Analysis of Queueing Models Using Simulation Technique

By

M. BHUVANA

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-43 IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY 1997

ANALYSIS OF QUEUEING MODELS USING SIMULATION TECHNIQUE

BY

M. BHUVANA

A Dissertation submitted to Avinashilingam Institute for HomeScience and

Higher Education for Women, (Deemed University)

Coimbatore - 641 043.

In partial fulfilment of the requirements

for the degree of

MASTER OF SCIENCE IN MATHEMATICS

MAY 1997.

Certified as Bonafide Research Work.

K. N. Meenakshi
Signature of the
Head of the Department.
9.5.97.

9/5/97
Signature of the
Dean of Science

9/5/97
Signature of the Guide.

ACKNOWLEDGEMENT

ACKNOWLEDGEMENT

The investigator is grateful to PADMASHREE COLONEL **Mrs. RAJAMMAL P. DEVADAS,** *M.A., M.Sc., Ph.D. (Ohio State), D.Sc.(Madras), Hon. D.H.L. (Oregon State), Hon. D.H.L. (Ohio State), Hon. D.Sc. (C.A. Agri., Kanpur), Chancellor, Avinashilingam Institute for HomeScience and Higher Education for Women (Deemed University), Coimbatore, for providing the opportunity to conduct her investigation in this esteemed university.*

The author expresses her gratitude to **Mrs. LAKSHMI SANTA RAJAGOPAL,** *M.S(Tennessee), Ph.D. (Madras), Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University), Coimbatore, for providing facilities to carry out the study.*

The author extends her deep sense of gratitude to **Mrs. SAROJA PRABHAKAR,** *M.A., Diploma in Education(Madras), Ph.D.(Mother Teresa), Registrar and* **Mrs. C.V.R. INDIRA** *M.Sc., Ph.D (Madras), Dean, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University), Coimbatore, for facilitating the conduct of the study.*

The author wishes to express her sincere gratitude to **Miss. K. N. MEENAKSHI,** *M.Sc., Ph.D., Professor and Head of the Department of Mathematics, Avinashilingam Institute for HomeScience and Higher*

Education for Women (Deemed University), Coimbatore, for the encouragement extended by her, during the entire course of the study.

The author wishes to place on record her indebtedness to **Mrs. S. MUTHULAKSHMI** *M.Sc., (Madras), M. Phil (Bharathiar)., Dip. H.Ed. (Madras), Selection Grade Lecturer, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women (Deemed University), Coimbatore, for her constant benevolent help, invaluable advice and illuminating guidance. She gratefully acknowledges her persistent efforts and affectionate endeavours.*

The author would like to extend her thanks to other members of the staff and also to her dear friends who were responsible for the good finish of this dissertation.

The author is greatly indebted to her loving Parents and Brother for being the motivating forces behind this dissertation and providing moral support in carrying out the work.

CONTENTS

CONTENTS

INTRODUCTION

INTRODUCTION

"Delay is the enemy of efficiency and

waiting is the enemy of utilisation"

Queueing theory attempts to find an optimal balance between delay and waiting. The theory of queues originated in 1909 with the publication of the theory of probabilities and telephone conversations by the Danish Mathematican A.K.Erlang (1878 - 1929). Queueing theory attempts to predict fluctuating demands and to provide adequate service which enables to minimize the delay and waiting.

A variety of mathematical models designed to solve various problems. In building these models assumptions are made with restricted situations. The occasions of the physical system or real life system is either too complex to build a mathematical model or to get an answer with reasonable accuracy.

The Queueing Models like other Mathematical models has its own set of requirements and assumptions. This Queueing system can be analysed analytically. However if the assumptions are not fulfilled, analytical methods are no longer applicable. Even if the mathematical model can be constructed, available quantitative techniques may not be amenable to solve the resulting model. In this juncture other means are resorted to analyse this system. One such means is "The Technique of Simulation". Simulation was originated in 1940 with the analysis of Nuclear Shielding problems by John Von

Neumann and Stanislaw Illam. Simulation has attained a substantial progress with the advent of computer.

Simulation is a quantitative technique that utilizes a computerized mathematical model inorder to represent actual decision making under the conditions of uncertainity for evaluating course of action based upon facts and assumptions.

Review of Literature

The original work in simulation is due to John Von Neumann and Stanislaw Ullam (1940) using Monte Carlo analysis in conjunction with the mathematical technique (Thierauf, R.J.).

Monte Carlo method for solving theoretical problems originated in early 1960's. This is based on estimating the output of a system through sampling. In this respect, many ideas were developed in conjunction with Monte Carlo for the application of simulation. These ideas include the use of random numbers to get samples and to estimate the desired result. (Taha, H.A ., 1987).

Gillett, B.E., (1989) as given analysis of queueing models involving sampling from a known probability distribution.

This dissertation brings out the application of simulation to queueing models.

There are several types of simulation but we will restrict our attention to the simulation of mathematical models of system in which the variables involved are subject to random variation. Many system can be modelled as queueing system. But simulation of only a single server queueing system and multiserver queueing system with Poisson and NonPoisson inputs are considered.

SYNOPSIS

This dissertation traces the development of the available queueing models and various methods of simulation. It describes the application of simulation in solving problems relating to waiting line.

The contents of the dissertation have been divided into four chapters. In Chapter I, general concepts essential to develop the material are presented. In Chapter II, section 1 gives types of simulation and Section 2 includes the various method of simulation and its application in general. In Chapter III, Section 1 deals with various queueing models and Section 2 gives the simulated model of single server and multiserver queues based on distribution. In the same section MonteCarlo approach to single server queueing model is described. Chapter IV deals with the application of simulation for

1) Optimum Allocation of Doctor's to outpatient clinic.
2) Determination of parameters to a Shipping problem.

CHAPTER - I

CHAPTER - I

BASIC CONCEPTS

Terms used in queueing model.

The following terms are mainly used in developing the queueing model.

Customer

The arriving unit at a station for service. The customer may be either person, machine, vehicle or other items.

Service Station (or) Channel

Point where service is provided.

Waiting time

It is a time a customer spends in the queue before being serviced.

Queue

Stands for a number of customer waiting to be serviced. The basic elements of queueing model depend on the following factors.

a) Arrival distribution.

b) Service time distribution.

c) Design of service facility.

d) Queue Discipline. (FCFS, LCFS)

Entity

An object of interest in the system.

Attribute

A property of an entity.

Activity

Any process that causes a change in the system.

Event

The start or completion of an activity; the occurrence of a change at a point in time.

Poisson Distribution

Poisson distribution is defined as

$$P(r) = \frac{e^{-m} m^{r}}{r!} \quad \text{Where } r = 0, 1, 2, 3, 4.......$$

m = mean of the Poisson distribution

Normal distribution

Normal distribution is defined as

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{\frac{-(x-\mu)}{2\sigma^2}} \quad \text{where } -\infty < x < \infty$$

where x = values of the continous random variable

μ = mean of the normal distribution

σ^2 = Variance of the normal distribution

Random Number

A Random number is a sequence of numbers whose probability of occurrence is the same as that of any other number in the sequence. Random numbers can be generated by 4 methods.

