2. Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Starting Failure and Feedback

Bulk arrival retrial G-queue with two phase service and feedback is considered. The system consists of positive and negative customers. Arrival of negative customer brings the busy server down. All the arriving positive customers receive the first phase of essential service, whereas only some of them opt the second phase of multistages of service. In each stage there are multi-optional heterogeneous services. If the arriving batch of customers finds the server idle, one of the arriving customers begins essential service and the rest joins the orbit. On the other hand if the server is unavailable, all the customers of the batch enter the orbit. After completing essential service the customer moves to second phase, leaves the system or joins the orbit as feedback customer. In second phase the customers move the stages in succession. After service completion from a stage, the customer moves to the next stage, leaves the system or joins the orbit as a feedback customer. It is assumed that, the server is subject to starting failure in addition with failure due to negative arrivals. The service time, retrial time and repair time are assumed to be arbitrarily distributed. A mathematical model is constructed and the steady state distributions of the server state including the average number of customers in the system are derived. Numerical results are presented to explore the effect of several parameters on the system measures.

2.1 Model Description

A single server batch arrival retrial queue where the server is subject to starting failure is considered. The server provides M-stages of sequential services followed by essential service. Each stage contains multi-optional heterogeneous services. Stage i (i=1,2,.....,M) consists of k_i optional services. Arriving customers are of two types, positive and negative. The positive customers arrive in batches of size Y according to the Poisson process with arrival rate λ^+ having distribution function $P(Y=k) = C_k$, k=1,2,3,...., probability generating function C(z) and first two moments m_1 and m_2. Upon arrival, if the server is busy, all the customers join the orbit. Otherwise, one of the customers in the batch starts the service. The inter-retrial times form an arbitrary distribution $A(x)$ with Laplace-Stieltjes transform $A^*(s)$ and the hazard rate function $\eta(x) = \frac{dA(x)}{1 - A(x)}$. If the server is started successfully, the customer gets the essential service immediately. Otherwise, the customer joins the orbit and the server is sent for repair. The probability of getting successful commencement of essential service is α. The repair time is generally distributed with distribution function $S(x)$, Laplace-Stieltjes transform $S^*(s)$, first two moments γ_1 and γ_2 and the conditional completion rate $\gamma(x) = \frac{dS(x)}{1 - S(x)}$.

After completing the essential service, the customer proceeds to first stage in second phase and opts j_1^{th} (j_1=1,2,....,k_1) option with probability p_{j_1}, joins the orbit as a feedback customer with probability δ_0 or departs the system with probability $q_0 \left(= 1 - \delta_0 - \sum_{j_1=1}^{k_1} p_{j_1} \right)$.

After the completion of first stage, the customer moves to second stage and opts j_2^{th} (j_2=1,2,....,k_2) option with probability p_{j_2}, joins the orbit as a feedback customer with probability δ_1 or departs the system with probability q_1. In general, after the completion of i^{th} stage (i=1,2,….,M) service, the customer opts j_{i+1}^{th} (j_{i+1}=1,2,....,k_{i+1}) option in $(i+1)^{th}$ stage with probability $p_{j_{i+1}}$, joins the orbit with probability δ_i or departs the system with probability $q_i \left(= 1 - \delta_i - \sum_{j_{i+1}=1}^{k_{i+1}} p_{j_{i+1}} \right)$. After the final stage (M^{th} stage) service, the customer may leave the system with probability q_M or join the orbit with probability δ_M $(= 1 - q_M)$. The essential service time is generally distributed with distribution function $B_0(x)$, Laplace-Stieltjes transform $B_0^*(s)$, n^{th} factorial moment $\mu_0^{(n)}$ and the conditional completion rate $\mu_0(x) = \frac{dB_0(x)}{1 - B_0(x)}$. The service time of i^{th} stage (i =1,2,….,M), j_i^{th} (j_i =1,2,....,k_i) optional service is generally distributed with distribution function $B_{i,j_i}(x)$, Laplace-

Stieltjes transform $B^*_{i,j_i}(s)$, n^{th} factorial moment $\mu^{(n)}_{i,j_i}$ and the conditional completion rate $\mu_{i,j_i}(x) = \frac{dB_{i,j_i}(x)}{1 - B_{i,j_i}(x)}$.

The negative customers arrive in single according to the Poisson process with rate λ^-. The arrival of negative customer in the system makes the server down and pushes out the customer being in service from the system. The failed server is sent for repair immediately. The repair time of the server failed during essential service is generally distributed with distribution function $R_0(x)$, Laplace-Stieltjes transform $R^*_0(s)$, n^{th} factorial moment $\beta^{(n)}_0$ and the conditional completion rate $\beta_0(x) = \frac{dR_0(x)}{1 - R_0(x)}$. The repair time of the server failed during i^{th} stage ($i = 1,2,\ldots,M$), j_i^{th} ($j_i = 1,2,\ldots,k_i$) optional service is generally distributed with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R^*_{i,j_i}(s)$, n^{th} factorial moment $\beta^{(n)}_{i,j_i}$ and the conditional completion rate $\beta_{i,j_i}(x) = \frac{dR_{i,j_i}(x)}{1 - R_{i,j_i}(x)}$. The diagrammatic representation of the proposed model is shown in Fig. 2.1.

