C$ then by hypothesis there exist a fuzzy soft open set S_A such that $L_A \tilde{\subseteq} S_A$ and $Cl(S_A) \tilde{\subseteq} K_A^C \implies K_A \tilde{\subseteq} Cl(S_A)^C \implies S_A$ and $Cl(S_A)^C$ are disjoint fuzzy soft open sets such that $L_A \tilde{\subseteq} S_A$ and $K_A \tilde{\subseteq} Cl(S_A)^C$.

Theorem 4.1.18

A fuzzy soft closed subspace of a fuzzy soft normal space fuzzy soft normal.

SECTION 4.2

FUZZY SOFT COMPACTNESS

Definition 4.2.1

A family ψ of fuzzy soft sets is a cover of a fuzzy soft set (F, A) if

$$(F, A) \subseteq \cup \{ ((F_i, A): (F_i, A) \in \psi, i \in I \}$$

It is a **fuzzy soft open cover** if each member of ψ is a fuzzy soft open set. A subcover of ψ is a subfamily of ψ which is also a cover.

Definition 4.2.2

Let (U, E, τ) be fuzzy soft topological space and $(F, A) \in FS(U, E)$. Fuzzy soft set (F, A) is called **compact** if each fuzzy soft open cover of (F, A) has a finite subcover. Also fuzzy soft topological space (U, E, τ) is called compact if each fuzzy soft open cover of $\tilde{1}_E$ has a finite subcover.

Result 4.2.3

A fuzzy soft topological space (U, E, τ) is compact if U is finite.

Result 4.2.4

Let (U, E, τ_1) and (V, E, τ_2) be two fuzzy soft topological spaces and $\tau_1 \tilde{\subseteq} \tau_2$. Then, fuzzy soft topological space (U, E, τ_1) is compact if (V, E, τ_2) is compact.

Theorem 4.2.5

Let (G, B) be a fuzzy soft closed set in fuzzy soft compact space (U, E, τ). Then (G, B) is also compact.

Proof

Let (F_i, A) be any open covering of(G, B). Then $\tilde{1}_U \subseteq (\cup_{i \in I} ((F_i, A) \cup (G, B)^C$, that is (F_i, A) together with fuzzy soft open set $(G, B)^C$ is a open covering of $\tilde{1}_U$. Therefore there exists a finite subcovering $(F_1, A), (F_2, A) \ldots (F_n, A), (G, B)^C$. Hence we obtain $\tilde{1}_U \subseteq (F_1, A) \cup (F_2, A) \cup \ldots \cup (F_n, A) \cup (G, B)^C$. Therefore, we get $(G, B) \subseteq (F_1, A) \cup (F_2, A) \cup \ldots \cup (F_n, A) \cup (G, B)^C$ which clearly implies $(G, B) \subseteq (F_1, A) \cup (F_2, A) \cup \ldots \cup$

(F_n, A) since $(G, B) \cap (G, B)^C = \phi$. Hence (G, B) has a finite subcovering and so is compact.

Definition 4.2.6

Let (U, E, τ) be a fuzzy soft topological space over (U, E) and $x, y \tilde{\in} (U, E)$ such that $x \neq y$. If there exists fuzzy soft sets open sets (F, A) and (G, A) such that $x \in (F, A)$, $y \in (G, A)$ and $(F, A) \cap (G, A) = \phi$, then (U, E, τ) is called a **fuzzy soft Hausdorff space.**

Theorem 4.2.7

Let (G, B) be a fuzzy soft compact set in fuzzy soft Hausdorff space (U, E, τ). Then (G, B) is closed.

Proof

Let $x \in (G, B)^C$. For each $y \in (G, B)$, we have $x \neq y$, so there are disjoint fuzzy soft open sets (F_y, A) and (H_y, A) so that $x \in (F_y, A)$ and $y \in (H_y, A)$. Then $\{ (H_y, A): y \in (G, B)\}$ is an fuzzy soft open cover of(G, B). Let $\{ (H_{y_1}, A), (H_{y_2}, A), \dots, (H_{y_n}, A)\}$ be a finite subcover. Then $\cap_{i=1}^{n}(F_y, A)$ is an open set containing x and contained in $(G, B)^C$. Thus $(G, B)^C$ is fuzzy soft open and (G, B) is fuzzy soft closed.

Theorem 4.2.8

Let (U, E, τ_1) and (V, E, τ_2) be fuzzy soft topological spaces and $(\varphi, \psi): (U, E, \tau_1) \to (V, E, \tau_2)$ continuous and onto fuzzy soft function. If (U, E, τ_1) is fuzzy soft compact then (V, E, τ_2) is fuzzy soft compact.