I) Manual generation

II) Random Number table in the memory of the computer.

III) Computer Methods.

IV) Recursive Equation.

Classification of queues

Queueing model is specified in the following symbolic form

(a/b/c) ; (d/e)

The first and second symbol denote the types of distributions of interarrival and interservice times respectively.

c - specifies the number of servers

d - capacity of the system

e - queue discipline.

Also we designate,

M ≡ Poisson arrival or departure distribution

E_k ≡ Erlangian or Gamma interarrival for service time distribution

GI ≡ General input distribution

G ≡ General service time distribution

CHAPTER - II

CHAPTER - II

METHODS OF SIMULATION

In this Chapter, the general concepts of simulation technique are presented. Section 1 gives the three types of simulation and Section 2 presents the various methods of simulation and general applications of simulation.

Simulation has made substantial progress with the development of digital computers. Recent advances in simulation methodologies, software availability and technical developments have made simulation one of the most widely used and popularly accepted tool in Operations Research and System Analysis.

Simulation is a method of solving decision making problems by designing constructing and manipulating a model of the real system. It is useful in solving problems where all the values of the variable are not known or partly known in the advance. Simulation studies are valuable and convenient method for breaking down a complicated system into subsystems This type of simulation allows to gain increased knowledge of the operating system. It also allows to observe a cause and effect relationship of the system and its subsystem.

It is possible to study the behaviour of the system inorder to determine the major parameters governing it and setforth recommendations which will improve the overall performance of the system.

Simulation models are handy for analysing proposed system in which information is sketched at best. The effect of using the simulation model can be observed without actually using it in the real situation.

Advantages of simulation

Data for further analysis can be generated from a simulation model.

Simulation models are easier to apply than pure analytical methods.

Simulation allows for time compression and simulation can be used for pedagogical methods.

Simulation tends to be relatively straight forward. In large scale and complex systems it is evident that simulation leads to flexibility, comprehensiveness and frequent.

The model of the system once when constructed may be employed as often as desired to analyse different situations.

By experimenting with the real system a number of times will be costly and sometimes results may be inappropriate.

SECTION 1

Categories of Simulation

The behaviour of systems is analysed as a function of time under the following categories.

(a) Analogue Model.

(b) Discrete Model.

(c) Continous Model.

Analogue Model

The physical (original) system is replaced by a model using an analogy which is easier to test the ability for manipulation. for example Vibration phenomenon in Mechanics is represented by an analogue electrical system.

Continous Model

The simulated system is monitored at every point of time. Observations of the variations of different characteristics with time is the object of this type of simulation model. for example Flow of liquid in a pipeline.

Discrete Model

The simulated system is looked only at selected points of time.

In discrete systems, the statistics of the situation can change only when certain events take place. And it is collected by jumping from one point to another on the time scale. For example, A waiting line in which customers either join a queue or enter service and then leave the service facility after the service is completed.

SECTION 2

In this section, methods and application of simulation are presented.

Methods of Simulation

Available methods of simulation are,

Monte Carlo Simulation

System Simulation and

Computer Simulation .

Monte Carlo Simulation

The term Monte Carlo was coined by Von Neumann. Inorder to design nuclear shields, various materials were tested based upon penetration power of Neutrons. They simulated the experiment using random numbers because of the difficulty of solving analytically. This technique was given the code name Monte Carlo.

Monte Carlo method is applicable under the following situations.

(i) When we deal with a long sequence of probability determined steps where we can write only mathematical formulas but not the equation.

(ii) Where physical experimentation is impracticable.

(iii)The creation of an exact formula is impossible.

(iv)Monte Carlo involves determining the probability distribution of the variables under consideration and then sampling from this distribution by means of random numbers to obtain data.

A set of random numbers generates a set of values that have the same distributional characteristics as the real population.

Monte Carlo simulation yields a solution which is closed to the optimal solution though not the exact solution.

Procedure of Monte Carlo Method

(1) Determination of the probability distribution.

(2) Construction of the cumulative probability distribution.

(3) Assignment of Monte Carlo members to the cumulative probability distribution.

(4) Selection of a random sample from the Monte Carlo distribution.

(5) Determination of simulated values of the actual random variable.

System Simulation Method

System simulation is a process in which real data is used in this analysis of a complex problem which is processed through a model which reproduces the operating experiment. The simulation model allows an analysis of the system response to alternative management actions, providing a sound basis for decision.

This method generally draws samples from a real population. No theoretical counter part of the actual population is used in this system.

Simulation model makes use of a mathematical or logical model which can be analytically solved to assist an individual in reaching a decision. For example, we can take the condition of an uncontrollable input to a system whose probability distribution is known and can be handled analytically but whose sequential pattern cannot be expressed for an analytical solution.

A system simulation differs from Monte Carlo in two aspects namely need for data which is not found in random tables and the use of a mathematical model.

Computer Simulation

Computer simulation emerge as one of the techniques most frequently employed by staff analysts in production and operations functions today. Performing a computer simulation is not usually mathematically complex, but it may be very time consuming.

Procedure of Computer Simulation

(i) Define the problem under consideration its nature and scope.

(ii) Build a mathematical model of the problem.

(iii)Write a computer program of the model and summary procedures

(iv)Process the program on the computer.

(v) Evaluate the results of the computer simulation, modify parameter values, and return the program until a range of parameter values has been evaluated.

Applications of simulation in various fields

(1) The operations at a large airport.

(2) The intensity of radiation.

(3) Optimum maintenance operation.

(4) Analysis of Assembly lines.

(5) Improvement of Budgetary process.

(6) Optimum Allocation of warehouses.

CHAPTER - III

CHAPTER - III

ANALYSIS OF QUEUEING MODELS

In this Chapter, the techniques of simulation with special emphasize to queueing theory is provided. Section 1 is considered for basic model with the assumption and parameters of various Queueing models. Section 2 highlights the application of simulation techniques to queueing models.

The primary motivation for simulating queueing system is the inability to generate meaningful analytic solutions for complex queueing structures.

Any queueing system for which data on arrival or service times are available can be simulated. Because most queueing problems involves determination of the optimal members of facilities which on many occasions can be estimated fairly. Smaller queueing systems can be hand simulated.

SECTION 1

In this section various available queueing systems and their related models are presented.

M /M /1 Queueing System

This system deals with the process in which arrivals and departures occur randomly. Arrivals are considered as births, since when an arrival occurs, when the system is in the state E_n it is changed to E_{n+1} and the departure occuring while the system in state E_n sends the system down to E_{n-1}, is called as deaths. This type of process is called as birth-death process

Model I (M /M /1) : (∞/ F IFO).

This model deals with a queueing system with

1) Single service channel
2) Poisson Input
3) Exponential Service
4) Infinite capacity
5) First in first out

$P_n(t)$ denote the probability of n customers in the system at time (t).