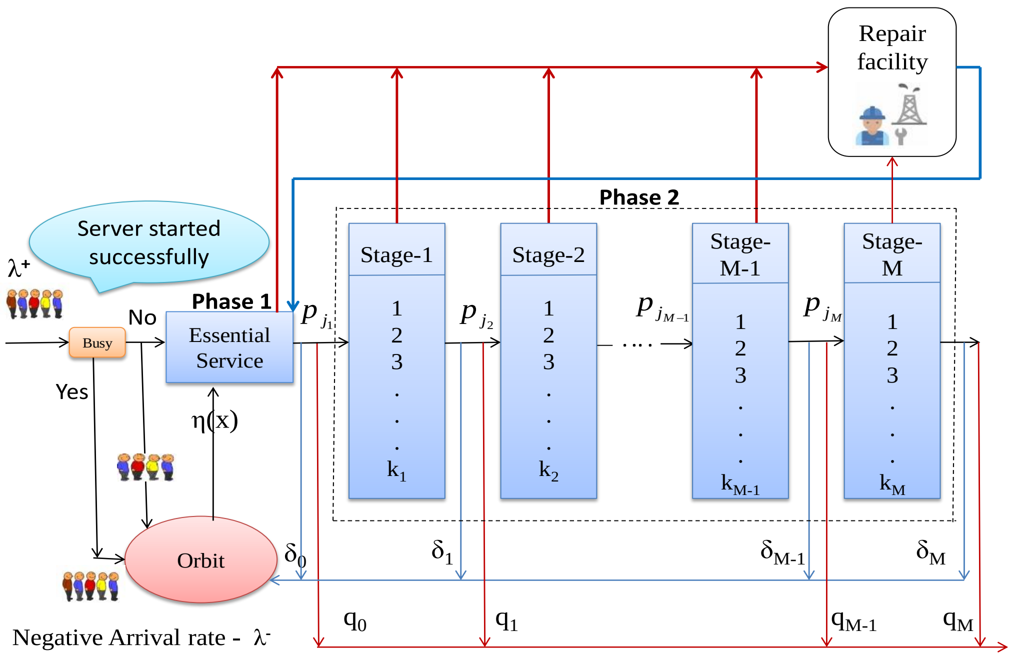

Fig. 2.1 Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Starting Failure and Feedback

2.2 Mathematical Analysis of the System

In this section, the steady state difference differential equations for the retrial system under consideration are formulated by considering the elapsed retrial time, the elapsed service time and the elapsed repair time as supplementary variables. Further the probability generating functions for the server state and the number of customers in the orbit and in the system are derived.

2.2.1 Server State Probabilities and Notations

The server state S(t) at time t is defined as

$$S(t)=\begin{cases} 0, & \text{server is idle} \\ 1, & \text{server is busy in essential service} \\ 2, & \text{server is busy in } i^{th} \text{ stage } {j_i}^{th} \text{ optional service} \\ 3, & \text{server is under repair due to starting failure} \\ 4, & \text{server in essential service is under repair due to negative arrival} \\ 5, & \text{server in } i^{th} \text{ stage } {j_i}^{th} \text{ optional service is under repair} \\ & \text{due to negative arrival} \end{cases}$$

Let $\xi_r(t)$, r = 0, 1 or 2 respectively denote the elapsed retrial time, service time or repair time.

The supplementary variables are introduced in order to obtain a bivariate Markov process { N(t), S(t), $t \geq 0$ }, where N(t) denotes the number of customers in the orbit at time t and S(t) denotes the server state.

The joint distributions of the server state and queue size are defined as

$$I_0(t) = P\{S(t)=0, N(t)=0\}$$

$$I_n(x,t) = P\{S(t)=0, N(t)=n,\ x<\xi_0(t)\leq x+dx\}, n\geq 1$$

$$P_{0,n}(x,t)dx = P\{S(t)=1,\ N(t)=n,\ x<\xi_1(t)\leq x+dx\}, n\geq 0$$

$$P_{i,j_i,n}(x,t)dx = P\{S(t)=2,\ N(t)=n,\ x<\xi_1(t)\leq x+dx\},\ n\geq 0,$$
$$i=1,2,\ldots,M,\ j_i=1,2,\ldots,k_i$$

$$S_n(x,t)dx \quad = \quad P\{S(t)=3,\ N(t)=n,\ x<\xi_2(t)\le x+dx\},\ n\ge 1$$

$$R_{0,n}(x,t)dx \quad = \quad P\{S(t)=4,\ N(t)=n,\ x<\xi_2(t)\le x+dx\},\ n\ge 0$$

$$R_{i,j_i,n}(x,t)dx = P\{S(t)=5,\ N(t)=n,\ x<\xi_2(t)\le x+dx\},\ n\ge 0,$$
$$i=1,2,\ldots,M,\ j_i=1,2,\ldots,k_i$$

2.2.2 Governing Equations

The system of equations that governs the model under consideration is given below.

$$\frac{d}{dt}I_0(t) = -\lambda^+ I_0 + q_0\int_0^\infty P_{0,0}(x,t)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,0}(x,t)\mu_{i,j_i}(x)dx$$
$$+\int_0^\infty R_{0,0}(x,t)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,0}(x,t)\beta_{i,j_i}(x)dx \qquad (2.1)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) I_n(x,t) = -(\lambda^+ + \eta(x))\ I_n(x,t),\quad n\ge 1 \qquad (2.2)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) P_{0,n}(x,t) = -(\lambda^+ + \lambda^- + \mu_0(x))\ P_{0,n}(x,t) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{0,n-k}(x,t),\quad n\ge 0$$
$$(2.3)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) P_{i,j_i,n}(x,t) = -(\lambda^+ + \lambda^- + \mu_{i,j_i}(x))\ P_{i,j_i,n}(x,t) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{i,j_i,n-k}(x,t), \qquad (2.4)$$
$$n\ge 0,\ i=1,2,\ldots,M,\ j_i=1,2,\ldots,k_i$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) S_1(x,t) = -(\lambda^+ + r(x))\ S_1(x,t) \qquad (2.5)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) S_n(x,t) = -(\lambda^+ + r(x))\ S_n(x,t) + \lambda^+\sum_{k=1}^{n} C_k S_{n-k}(x,t),\ n\ge 2 \qquad (2.6)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) R_{0,n}(x,t) = -(\lambda^+ + \beta_0(x))\ R_{0,n}(x,t) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{0,n-k}(x,t),\ n\ge 0$$
$$(2.7)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right) R_{i,j_i,n}(x,t) = -(\lambda^+ + \beta_{i,j_i}(x))\ R_{i,j_i,n}(x,t) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{i,j_i,n-k}(x,t), n\ge 0,$$
$$i=1,2,\ldots,M,\ j_i=1,2,\ldots,k_i \qquad (2.8)$$