Proof

Let (F_i, A) be any open covering of $\tilde{1}_V$ i.e., $\tilde{1}_V \subseteq \cup_{i \in I}(F_i, A)$. Then $(\varphi, \psi)^{-1}(\tilde{1}_V) \subseteq (\varphi, \psi)^{-1}(\cup_{i \in I}(F_i, A))$ and $\tilde{1}_U \subseteq \cup_{i \in I}(\varphi, \psi)^{-1}(F_i, A)$. So $(\varphi, \psi)^{-1}((F_i, A))$ is an open covering of $\tilde{1}_U$. As (U, E, τ) is compact, there are 1,2,…n in I such that

$$\tilde{1}_U \subseteq (\varphi, \psi)^{-1}(F_1, A) \cup (\varphi, \psi)^{-1}(F_2, A) \cup \dots \cup (\varphi, \psi)^{-1}(F_n, A).$$

Since (φ, ψ) is surjective, we have

$$\tilde{1}_V = (\varphi, \psi)(\tilde{1}_U)$$
$$\subseteq (\varphi, \psi)((\varphi, \psi)^{-1}(F_1, A) \cup (\varphi, \psi)^{-1}(F_2, A) \cup \dots \cup (\varphi, \psi)^{-1}(F_n, A))$$

$= (\varphi, \psi)((\varphi, \psi)^{-1}(F_1, A)) \tilde{\cup}, \ldots, \tilde{\cup} (\varphi, \psi)((\varphi, \psi)^{-1}(F_n, A))$

$= (F_1, A) \cup (F_2, A) \cup \ldots \cup (F_n, A)$.

So we have $\tilde{1}_V \tilde{\subseteq} (F_1, A) \cup (F_2, A) \cup \ldots \cup (F_n, A)$, i.e., $\tilde{1}_V$ is covered by a finite number of (F_i, A).

Hence (V, E, τ_2) is compact.

Definition 4.2.9

Let (U, E, τ_1) and (V, E, τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $(\varphi, \psi): (U, E, \tau_1) \rightarrow (V, E, \tau_2)$ is called **fuzzy soft closed** if $(\varphi, \psi)(F, A)$ is fuzzy soft closed set in (V, E, τ_2), for all fuzzy soft closed set (F, A) in (U, E, τ_1).

Theorem 4.2.10

Let (U, E, τ_1) be a fuzzy soft topological spaces and (V, E, τ_2) be a fuzzy soft Hausdroff space. Fuzzy soft mapping (φ, ψ) is closed if fuzzy soft mapping $(\varphi, \psi): (U, E, \tau_1) \rightarrow (V, E, \tau_2)$ is continuous.

Proof

Let (G, B) be any fuzzy soft closed set in (U, E, τ_1). Therefore, we have (G, B) is compact. Since fuzzy soft mapping (φ, ψ) is continuous, fuzzy soft set $(\varphi, \psi)(G, B)$ is compact in (V, E, τ_2). As (V, E, τ_2) is fuzzy soft Hausdorff space, fuzzy soft set $(\varphi, \psi)(G, B)$ is closed. Then fuzzy soft mapping(φ, ψ) is closed.

Definition 4.2.11

A family ψ of fuzzy soft sets has the **finite intersection property** if the intersection of the members of each finite subfamily of ψ is not the null fuzzy soft set.

Theorem 4.2.12

A fuzzy soft topological space is compact if and only if each family of fuzzy soft closed sets with the finite intersection property has a nonnull intersection.

Proof

Let ψ be any family of fuzzy soft closed subset such that

$\cap \{ (F_i, A): (F_i, A) \in \psi, i \in I\} = \tilde{0}_E$. Consider $\Omega = \{(F_i, A)^C : (F_i, A) \in \psi, i \in I\}$. So Ω is a fuzzy soft open cover of $\tilde{1}_E$. As fuzzy soft topological space is compact, there exists a finite sub covering $(F_1, A)^C$, $(F_2, A)^C, \dots, (F_n, A)^C$Then $\cap_{i=1}^{n}(F_i, A) = \tilde{1}_E - \cup_{i=1}^{n}(F_i, A)^C = \tilde{1}_E - \tilde{1}_E = \tilde{0}_E$. Hence ψ cannot have finite intersection property.

Conversely assume that a fuzzy soft topological space is not compact. Then any fuzzy soft open cover of $\tilde{1}_E$ has not a finite sub cover. Let $\{(F_i, A): i \in I\}$ be fuzzy soft open cover of $\tilde{1}_E$. So $\cup_{i=1}^{n}(F_i, A) \neq \tilde{1}_E$. Therefore $\cap_{i=1}^{n}(F_i, A)^C \neq \tilde{0}_E$. Thus $\{(F_i, A)^C: i = 1,2, \dots, n\}$ have a intersection property. By using hypothesis
$\cap (F_i, A)^C \neq \tilde{0}_E$ and we have this is a contradiction. Thus the fuzzy soft topological space is compact.