Then we have

$P_n(t+\Delta t) = P_n(t)$. P[no arrival in Δt]. P[no service completion in Δt)

+ $P_{n-1}(t)$ P[one service completion in Δt]. P[no arrival in Δt]

+ $P_{n+1}(t)$ P[one service completion in Δt]. P[one arrival in Δt]

+ $P_{n-1}(t)$ P[one arrival in Δt]. P[no service in Δt]

$+O(\Delta t), \quad n \geq 1.$

$P_n(t+\Delta t) = P_n(t)\,[1-\lambda\Delta t - \mu\,\Delta t] + P_{n+1}(t)\,[\mu\Delta t] + P_{n+1}(t)\,[\lambda\,\Delta t] + 0(\Delta t)\,,\ n \geq 1.$

When the system has no arrival but one service completed in Δt we have

$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_1(t)\,\mu\,\Delta t + 0(\Delta t)$

Steady state solutions are

$-(\lambda+\mu)\,P_n + \mu P_{n+1} + \lambda P_{n-1} = 0$ if $n \geq 1$.

$-\lambda\,P_0 + \mu\,P_1 = 0$

Finally we have

$$P_n = (\lambda/\mu)^n \left(1 - \lambda/\mu\right)$$

$$= \rho^n (1-\rho) ; \qquad \rho = \lambda/\mu < 1 \text{ and } n \geq 0.$$

This expression is the probability distribution of queue length.

i) Probability of queue size being greater than or equal to n is given by

$$(P \geq n) = \sum_{k=n}^{\infty} P_k = \rho^n$$

ii) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{\infty} nP_n = \frac{1}{1-\rho}$$

iii) Average queue length is given by

$$E(m) = \sum_{m=0}^{\infty} mP_n$$

$$= \frac{\rho^2}{1-\rho}$$

iv) Average length of nonempty queue is given by

$$E(m \,/\, m > 0) = \frac{\mu}{\mu - \lambda}$$

Let $P_n(w)$ denote the probability that the waiting time lies between W and W+dw when there are n customer in the system.

i) Average waiting time of an arrival is given by

$$E(w) = \frac{\rho}{\mu(1-\rho)}.$$

ii) Average waiting time of an arrival who has to wait is given by

$$E(w / w > 0) = \frac{1}{\mu(1-\rho)}$$

iii) Average waiting time that an arrival spends in the system is given by

$$E(v) = \frac{1}{\mu(1-\rho)}$$

Model II (M/ M/ 1) : (∞/ SIRO)

Service discipline is SIRO-service in Random order

$$P_n = (1-\rho)\, \rho^n, \qquad n \geq 0$$

Average Number of customers remains the same as in (M/M/1):(∞/FIFO)

Model III (M/ M/ 1) : (N/ FIFO)

Here the maximum number of customers in the system is limited to N.

Difference equations is given as

$$P_N(t+\Delta t) = P_N(t)\,[1-\mu\,\Delta t] + P_{N-1}(t)\,[\lambda\Delta t]\,[1-\mu\Delta t] + 0(\Delta t)$$

We obtain the steady state solution as

$$P_n = (1-\rho)\, \rho^n$$

i) Average Number of customers in the system is given by

$$E(n) = \sum_{n=0}^{N} nP_n$$

$$= \frac{P_0\, \rho\left[1 - (N+1)\rho^N + N\rho^{N+1}\right]}{(1-\rho)\left(1-\rho^{N+1}\right)}$$

ii) Average queue length is given by

$$E(m) = \sum_{n=1}^{N}(n-1)P_n$$
$$= \frac{\rho^2\left[1 - N\rho^{N-1} + (N-1)\rho^N\right]}{(1-\rho)\left(1-\rho^{N+1}\right)}$$

iii) Average waiting time in the queue is given by the relation

$$E(w) = E(v) - 1/\mu$$
$$= \{E(m)\}/\lambda^1$$

where λ^1-meanrate of customers entering the system.

Model IV (Birth death process).

Assume the system is in E_n. Then the difference equation is given by

$$P_n(t+\Delta t) = P_n(t)\{1-(\lambda_n+\mu_n)\Delta t\} + \mu_{n+1}P_{n+1}(t)\,\Delta t + \lambda_{n-1}P_{n-1}(t)(\Delta t) + 0(\Delta t). \qquad n \geq 1$$

and

$$P_0(t+\Delta t) = P_0(t)\{1-\lambda_0\Delta t - O(\Delta t)\} + P_1(t)\{\mu_1\Delta t + 0(\Delta t)\}\{1-\lambda_1\,\Delta t + 0(\Delta t)\} + 0(\Delta t) \qquad n=0$$

Difference differential equation is given by

$$-(\lambda_n+\mu_n)P_n + \mu_{n+1}P_{n+1} + \lambda_{n-1}P_{n-1} = 0 \qquad n \geq 1$$

$$-\lambda_0 P_0 + \mu_1 P_1 = 0$$

In general we have the formula

$$P_n = P_0 \prod_{i=0}^{n-1} \frac{\lambda i}{\mu_{i+1}}, \qquad n \geq 1.$$

Special cases

(i) When $\lambda_n = \lambda$, for $n \geq 0$

and $\mu_n = \mu$ for $n > 1$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n\right]^{-1} = 1-\rho.$$

Therefore $P_n = \rho^n(1-\rho)$ for $n \geq 0$.

(ii) When $\lambda_n = \dfrac{\lambda}{n+1}$ for $n \geq 0$ and $\mu_n = \mu$ for $n > 1$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!\,\mu^n}\right] = e^{-\rho}$$

$$P_n = \left[\frac{1}{n!}\rho^n\right] e^{-\rho} \text{ for } n \geq 0.$$

(iii) when $\lambda_n = \lambda$ for $n \geq 0$ and $\mu_n = \mu$ for $n > 1$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!\,\mu^n}\right]^{-1} = e^{-\rho}$$

$$P_n = \left[\frac{1}{n!}\rho^n\right] e^{-\rho} \qquad \text{for } n \geq 0$$

$M/M/c$ *Queueing system*

This queueing system deals with queue which are served by parallel service channels(c) in which each server has an identically and independently distributed exponential service and the arrival is poisson.

Model V (M/ M/ c) : (∞/ FIFO).

We consider here C service channels. Input is poisson and service is exponential, hence it is an Birth-Death process.

Mean arrival rate is $\lambda_n = \lambda$ for all n.