where δ_{0n} is the Kronecker delta with boundary conditions

$$I_n(0,t) = \delta_0\int_0^\infty P_{0,n-1}(x,t)\mu_0(x)dx + \sum_{i=1}^{M}\delta_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-1}(x,t)\mu_{i,j_i}(x)dx + q_0\int_0^\infty P_{0,n}(x,t)\mu_0(x)dx$$
$$+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x,t)\mu_{i,j_i}(x)dx + \int_0^\infty S_n(x,t)r(x)dx$$
$$+\int_0^\infty R_{0,n}(x,t)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x,t)\beta_{i,j_i}(x)dx,\ n\geq 1 \qquad (2.9)$$

$$P_{0,0}(0,t) = \alpha\lambda^+C_1I_0(t) + \alpha\int_0^\infty I_1(x,t)\eta(x)dx \qquad (2.10)$$

$$P_{0,n}(0,t) = \alpha\lambda^+C_{n+1}I_0(t) + \alpha\lambda^+\sum_{k=1}^{n}C_k\int_0^\infty I_{n-k+1}(x,t)dx + \alpha\int_0^\infty I_{n+1}(x,t)\eta(x)dx,\ n\geq 1 \qquad (2.11)$$

$$P_{1,j_1,n}(0,t) = p_{j_1}\int_0^\infty P_{0,n}(x,t)\mu_0(x)dx,\ n\geq 0,\ j_1 = 1,2,....,k_1 \qquad (2.12)$$

$$P_{i,j_i,n}(0,t) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1},n}(x,t)\mu_{i-1,j_{i-1}}(x)dx,\ n\geq 0,\ i = 2,3,......,M,\ j_i = 1,2,....,k_i \qquad (2.13)$$

$$S_1(0,t) = \bar{\alpha}\lambda^+C_1I_0(t) + \bar{\alpha}\int_0^\infty I_1(x,t)\eta(x)dx \qquad (2.14)$$

$$S_n(0,t) = \bar{\alpha}\lambda^+C_nI_0(t) + \bar{\alpha}\lambda^+\sum_{k=1}^{n}C_k\int_0^\infty I_{n-k}(x,t)dx + \bar{\alpha}\int_0^\infty I_n(x,t)\eta(x)dx,\ n\geq 2 \qquad (2.15)$$

$$R_{0,n}(0,t) = \lambda^-\int_0^\infty P_{0,n}(x,t)dx,\ n\geq 0 \qquad (2.16)$$

$$R_{i,j_i,n}(0,t) = \lambda^-\int_0^\infty P_{i,j_i,n}(x,t)dx,\ n\geq 0,\ i = 1,2,3,......,M,\ j_i = 1,2,....,k_i \qquad (2.17)$$

Assuming the existence of steady state probabilities, the steady state equation corresponding to the equations (2.1) to (2.17) are

$$\lambda^+I_0 = q_0\int_0^\infty P_{0,0}(x)\mu_0(x)dx + \sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,0}(x)\mu_{i,j_i}(x)dx$$
$$+\int_0^\infty R_{0,0}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,0}(x)\beta_{i,j_i}(x)dx \qquad (2.18)$$

$$\frac{d}{dx}I_n(x) = -(\lambda^+ + \eta(x))\,I_n(x),\quad n\geq 1 \qquad (2.19)$$

$$\frac{d}{dx}P_{0,n}(x) = -(\lambda^+ + \lambda^- + \mu_0(x))\,P_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{0,n-k}(x)\,,\; n \ge 0 \qquad (2.20)$$

$$\frac{d}{dx}P_{i,j_i,n}(x) = -(\lambda^+ + \lambda^- + \mu_{i,j_i}(x))\,P_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{i,j_i,n-k}(x)\,, \qquad (2.21)$$
$$n \ge 0,\; i = 1,2,\ldots,M,\; j_i = 1,2,\ldots,k_i$$

$$\frac{d}{dx}S_1(x) = -(\lambda^+ + r(x))\,S_1(x) \qquad (2.22)$$

$$\frac{d}{dx}S_n(x) = -(\lambda^+ + r(x))\,S_n(x) + \lambda^+\sum_{k=1}^{n} C_k S_{n-k}(x),\; n \ge 2 \qquad (2.23)$$

$$\frac{d}{dx}R_{0,n}(x) = -(\lambda^+ + \beta_0(x))\,R_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{0,n-k}(x),\; n \ge 0 \qquad (2.24)$$

$$\frac{d}{dx}R_{i,j_i,n}(x) = -(\lambda^+ + \beta_{i,j_i}(x))\,R_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{i,j_i,n-k}(x), \qquad (2.25)$$
$$n \ge 0,\; i = 1,2,\ldots,M,\; j_i = 1,2,\ldots,k_i$$

$$I_n(0) = \delta_0\int_0^\infty P_{0,n-1}(x)\mu_0(x)dx + \sum_{i=1}^{M}\delta_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-1}(x)\mu_{i,j_i}(x)dx + q_0\int_0^\infty P_{0,n}(x)\mu_0(x)dx$$
$$+\sum_{i=1}^{M} q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx + \int_0^\infty S_n(x)r(x)dx$$
$$+\int_0^\infty R_{0,n}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx,\; n \ge 1 \qquad (2.26)$$

$$P_{0,0}(0) = \alpha\lambda^+ C_1 I_0 + \alpha\int_0^\infty I_1(x)\eta(x)dx \qquad (2.27)$$

$$P_{0,n}(0) = \alpha\lambda^+ C_{n+1} I_0 + \alpha\lambda^+\sum_{k=1}^{n} C_k\int_0^\infty I_{n-k+1}(x)dx + \alpha\int_0^\infty I_{n+1}(x)\eta(x)dx,\; n \ge 1 \qquad (2.28)$$

$$P_{1,j_1,n}(0) = p_{j_1}\int_0^\infty P_{0,n}(x)\mu_0(x)dx,\; n \ge 0,\; j_1 = 1,2,\ldots,k_1 \qquad (2.29)$$

$$P_{i,j_i,n}(0) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1},n}(x)\mu_{i-1,j_{i-1}}(x)dx,\; n \ge 0,\; i = 2,3,\ldots\ldots,M,\; j_i = 1,2,\ldots,k_i \qquad (2.30)$$

$$S_1(0) = \bar{\alpha}\lambda^+ C_1 I_0 + \bar{\alpha}\int_0^\infty I_1(x)\eta(x)dx \qquad (2.31)$$

$$S_n(0) = \bar{\alpha}\,\lambda^+ C_n I_0 + \bar{\alpha}\,\lambda^+\sum_{k=1}^{n} C_k\int_0^\infty I_{n-k}(x)dx + \bar{\alpha}\int_0^\infty I_n(x)\eta(x)dx,\; n \ge 2 \qquad (2.32)$$

$$R_{0,n}(0) = \lambda^{-}\int_0^\infty P_{0,n}(x)dx\ ,\ n \ge 0 \tag{2.33}$$

$$R_{i,j_i,n}(0) = \lambda^{-}\int_0^\infty P_{i,j_i,n}(x)dx\ ,\ n \ge 0,\ i = 1,2,3,......,M,\ j_i = 1,2,....,k_i \tag{2.34}$$

The normalizing condition is

$$I_0 + \sum_{n=1}^{\infty}\int_0^\infty I_n(x)\ dx + \sum_{n=0}^{\infty}\int_0^\infty P_{0,n}(x)\ dx + \sum_{n=0}^{\infty}\int_0^\infty \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i,n}(x)\ dx + \sum_{n=1}^{\infty}\int_0^\infty S_n(x)\ dx$$

$$+ \sum_{n=1}^{\infty}\int_0^\infty R_{0,n}(x)\ dx + \sum_{n=0}^{\infty}\int_0^\infty \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i,n}(x)\ dx\ = 1 \tag{2.35}$$

2.2.3 Orbit Size Distribution at Random Epoch

To analyse the model, the following probability generating functions are defined.

$$\left.\begin{aligned}
I(x,z) &= \sum_{n=1}^{\infty} I_n(x)z^n\ ; & P_0(x,z) &= \sum_{n=0}^{\infty} P_{0,n}(x)z^n \\
P_{i,j_i}(x,z) &= \sum_{n=0}^{\infty} P_{i,j_i,n}(x)z^n\ ; & S(x,z) &= \sum_{n=1}^{\infty} S_n(x)z^n \\
R_0(x,z) &= \sum_{n=0}^{\infty} R_{0,n}(x)z^n \quad \text{and} & R_{i,j_i}(x,z) &= \sum_{n=0}^{\infty} R_{i,j_i}(x)z^n
\end{aligned}\right\} \tag{2.36}$$

Multiplying the equations (2.18) to (2.25) by z^n and summing over all possible values of n, we get

$$\left(\frac{\partial}{\partial x} + \lambda^{+} + \eta(x)\right) I(x,z) \ = \ 0 \tag{2.37}$$

$$\left(\frac{\partial}{\partial x} + \lambda^{+} + \lambda^{-} + \mu_0(x)\right) P_0(x,z) \ = \ 0 \tag{2.38}$$

$$\left(\frac{\partial}{\partial x} + \lambda^{+} + \lambda^{-} + \mu_{i,j_i}(x)\right) P_{i,j_i}(x,z) \ = \ 0,\ \ i = 1,2,....,M,\ j_i = 1,2,....,k_i \tag{2.39}$$

$$\left(\frac{\partial}{\partial x} + \lambda^{+} + r(x)\right) S(x,z) \ = \ 0 \tag{2.40}$$

$$\left(\frac{\partial}{\partial x} + \lambda^{+} + \beta_0(x)\right) R_0(x,z) \ = \ 0 \tag{2.41}$$

$$\left(\frac{\partial}{\partial x}+\lambda^{+}+\beta_{i,j_i}(x)\right)R_{i,j_i}(x,z) \;=\; 0,\; i=1,2,....,M,\;\; j_i=1,2,....,k_i \tag{2.42}$$