C servers will remain busy if there are more than C customers with mean rate μ is given by Cμ. n of C servers will be busy if there are fewer than C customers with mean service rate nμ.

$$\mu_n = \begin{cases} n\mu & 1 \le n < C \\ c\mu & n \ge C \end{cases}$$

$$P_n = \frac{\lambda^n P_0}{n!\mu^n} \quad ; \quad 1 \le n < C$$

$$= \frac{\lambda^n P_0}{C^{n-C} C! \mu^n} \quad ; \quad n \ge C$$

i) Probability that an arrival has to wait is given by

$$P(n \ge C) = \sum_{n=C}^{\infty} P_n$$

$$= \frac{(\lambda/\mu)^C}{C!(C\mu - \lambda)} P_0$$

ii) Probability that an arrival enters the service without wait is given by

$$1 - P(n \ge C) = 1 - \frac{C(\lambda/\mu)^C}{C!(C - \lambda/\mu)} P_0$$

iii) Average Queue length is given by

$$E(m) = \sum_{n=c}^{\infty}(n-C)P_n$$

$$= \frac{\lambda\mu(\lambda/\mu)^C P_0}{(C-1)!\,(C\mu-\lambda)^2}$$

iv) Average number of customers in the system is given by

$$E(n) = \lambda/\mu + E(m)$$

$$= \frac{(\lambda\mu)(\lambda/\mu)^C P_0}{(C-1)!(C\mu-\lambda)^2} + \frac{\lambda}{\mu}$$

v) Average waiting time of an arrival is given by

$$E(w) = \frac{1}{\lambda}E(m) = \frac{\mu(\lambda/\mu)^C . P_0}{(C-1)!(C\mu-\lambda)^2}$$

vi) Average waiting time an arrival spends in the system is given by

$$E(v) = E(w) + \frac{1}{\mu} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^C P_0}{(C-1)!(C\mu-\lambda)^2} + \frac{1}{\mu}$$

Model VI. (M/ M/ C) : (N/ FIFO)

Maximum number in the system is limited to N where N>C.

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \qquad ;\ 0 \le n < C$$

$$= \frac{1}{C^{n-C}C!}\left(\frac{\lambda}{\mu}\right)^n P_0 \qquad ;\ C \le n \le N$$

i) Average queue length is given by

$$E(m) = \sum_{n=c}^{N}(n - C)P_n$$

$$= \frac{P_0(C\rho)^C \rho}{C!(1-\rho)^2}\left[1 - \rho^{N-C+1} - (1-\rho)(N-C+1)\rho^{N-C}\right]$$

ii) Average Number of customers in the system is given by

$$E(n) = \sum_{n=0}^{N} nP_n$$

$$= E(m) + C - P_0 \sum_{n=0}^{C-1} \frac{(C-n)(\rho C)^n}{n!}$$

iii) Average waiting time in the system is

$$E(v) = \frac{[E(n)]}{\lambda^1} \text{ where } \lambda^1 = \lambda(1 - P_N)$$

Average waiting time in the queue is

$$E(w) = \frac{[E(m)]}{\lambda^1}$$

Model VII (M/ M/ C) : (C / FIFO)

The result obtained here is same as model VI except with N=C. No waiting queue is allowed.

$$P_n = \begin{cases} \frac{1}{n!}(\lambda/\mu)^n P_0 & ; \ 0 \le n \le c \\ 0 & \text{otherwise} \end{cases}$$

$$\text{where } P_0 = \left[\sum_{n=0}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

Model VIII (Power Supply Model)

Consider an electric circuit which supplies power to 'a' customers.

If there are n customers in the queue, then

$$\lambda_n = (a-n)\lambda \quad 0 \leq n \leq a$$

$$\mu_n = n\mu.$$

$$P_n = \binom{a}{n}\left(\frac{\lambda}{\lambda+\mu}\right)^n\left(\frac{\mu}{\lambda+\mu}\right)^{a-n}$$

Non poisson Queueing Systems

Queues in which arrivals and\or departures doesnot follow the poisson are called non-poisson queues. The techniques used to study this non-poisson queues are

(a) Phase technique

(b) The technique of Imbedded Markov Chain, and

(c) The supplementary variable technique.

M/ E_k/ 1 Queueing System

In this type of system, service time has an Erlang type k distribution. Eventhough the service maynot actually consist of k phases, we consider being made up of k exponential phases with mean $\frac{1}{k\mu}$.

Model I (M/ E_k/ 1) : (∞/ FIFO)

This model deals with single service with n phases in the system. Assume that a new arrival creates k-phases of service and departure of one customer reduces k phase of service.

Steady state difference equations are

$$-(\lambda + k\mu)P_n + k\mu P_{n+1} + \lambda P_{n-k} = 0 \qquad \text{for } n \geq 1$$

$$-\lambda P_0 + k\mu P_1 = 0$$

$$P_n = (1-k\rho)\sum_{i,j,k} \rho^m (-1)^i \binom{m}{i}\binom{m+j-1}{j} \qquad \text{for } j + ik + m = n$$

(i) Average number of phases in the system $E(n_p)$ is given by

$$E(n_p) = \frac{k(k+1)}{2} \cdot \frac{\frac{\lambda}{k\mu}}{1-\frac{k\lambda}{k\mu}} = \left(\frac{k+1}{2}\right)\frac{\lambda}{\mu-\lambda}$$

(ii) Average waiting time of the phase in the system is

$$E(w_p) = \frac{E(n_p)}{\mu} = \left(\frac{k+1}{2\mu}\right)\frac{\lambda}{\mu-\lambda}$$

(iii)Average waiting time of an arrival is given by

$$E(w) = \frac{E(w_p)}{K} = \left(\frac{k+1}{2k}\right)\frac{\lambda}{\mu(\mu-\lambda)}$$

(iv)Average time an arrival spends in the system is given by

$$E(v) = E(w) + \frac{1}{\mu}$$

$$= \left(\frac{k+1}{2k}\right)\frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}.$$

(v) Average number of units in the system is given by

$$E(n) = \lambda E(V) = \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

(vi)Average Queue length is given by

$$E(m) = E(n) - \frac{\lambda}{\mu} = \left(\frac{k+1}{2k}\right) \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Model II :(M/ E_k/ 1) : (1 / FIFO)

Capacity of the system is unity. There is no queue and contains only one customer.

The customer is in the n[th] phase. Steady state difference equations are given by

$$-k\mu P_n + k\mu P_{n+1} = 0 \qquad 1 \le n \le k$$

$$-k\mu P_k + \lambda P_0 = 0 \qquad n = k$$

$$-\lambda P_0 + k\mu P_1 = 0 \qquad n = 0$$

$$P_n = \frac{1}{k} \cdot \frac{\lambda}{\lambda+\mu}$$

M/ G/ 1 Queueing system

This system consists of single server, Poisson arrival and general service time distribution.

The various formula for (M/ G/ 1) : (∞/ GD),

(i) Average Number of customers in the system

$$= \frac{\lambda^2\sigma^2 + \rho^2}{(2(1-\rho))} + \rho$$

(ii) Average Queue length

$$= \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

(iii)Average waiting time of a customer in the queue

$$= \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}$$

(iv)Average waiting time that a customer spends in the system

$$\frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$$

Other Queueing Models

Inspite of all above models, we have several other models by changing the queue discipline or considering different patterns of arrival and service rate other than poisson and exponential we have finite or infinite queue, circular Queues, queues in service, queue with balking and reneging. Some of them are

(i) Queues in series where input of a queue is got from the output of another queue.