Using the definition in (2.36), the equations (2.26) to (2.34) yield

$$I(0,z)=(q_0+\delta_0\, z)\int_0^{\infty}P_0(x,z)\,\mu_0(x)dx+\sum_{i=1}^{M}(q_i+\delta_i\, z)\sum_{j_i=1}^{k_i}\int_0^{\infty}P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx+\int_0^{\infty}S(x,z)\,r(x)dx$$
$$+\int_0^{\infty}R_0(x,z)\beta_0(x)dx+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty}R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx-\lambda^{+}I_0 \tag{2.43}$$

$$P_0(0,z)=\frac{\alpha}{z}[\;\lambda^{+}C(z)I_0+\int_0^{\infty}I(x,z)\eta(x)dx\;+\lambda^{+}C(z)\int_0^{\infty}I(x,z)dx] \tag{2.44}$$

$$P_{1,j_1}(0,z)=p_{j_1}\int_0^{\infty}P_0(x,z)\mu_0(x)dx,\;\; j_1=1,2,......,k_1 \tag{2.45}$$

$$P_{i,j_i}(0,z)=p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^{\infty}P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx,\quad i=2,3,......,M,\;\; j_i=1,2,......,k_i \tag{2.46}$$

$$S(0,z)=\frac{\bar{\alpha}}{z}[\;\lambda^{+}C(z)I_0+\int_0^{\infty}I(x,z)\eta(x)dx\;+\lambda^{+}C(z)\int_0^{\infty}I(x,z)dx] \tag{2.47}$$

$$R_0(0,z)=\lambda^{-}\int_0^{\infty}P_0(x,z)dx \tag{2.48}$$

$$R_{i,j_i}(0,z)=\lambda^{-}\int_0^{\infty}P_{i,j_i}(x,z)dx,\qquad i=1,2,3,......,M,\quad j_i=1,2,......,k_i \tag{2.49}$$

The solution of the partial differential equations (2.37) to (2.42) are obtained respectively as

$$I(x,z)=I(0,z)\;e^{-\lambda^{+}x}(1-A(x)) \tag{2.50}$$

$$P_0(x,z)=P_0(0,z)e^{-(\lambda^{+}+\lambda^{-}-\lambda^{+}C(z))x}(1-B_0(x)) \tag{2.51}$$

$$P_{i,j_i}(x,z)=P_{i,j_i}(0,z)e^{-(\lambda^{+}+\lambda^{-}-\lambda^{+}C(z))x}(1-B_{i,j_i}(x)),\; i=1,2,......,M,\;\; j_i=1,2,.....,k_i \tag{2.52}$$

$$S(x,z)=S(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-S(x)) \tag{2.53}$$

$$R_0(x,z)=R_0(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-R_0(x)) \tag{2.54}$$

$$R_{i,j_i}(x,z)=R_{i,j_i}(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-R_{i,j_i}(x)),\; i=1,2,......,M,\;\; j_i=1,2,.....,k_i \tag{2.55}$$

Using the equations (2.50) to (2.55) in the equations (2.43) to (2.49) and simplifying, we have

$$I(0,z) = (q_0+\delta_0 z)\,P_0(0,z)\,B_0^*(g(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}(q_i+\delta_i z)\,P_{i,j_i}(0,z)\,B_{i,j_i}^*(g(z))$$
$$+ S(0,z)\,S^*(h(z)) + R_0(0,z)\,R_0^*(h(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)\,R_{i,j_i}^*(h(z)) - \lambda^+ I_0 \qquad (2.56)$$

$$P_0(0,z) = \frac{\alpha}{z}\left[\lambda^+ C(z) I_0 + \left(A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))\right) I(0,z)\right] \qquad (2.57)$$

$$P_{1,j_1}(0,z) = p_{j_1}\,P_0(0,z)\,B_0^*(g(z)), \quad j_1 = 1,2,\ldots\ldots,k_1 \qquad (2.58)$$

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} P_{i-1,j_{i-1}}(0,z)\,B_{i-1,j_{i-1}}^*(g(z)),\ i = 2,3,\ldots\ldots,M,\ j_i = 1,2,\ldots\ldots,k_i \qquad (2.59)$$

$$S(0,z) = \bar{\alpha}\left[\lambda^+ C(z) I_0 + \left(A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))\right) I(0,z)\right] \qquad (2.60)$$

$$R_0(0,z) = \lambda^-\,P_0(0,z)\,(1 - B_0^*(g(z)))\big/g(z) \qquad (2.61)$$

$$R_{i,j_i}(0,z) = \lambda^-\,P_{i,j_i}(0,z)\,(1 - B_{i,j_i}^*(g(z)))\big/g(z),\ i = 1,2,3,\ldots\ldots,M,\ j_i = 1,2,\ldots\ldots,k_i \qquad (2.62)$$

where

$$g(z) = \lambda^+(1 - C(z)) + \lambda^- \quad \text{and} \quad h(z) = \lambda^+(1 - C(z))$$

Equation (2.59) recursively leads to

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\sum_{j_{i-2}=1}^{k_{i-2}} P_{i-2,j_{i-2}}(0,z)\,B_{i-2,j_{i-2}}^*(g(z))\,B_{i-1,j_{i-1}}^*(g(z))$$

$$= p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}}\sum_{j_{i-3}=1}^{k_{i-3}} P_{i-3,j_{i-3}}(0,z)\,B_{i-3,j_{i-3}}^*(g(z))\,B_{i-2,j_{i-2}}^*(g(z))\,B_{i-1,j_{i-1}}^*(g(z))$$

$$= p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}}\cdots\sum_{j_2=1}^{k_2} p_{j_2}\sum_{j_1=1}^{k_1} p_{j_1} P_0(0,z)\,B_0^*(g(z))\,B_{1,j_1}^*(g(z))$$
$$B_{2,j_2}^*(g(z))\ldots\ldots B_{i-2,j_{i-2}}^*(g(z))\,B_{i-1,j_{i-1}}^*(g(z))$$

$$= p_{j_i}\left[\sum_{j_1=1}^{k_1} p_{j_1}\,B_{1,j_1}^*(g(z))\sum_{j_2=1}^{k_2} p_{j_2}\,B_{2,j_2}^*(g(z))\ldots\ldots\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\,B_{i-1,j_{i-1}}^*(g(z))\right] B_0^*(g(z))\,P_0(0,z)$$

$$= p_{j_i}\left[\prod_{l=1}^{i-1}\sum_{j_l=1}^{k_l} p_{j_l} B^*_{l,j_l}(g(z))\right] B^*_0(g(z))\; P_0(0,z)$$

$$= p_{j_i}\Lambda^*_{i-1}(g(z))B^*_0(g(z))\; P_0(0,z)\;,\;\; i=2,3,......,M,\;\; j_i=1,2,......,k_i \qquad (2.63)$$

where

$$\Lambda^*_0(g(z))=1\;, \qquad \Lambda^*_i(g(z))=\prod_{l=1}^{i}\sum_{j_l=1}^{k_l} p_{j_l} B^*_{l,j_l}(g(z))$$

Substituting the expression (2.63) in the equation (2.62), we obtain

$$R_{i,j_i}(0,z) \;=\; \lambda^-\; p_{j_i}\Lambda^*_{i-1}(g(z))B^*_0(g(z))\; P_0(0,z)\;(1-B^*_{i,j_i}(g(z)))\big/g(z), \qquad i=1,2,3,......,M,\quad j_i=1,2,......,k_i \qquad (2.64)$$

Using the equations (2.60), (2.61), (2.63) and (2.64) in the equation (2.56) and simplifying, we get

$$I(0,z) \;=\; \frac{T_1(z)\;P_0(0,z)\;-\;\lambda^+ I_0\left(1-\bar{\alpha}\;C(z)S^*(h(z))\right)}{1-\bar{\alpha}\left(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))\right)S^*(h(z))} \qquad (2.65)$$

where

$$T_1(z)=(q_0+\delta_0\,z)\,B^*_0(g(z))+\sum_{i=1}^{M}(q_i+\delta_i\,z)\,\Lambda^*_i(g(z))B^*_0(g(z))+\lambda^-((1-B^*_0(g(z)))/g(z))$$
$$R^*_0(h(z))+\lambda^-\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda^*_{i-1}(g(z))((1-B^*_{i,j_i}(g(z)))/g(z))R^*_{i,j_i}(h(z))B^*_0(g(z))$$

Using the expression of I(0,z), the equation (2.57) yields

$$P_0(0,z) \;=\; \frac{\alpha\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)}{z-[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]\;(\bar{\alpha}\,z\,S^*(h(z))+\alpha\;T_1(z))} \qquad (2.66)$$

Substituting the expression of $P_0(0,z)$ in equations (2.61), (2.63) to (2.65), we obtain respectively

$$I(0,z) \;=\; \frac{\lambda^+ I_0\,[\alpha A^*(\lambda^+)(C(z)-1)T_1(z)-D(z)(1-\bar{\alpha}C(z)\,S^*(h(z)))]}{D(z)[1-\bar{\alpha}(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))S^*(h(z))]} \qquad (2.67)$$

$$P_{i,j_i}(0,z) = \frac{p_{j_i}\alpha\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)\Lambda^*_{i-1}(g(z))B^*_0(g(z))}{D(z)}, \; i=1,2,3,......,M, \; j_i = 1,2,......,k_i \quad (2.68)$$

$$R_0(0,z) = \frac{\lambda^-\alpha\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)((1-B^*_0(g(z)))/g(z))}{D(z)} \quad (2.69)$$

$$R_{i,j_i}(0,z) = \frac{\lambda^-\lambda^+ I_0\,\alpha\, A^*(\lambda^+)(C(z)-1)p_{j_i}\Lambda^*_{i-1}(g(z))((1-B^*_{i,j_i}(g(z)))/g(z))B^*_0(g(z))}{D(z)},$$

$$i=1,2,3,......,M, \; j_i = 1,2,......,k_i \quad (2.70)$$

where

$$D(z) = z-[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]\,(\bar{\alpha}\, z S^*(h(z))+\alpha\, T_1(z))$$

Using equation (2.67), the equation (2.60) yields

$$S(0,z) = \frac{\bar{\alpha}\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)[D(z)+\alpha(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))T_1(z)]}{D(z)[1-\bar{\alpha}(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))S^*(h(z))]} \quad (2.71)$$

Using the solutions the probability generating functions of the orbit size at different states of the server are derived.

- The probability generating function of the orbit size when the server is idle is given by

$$I(z) = \int_0^\infty I(x,z)dx$$

$$= I(0,z)\int_0^\infty e^{-\lambda^+ x}(1-A(x))\,dx$$

$$= \frac{I_0(1-A^*(\lambda^+))[A^*(\lambda^+)T_1(z)\alpha(C(z)-1)-(1-\bar{\alpha}C(z)S^*(h(z)))D(z)]}{D(z)[1-\bar{\alpha}(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))S^*(h(z))]} \quad (2.72)$$

- The probability generating function of the orbit size when the server is busy in first phase is

$$P_0(z) = \int_0^\infty P_0(x,z)dx$$

$$= P_0(0,z)\int_0^\infty e^{-(\lambda^+ +\lambda^- -\lambda^+ C(z))x}(1-B_0(x))\,dx$$

$$= \frac{\alpha\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)((1-B_0^*(g(z)))/g(z))}{D(z)} \qquad (2.73)$$