(ii) Circular queue where a customer joins the queue after the service.

(iii) Multichannel queue with different service rate at each channel.

(iv) Queues with priority or preemptive priority, etc.

Therefore, as the complexity of the system increases, the difficulty of analytical solution increases with faster rate. Therefore many realistic problems are solved through simulation with a great deal of effort spend on modelling the complex real life problem.

SECTION 2

In this section simulation of single server queue and multiserver queue are discussed.

Simulation of a single server queue

In this section simulation of the arrival and serving of N customers by a single server is considered. Let customers be marked with 1,2 ..... N.

Let the interarrival time AT_k denote the time gap between the arrivals of the $(K-1)^{th}$ customer and the k^{th} customer into the system. These times will be generated, as samples from some specified probability distribution, by means of an appropriate random number generator.

ST_k is the service time of the k^{th} customer K= 1,2........N.

CAT_k is cumulative arrival time of the k^{th} customer. Initially assume there is no queue and the server is free. As the first customer arrives at time zero, he goes directly to the service counter. After the service, the customer leaves the system at time ST_k .

The second customer will arrive at $CAT_2 = AT_2$.

If $ST_1 > CAT_2$, the second customer has to wait (forming a queue of length) for a period $WT_2 = ST_1 - CAT_2$.

WT_k denotes the waiting time of the K^{th} customer in the queue.

If $CAT_2 > ST_1$ then the departure of the first customer takes place before the arrival of the second. Thus the service counter remains idle awaiting the arrival of the second customer for a period.

$IDT_2 = CAT_2 - ST_1$

IDT_1 denotes the idle time of the server awaiting the arrival of the K^{th} customer.

Consider the general situation, Assume that (i-1) customers have arrived into the system and (j-1) customer have left the system.

The next customer due to arrive is the i^{th} customer and the next customer due to depart is the j th customer

Clearly $i \leq j \leq n$ and the queue length is given by (i - j-1) , if $i > j$

The next arrival time $NAT = CAT_i$.

The next departure time, the cumulated departure time CDT_j of the jth customer is given by

$NDT = CDT_j$ =cumulative arrival time of j + waiting time of j + service time of j

$= CAT_j + WT_j + ST_j$

By comparing NAT with NDT we may determine whether i would arrive first or j would depart first

The difference is NAT-NDT, which is denoted by DIF

If DIF is negative, an arrival would take place first and the length of the queue would increase by 1.

If the DIF is positive and the queue length (i-j+1) is also positive then a departure would take place first, reducing the length of queue by 1

In both cases there is no idle time for the service counter.

If DIF is positive and queue is zero, then the service counter will lie idle waiting for the ith customer for the duration.

$IDT_i = DIF$.

Finally, DIF = 0 implies that both events the next arrival and departure will take place simultaneously, having the queue length unaltered.

Simulation for this queueing situation is shown in the flow chart

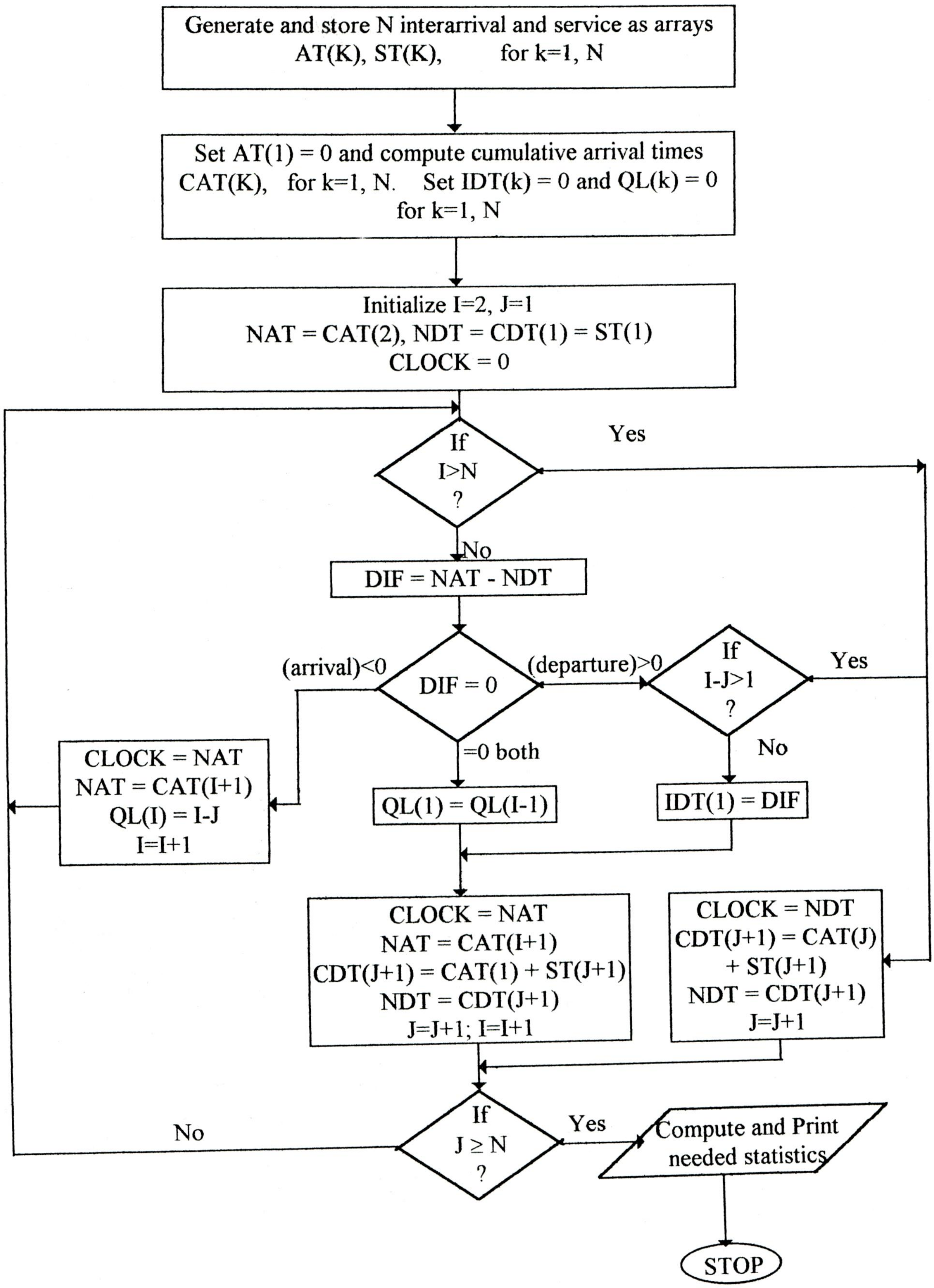
In the flow chart, the interarrival times AT_i's and the service time ST_i can be generated by calling the suitable subroutines. From the interarrival time, the cumulative arrival times are calculated as follows :

$$CAT_k = CAT_{k-1} + AT_k$$

$$CAT_1 = AT_1 = 0$$

The event times are indicated by the variable clock.

Flowchart for 1 server-queue simulation



Using the flowchart in computation, we can calculate the following parameters.

1. Probability that there are m customer in the system
2. Maximum queue length
3. Average Queue length (averaged overall queue lengths including zero)
4. Average length of non empty queue [arranged over only non zero queue lengths]
5. Average Number of customers in the system
6. Average waiting time (averaged over all customers)
7. Average waiting time of customers that has to wait (averaged over only those customers that wait in the queue)
8. Maximum waiting time
9. Average time spend by a customer in the system
10. Average Idle time
11. Percentage of the time the service counter is busy
12. Average fraction of the time a customer spends waiting in the queue.

Simulation of a two server queue

Customers arrive according to the given probability distribution for interarrival times.

The interarrival time AT_k denote the time gap between the arrivals of the (k-1)th customer and kth customer into the system.

$ST_{k,1}$ denote the service time for the kth customer at facility 1.

CAT_k denote the cumulative arrival time for the kth customer and $CDT_{k,1}$ denotes the cumulative departure, the time elapsed at the departure of the kth customer from facility 1.

Assume initially AT_1=0 and the cumulative arrival time is given by

$$CAT_k = CAT_{k-1} + AT_k$$

and $CAT_1 = AT_1 = 0$

As the first customer arrives at time zero, it goes directly to facility 1.

If the second customer arrives before the service is completed in the facility 1, it goes directly to the facility 2.

Let $IDT_{k,1}$ denotes the idle time of facility 1, while waiting for kth customer to arrive. When the third customer arrives, we have to check which of the two facilities is vacant or else we have to see which of it fall vacant earlier.

This is done by comparing the latest values of the cumulative departure times of facility 1 and facility 2. For these are the instants when the facilities would be free next.

The smaller of the two indicates the time when the next earliest departure from the system would take place.

Thus the next departure time is given by

$$MNDT = \min \{ CDT_{k,1}, CDT_{k,2} \}$$

This gives the earliest time when the service for the third customer to start.

If $CAT_k > MNDT$, the idle time facility 1 spends while waiting for the next customer to arrive is given by

$$IDT_{k,1} = CAT_k - MNDT$$

The cumulative departure time at facility 1 is given by

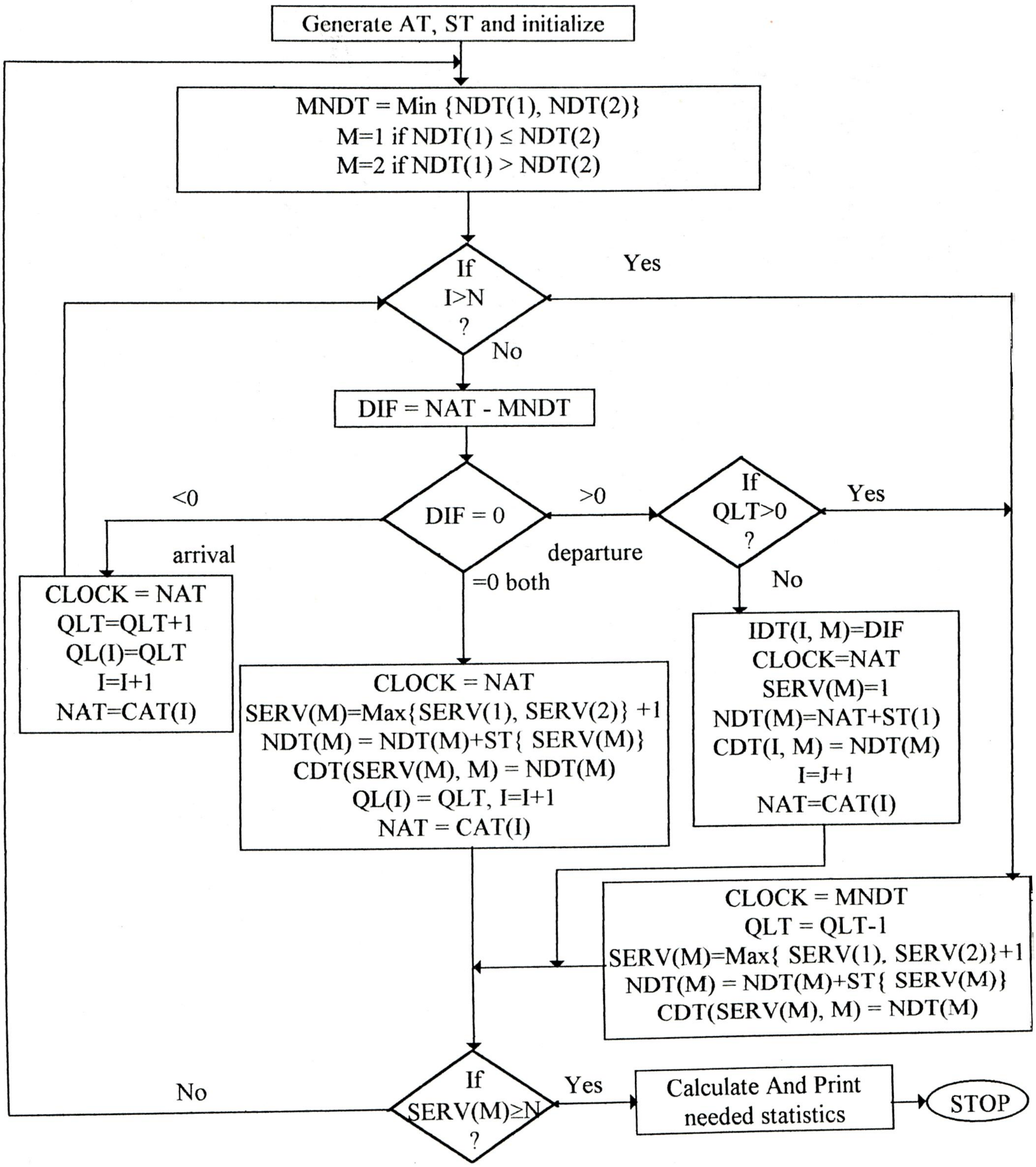
$CDT_{k,1}$ = cumulative arrival time for kth customer + service time

$$= CAT_k + STK_{k,1}$$

Now if, $MNDT > CAT_k$, then the customer will form a queue and wait for a period there and it is given by

$$MNDT - CAT_k = WT_k$$

Flowchart Of 2 Server-Queue



In the flow chart of two server, we require two next departure times NDT(1), NDT(2). The minimum of these two, MNDT tells us the time when a server would be free next. MNDT is compared with the next arrival time NAT to determine whether an arrival or a departure takes place. We should know which of the two facility will fall vacant at time MNDT and it is denoted by a variable M.