- The probability generating function of the orbit size when the server is busy in second phase service is

$$P(z) = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z)$$

$$= \int_0^\infty \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(x,z)\,dx$$

$$= \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(0,z)\int_0^\infty e^{-(\lambda^+ +\lambda^- -\lambda^+ C(z))x}(1-B_{i,j_i}(x))\,dx$$

$$= \frac{\alpha\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))}{D(z)} \qquad (2.74)$$

- The probability generating function of the orbit size when the server is under repair due to starting failure is

$$S(z) = \int_0^\infty S(x,z)\,dx$$

$$= S(0,z)\int_0^\infty e^{-(\lambda^+ -\lambda^+ C(z))x}(1-S(x))\,dx$$

$$= \frac{\bar{\alpha} I_0 A^*(\lambda^+)[\alpha(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))T_1(z)+D(z)](S^*(h(z))-1)}{D(z)[1-\bar{\alpha}(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))S^*(h(z))]} \qquad (2.75)$$

- The probability generating function of the orbit size when the server is under repair due to negative arrival during essential service is given by

$$R_0(z) = \int_0^\infty R_0(x,z)\,dx$$

$$= R_0(0,z)\int_0^\infty e^{-(\lambda^+ -\lambda^+ C(z))x}(1-R_0(x))\,dx$$

$$= \frac{\lambda^{-}\alpha I_0 A^*(\lambda^+)((1-B_0^*(g(z)))/g(z))(R_0^*(h(z))-1)}{D(z)} \tag{2.76}$$

- The probability generating function of the orbit size when the server in second phase is under repair is

$$R(z) = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \int_0^\infty \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(x,z)dx$$

$$= \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)\int_0^\infty e^{-(\lambda^+ - \lambda^+ C(z))x}(1-R_{i,j_i}(x))\,dx$$

$$= \frac{\lambda^{-}\alpha I_0 A^*(\lambda^+)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(g(z))\Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))(R_{i,j_i}^*(h(z))-1)}{D(z)} \tag{2.77}$$

Using the normalizing condition, the analytical expression for I_0 is obtained as

$$I_0 = (1-m_1(1-A^*(\lambda^+)) - \bar{\alpha}(1+\lambda^+ m_1\gamma_1) - \alpha T_1'(1))/\{\alpha A^*(\lambda^+)[1-B_0^*(\lambda^-)\sum_{i=0}^{M}\delta_i \Lambda_i^*(\lambda^-)$$
$$+\sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} - \sum_{i=1}^{M}(\delta_i + q_i)\left(M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}\right) + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)}$$
$$-h_2 B_0^*(\lambda^-) - h_1 f_0^{(1)}]\}$$

where

$$T_1'(1) = B_0^*(\lambda^-)\sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-) - \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} + \sum_{i=1}^{M}(\delta_i + q_i)[M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}]$$
$$+(1-B_0^*(\lambda^-))[(1/\lambda^-) + \beta_0^{(1)}] + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))$$
$$[(1/\lambda^-) + \beta_{i,j_i}^{(1)}] - B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} + h_1 f_0^{(1)} + h_2 B_0^*(\lambda^-)$$

$$h_1 = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))$$

$$h_2 = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))$$

$$f_0^{(1)} = \lambda^+ m_1 \int_0^\infty x e^{-\lambda^- x} b_0(x)dx \ , \qquad f_{i,j_i}^{(1)} = \lambda^+ m_1 \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

$$M_i^{(1)} = \lim_{z\to 1} \Lambda_i^{*\prime}(g(z)) = \sum_{m=1}^{i}\left[\sum_{j_m=1}^{k_m} p_{j_m} B_{m,j_m}^{*\prime}(\lambda^-)\left(\prod_{\substack{n=1\\ n\neq m}}^{i}\sum_{j_n=1}^{k_n} p_{j_n} B_{n,j_n}^{*}(\lambda^-)\right)\right],$$

$$M_i^{(2)} = \lim_{z\to 1} \Lambda_i^{*\prime\prime}(g(z))$$

$$= \sum_{m=1}^{i}\left[\sum_{j_m=1}^{k_m} p_{j_m} B_{m,j_m}^{*\prime\prime}(\lambda^-)\left(\prod_{\substack{n=1\\ n\neq m}}^{i}\sum_{j_n=1}^{k_n} p_{j_n} B_{n,j_n}^{*}(\lambda^-)\right)\right]$$

$$+ 2\sum_{m=1}^{i-1}\left[\left(\sum_{j_m=1}^{k_m} p_{j_m} B_{m,j_m}^{*\prime}(\lambda^-)\right)\left(\sum_{j_{m+1}=1}^{k_{m+1}} p_{j_{m+1}} B_{m+1,j_{m+1}}^{*\prime}(\lambda^-)\right)\left(\prod_{\substack{n=1\\ n\neq m\\ n\neq m+1}}^{i}\sum_{j_n=1}^{k_n} p_{j_n} B_{n,j_n}^{*}(\lambda^-)\right)\right]$$