In this case, we see that the customers need not depart in the same order in which they arrived. We consider two pair of variables SERV(1), & SERV(2) which tells which customer is currently being served by server1 & server 2 respectively the larger of these two number plus 1 is the customer to be served next.

ie) SERV(M) = Max{ SERV (1) , SERV(2) } +1

The current queue length is denoted by the variable QLT, which is increased by 1 if the next event is an arrival and decreased if it is departure and left unaltered if both arrival & departure occur simultaneously

Monte carlo (Random) Approach to Queueing

We see that in many cases, the observed distributions for arrival and service times cannot be filled to certain mathematical distribution like Poisson and Exponential. In addition with this, the first in and first out assumption may not be valid for a particular queueing problem. In multichannel queueing, departure from one queue may form the arrival for another. Under all such conditions, the Monte Carlo method is extremely useful since none of the previous queueing models perform adequately.

Basically the Monte Carlo method is a simulation technique in which statistical distribution function are created by using a series of random numbers and Monte Carlo is useful in solving single channel and multichannel waiting line problems.

Monte Carlo method is oriented towards computer processing and this approach can develop many months or years of data in a matter of a five minutes on a computer. Also Monte Carlo allows manipulation of those factors that are subject to control, such as adding another service station, without actually having to incur the expense of installing one. Changes can be tried without disrupting the actual process.

Even though Monte Carlo analysis does not require that the arrival and service time distributions obey certain theoretical forms, it does demand that the form and parameters of these distributions set forth the cumulative distribution which can then be developed are used as a means for generating arrival and service times.

Single Channel Queue Monte Carlo Method Using Random Number

First assume that the queueing process starts at certain period of time and continues for certain period of time.

An arrival moves immediately into the service if it is empty. Otherwise, if the service station is busy in the waiting line will enter the service facility on a first come first served basis.

In this method a table of random number is considered. The random number for arrival times and service times are arbitrarily taken. Let A_m denote the mean arrival time and S_m denote the mean service time.

Having generating arrival & service time from a table of random number, the next step is to list the waiting time in the appropriate column.

If the first arrival comes few minutes after the starting time the clerk has waited for few minutes. Waiting time for attendant is 1 while waiting time for customer is blank due to zero waiting time on the part of the first customer and the simulated service time for the first arrival is noted and with this we could find the service ending. In the similar manner we could find for a set of random number until the process is over under a certain period of time.

Based on simulated data we could calculate the following

i) Average length of the waiting line.

ii) Average time a customer waits before being served.

iii) Average time a customer spends in the system.

iv) Whether another attendant is added to system.

CHAPTER - IV

CHAPTER IV

APPLICATION

In this chapter, two applications are provided which are based on uniform arrival and normal service distributions and general probability distributions respectively.

Allocation of Doctors to outpatient Clinic

To design an improved system - The director of the clinic wants to identify a better staffing allocation in which the duty doctors are to be allocated so as to attend the patients without long delays and at the same timewithout much idle time for doctors.

Two doctors presently serve as the staff of the clinic Patients complained about excessive waiting times before being served and doctors complained about being over worked. So the director of the clinic wonders number of patient waiting time and doctor idle time is being experienced and the improvement is noticed when a third doctor is added to the staff.

Decision Rules

1) Patients arrive uniformly
2) Doctors serve the patients in first come first served basis.
3) Patients arrival patterns are assumed to be about the same for all hours of the day.

4) Patients waiting time or doctor idle time is given by

$$T_n = T_i - (60N - W_{n-1})$$

Where T_n = either patient waiting or doctor idle in time period n

T_i = Service times for the i[th] patient arriving in time period n

W_{n-1} = Patient waiting time in last period

N = Number of doctors on the staff

5. Number of patients arriving during any hour is selected from a table of uniformly distributed random numbers.
6. Service times are normally distributed with mean 18 minutes and a standard deviation of 4 minutes.

Two systems of allocation of doctors are compared for N=2 and N =3 doctors.

Table 1 presents the range of random numbers allocated each class according to the frequency of that class.

Random Number Ranges for each class in Discrete Arrival Distribution for Monte Carlo

Patient Arriving Per hour	Relative Frequency (percent)	Random Number Range
6	20	0 - 19
7	30	20-49
8	25	50-74
9	15	75-89
10	10	90-99

Table 2 gives the simulated number of patients arrived for each hour with random numbers.

Determining Number of Patient Arrivals for each hour of Simulation using Monte Carlo

Hour	Uniform Random Number(RN)	Patients Arrival
1	00	6
2	28	7
3	80	9
4	40	7
5	79	9
6	86	9
7	55	8
8	59	8
9	14	6

To find the service time for our patients

μ = Mean = 18 minutes

σ = Standard deviation = 4 minutes

Normally service time for each patient is computed using the formula

$$T_i = \mu + Z_i (\sigma)$$

Where Z_i is taken from the Z'scores - the number of standard deviation each time is from the mean.

$$t_i = 18 + Z_i(4)$$

Let us consider 1.21, -1.31, -1.12, 1.32, 0.86 and 0.31 are Z scores for the six patients in the first hour of our simulation, we compute the service times for these patients.

$t_1 =$ 18 + 1.21 (4) = 22.84 minutes

$t_2 =$ 18 - 1.31 (4) = 12.76 minutes

$t_3 =$ 18 - 1.12 (4) = 13.52 minutes

$t_4 =$ 18 - 1.32 (4) = 23.28 minutes

$t_5 =$ 18 + 0.86 (4) = 21.44 minutes

$t_6 =$ 18 + .31 (4) = 19.24 minutes

Total 113.08 minutes

By repeating this procedure, we can compute the service times for all patients and total these service times for each hour of the simulation.

They are found as

113.1, 135.1, 160.6, 112.8, 197.1, 180.3, 154.7, 159.2 and 98.8

Performance of simulation

Number of patients arriving and the total service time for each hour of the simulation is shown in the table given below.

Outpatient Clinic Simulation

The patient waiting time and doctor idle time are computed for each hour for the staffing arrangement are presented in Table 3.

Hour	Numbe-r of Patient arriving	Total service Time (Min)	Patient Waiting time (Min)	Doctor idle time (Min)	Patient Waiting time (Min)	Doctor idle time (Min)
1	6	113.1	0	6.9	0	66.9
2	7	135.1	15.1	0	0	44.9
3	9	160.1	55.7	0	0	19.4
4	7	112.8	48.5	0	0	67.2
5	9	197.1	125.6	0	17.1	0
6	9	180.3	185.6	0	17.4	0
7	8	154.7	220.6	0	0	7.9
8	8	159.2	259.8	0	0	20.8
9	6	98.8	238.6	0	0	81.2
Totals	69	1,311.5	1,149.8	6.9	34.5	308.3
Average per patient		19.0	16.7	0.1	0.5	4.5

The patient waiting time and doctor idle time are computed for each hour for the staffing arrangements.

Consider for Hour 4:

Two doctors :

$$T_n = T_i - (60N - W_{n-1})$$

$$T_4 = 112.8 - (120 - 55.7)$$

$$= 112.8 - 64.3$$

$= 48.5$

Because T_4 is positive, it represents patient waiting time.

Three doctors :

$$T_n = t_i - (60N - W_{n-1})$$

$$T_4 = 112.8 - (180 - 0)$$

$$T_4 = -67.2$$

Because T_4 is negative, it represents doctor idle time.

Result	**Two doctors**	**Three Doctors**
Average service time per patient	19.0 Minutes	19.0 Minutes
Average waiting time per patient	16.7 Minutes	0.5 Minutes
Average Doctor idle time between patients	0.1 Minutes	4.5 Minutes

We see that in 2-doctors arrangement, the patients waiting time is more with overwork for the doctors. But the three doctor arrangement compromise both problems. Hence the director agrees with three doctor arrangement.

Determination of Parameters to a Shipping Problem

Ships arrive at a harbour according to the given probability distribution for interarrival times. When a ship arrives it goes directly to the facility if it remains idle when the facility is free or else it joins the queue and wait till the facility is free.

Consider six ships arrive at times 0, 10, 48, 55, 65, 85 hours. The simulation starts at time zero. At that time there are no ships in the system.

		Facility 1					Facility 2			
k	AT_2	CAT_k	$ST_{k,1}$	$CDT_{k,1}$	$IDT_{k,1}$	$ST_{K,2}$	$CDT_{k,2}$	$IDT_{k,2}$	WT_k	QL_k
1	0	0	25	25	0	-	-	-	0	0
2	10	10	-	-	-	30	40	10	0	0
3	38	48	22	70	23	-	-	-	0	0
4	7	55	-	-	-	50	105	15	0	0
5	10	65	45	115	0	-	-	-	5	1
6	20	85	-	-	-	15	120	0	20	1

The first column in the table is ships serial number k. The second column gives the interarrival times AT_k. The third column gives the cumulative arrival time CAT_k. The fourth column gives the service time for k^{th} ship at facility 1 denoted by $ST_{k,1}$. Column five denotes the cumulative departure time of the k^{th} ship at facility 1 denoted by $CDT_{k,1}$. Column six denotes the idle time $IDT_{k,1}$ of facility 1 while waiting for ship k to arrive. The next column gives the corrosponding three numbers for facility 2. The tenth column gives the waiting time WT_k for the k^{th} ship. The last column contains the queue length QL_k immediately after the arrival of ship k.

First assume initially $AT_1 = 0$

From the chapter III, Section 2.2 we get

$$CAT_k = CAT_{k-1} + AT_k$$

$$CAT_1 = AT_1 = 0$$

As the first ship arrives at time zero, it goes directly to facility 1. The service time of ship 1 at facility 1 is 25 hours and therefore ship leaves the system at 25 hours.

At time 10 hours, the second ship arrives and it goes directly to facility 2. The service time is 30 hours for ship 2 at facility 2. And it leaves the system at 40.

$$CDT_{2,2} = 40$$

Before ship No. 2 arrived, facility 2 was idle for 10 hours. Now when the third ship arrives, we have to check to see which remains vacant or will fall vacant earlier. This is done by comparing the latest values of cumulative departure times. This will tell which facility will be freed next. The smaller value indicates the time when the next earliest departure from the system would take place.

The next departure time is

$$MNDT = \min \{ CDT_{k,1}, CDT_{k,2} \}$$

$$MNDT = \min \{25, 40\}$$

$$= 25.$$

This gives the earliest time when the service for the Ship No. 3 can be started.

Since $CAT_3 > MNDT$

Ship No.3 has not arrived at time 25, and

therefore CAT_3 - MNDT = 48-25

$= 23$

$= IDT_{3,1}$

This gives the amount of idle time of facility 1 while waiting for ship No.3 to arrive.

The cumulative departure time at facility 1 of ship No.3 is given by

$$CDT_{3,1} = CAT_3 + ST_{3,1}$$

$$= 48 + 22$$

$$= 70.$$

Before Ship No.4 arrives at time 55, facility 2 would have been released at time 40, beacuse

$$\min \{70, 40\} = 40$$

and therefore the idle time $IDT_{4,2} = 50$, Ship No. 4 departs from facility 2 at

$$CDT_{4,2} = CAT_4 + ST_{4,2}$$

$$= 55 + 50$$

$$= 105$$

Next, Ship No.5 arrives and both facilites are busy.

For $MNDT = \min \{70, 105\} = 70$

and $CAT_5 = 65$

Therefore, Ship No.5 will form a queue and wait there for a peroid

$$MNDT - CAT_5 = 70 - 65$$

= 5 hours.

At the end of this period it would be serviced by facility 1.

From time 70 to 105 hours, facilites 1 and 2 will both be busy servicing Ship No.5 and 4 respectively. Therefore Ship No.6, arriving at $CAT_6 = 85$, will have to wait in the queue for 20 hours to be serviced by facility 2 at time 105.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. BASU,S.K. AND BAGCHI, H. (1989) OPERATIONS RESEARCH FOR ENGINEERS. Oxford and IBH publishing company private limited .
2. DHARANI VENKATAKRISHNAN, S. (1992) OPERATIONS RESEARCH. Keerthi Publishing House (P) Limited.
3. ERLANG, A.K. (1909) THE THEORY OF PROBABILITIES AND TELEPHONE CONVERSATIONS NYT. Tiderskrift Mathematika,B.20, 33-39.
4. JACKSON, R.R.P AND HENDERSON, J.C. (1966) THE TIME-DEPENDENT SOLUTION TO THE MANY SERVER POISSON QUEUE. Opns. Res. 14, 720-722.
5. GILLETT, B.E. (1989) INTRODUCTION TO OPERATION RESEARCH. Tata McGraw Hill Publishing Company Limited. New Delhi.
6. GEOFFREY GORDON (1980) SYSTEM SIMULATION IBM Corporation. Prentice Hall of India Private Limited. New Delhi.
7. KENDALL, D.G. (1951) SOME PROBLEMS IN THE THEORY OF QUEUES. J. Roy. Statist. Soc. Series B, 13, 151-185.
8. LITTLE, J.D.C. (1961) A PROOF OF THE QUEUEING FORMULA $L = \lambda w$. Opns. Res. 9, 383-387.
9. MORSE, P.M. (1955) STOCHASTIC PROPERTIES OF WAITING LINES. Opns. Res. 3, 255-261.
10. NARSINGH DEO. (1996) SYSTEM SIMULATION WITH DIGITAL COMPUTER. Charles E. Millican Professor, Department of Computer Science, University of Central Florida.
11. PRITSKER, A. (1986) INTRODUCTION TO SIMULATION AND SLAM II Halsted Presswiley, New York.

12. SHENOY, G.V. SRIVASTAVA, U.K. AND SHARMA, S.C. (1988) — OPERATIONS RESEARCH FOR MANAGEMENT
Published by V.S.Johri for wiley Σ^n limited.

13. TAHA, H.A (1987) — OPERATIONS RESEARCH
MacMillan Publishing Company, New York.

14. THIERAUF, R.J. (1967) — AN INTRODUCTORY APPROACH TO OPERATION RESEARCH
Holden Day
Sanfrancisco