The probability generating function of the orbit size $P_q(z)$ and the system size $P_s(z)$ are respectively given by

$$P_q(z) = I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + S(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \alpha I_0 A^*(\lambda^+)\{ z - \sum_{i=0}^{M} (\delta_i z + q_i)\Lambda_i^*(g(z)) B_0^*(g(z))$$

$$+ (\lambda^+(C(z)-1) - \lambda^-)\,[((1 - B_0^*(g(z))/g(z))$$

$$+ B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))((1 - B_{i,j_i}^*(g(z))/g(z))))]\}/\ D(z) \qquad (2.78)$$

$$P_s(z) = I_0 + I(z) + zP_0(z) + z\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + S(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \alpha I_0 A^*(\lambda^+)\{ z - \sum_{i=0}^{M} (\delta_i z + q_i)\Lambda_i^*(g(z)) B_0^*(g(z))$$

$$+ (z\lambda^+(C(z)-1) - \lambda^-)\,[((1 - B_0^*(g(z))/g(z))$$

$$+ B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))((1 - B_{i,j_i}^*(g(z))/g(z))))]\}/\ D(z) \qquad (2.79)$$

2.3 Stability Condition

The necessary and sufficient condition for the system to be stable is

$$m_1(1-A^*(\lambda^+)) + \bar{\alpha}(1+\lambda^+ m_1\gamma_1) + \alpha T_1'(1) \; < \; 1$$

2.4 Performance Measures

- The probability that the server is idle in the non-empty system and the corresponding mean number of customers in the orbit are given by

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{(1-A^*(\lambda^+))\left(m_1 A^*(\lambda^+) - T_2\right)}{T_4} \tag{2.80}$$

$$L_I = \lim_{z\to 1} \frac{d}{dz} I(z)$$

$$= I_0(1-A^*(\lambda^+))\{\alpha^2 T_2 A^*(\lambda^+)[m_2 + 2m_1 T_1'(1)] - \alpha\, m_1 A^*(\lambda^+)[\alpha T_3 - 2T_2\bar{\alpha}(m_1(1-A^*(\lambda^+)) + \lambda^+ m_1\gamma_1)] + \alpha\bar{\alpha}\, T_2^{\,2}\, m_1 A^*(\lambda^+)/2(\alpha T_2)^2 \tag{2.81}$$

where

$$T_2 = 1 - m_1(1-A^*(\lambda^+)) - \bar{\alpha}(1+\lambda^+ m_1\gamma_1) - \alpha T_1'(1)$$

$$T_3 = m_2(1-A^*(\lambda^+)) + 2m_1(1-A^*(\lambda^+))[\,\bar{\alpha} + \bar{\alpha}\lambda^+ m_1\gamma_1 + \alpha T_1'(1)] + 2\bar{\alpha}\lambda^+ m_1\gamma_1 + \bar{\alpha}((\lambda^+ m_1)^2\gamma_2 + \lambda^+ m_2\gamma_1) + \alpha T_1''(1)$$

$$T_4 = \alpha A^*(\lambda^+)\{1 - \sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-) - \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} - \sum_{i=1}^{M}(\delta_i + q_i)[M_i^{(1)}B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-)f_0^{(1)}] + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))f_{i,j_i}^{(1)} - h_1 f_0^{(1)} - h_2 B_0^*(\lambda^-)\}$$

$$T_1''(1) = 2\,[\sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-)f_0^{(1)} + \sum_{i=1}^{M}\delta_i M_i^{(1)}B_0^*(\lambda^-)] - \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(2)} + \sum_{i=1}^{M}(\delta_i + q_i)[M_i^{(2)}B_0^*(\lambda^-) + M_i^{(1)}f_0^{(1)} + \Lambda_i^*(\lambda^-)f_0^{(2)}] + 2(\lambda^+ m_1/\lambda^-)[-f_0^{(1)} + h_2 B_0^*(\lambda^-) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)f_{i,j_i}^{(1)}B_0^*(\lambda^-) + h_1 f_0^{(1)}] - 2\lambda^+ m_1[-f_0^{(1)}\beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)} (1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)\beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)}B_0^*(\lambda^-) + (1-B_{i,j_i}^*(\lambda^-))$$

$$\beta_{i,j_i}^{(1)} f_0^{(1)}] + [2((\lambda^+ m_1)^2/\lambda^-) + \lambda^+ m_2]\left(T_5 + T_6\right) + (\lambda^+)^2 m_1^2 T_7 + h_1 f_0^{(2)}$$

$$-2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} f_0^{(1)} - 2B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)} f_{i,j_i}^{(1)} + 2h_2 f_0^{(1)}$$

$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(2)}(1 - B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(2)} B_0^*(\lambda^-)$$

$$T_5 = (1/\lambda^-)[1 - B_0^*(\lambda^-) + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)))]$$

$$T_6 = (1 - B_0^*(\lambda^-))\beta_0^{(1)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)})$$

$$T_7 = (1 - B_0^*(\lambda^-)))\beta_0^{(2)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(2)}$$

$$f_0^{(2)} = (\lambda^+ m_1)^2 \int_0^\infty x^2 e^{-\lambda^- x} b_0(x)dx + \lambda^+ m_2 \int_0^\infty x e^{-\lambda^- x} b_0(x)dx$$

$$f_{i,j_i}^{(2)} = (\lambda^+ m_1)^2 \int_0^\infty x^2 e^{-\lambda^- x} b_{i,j_i}(x)dx + \lambda^+ m_2 \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

- The probability that the server is busy in essential service and the corresponding mean number of customers in the orbit are given by

$$P_0 = \lim_{z\to 1} P_0(z)$$

$$= \frac{\alpha \lambda^+ m_1 A^*(\lambda^+)(1 - B_0^*(\lambda^-))}{\lambda^- T_4} \qquad (2.82)$$

$$L_{P_0} = \lim_{z\to 1} \frac{d}{dz} P_0(z)$$

$$= I_0 \alpha A^*(\lambda^+)\{T_2[\lambda^+ m_2(1 - B_0^*(\lambda^-)) - 2\lambda^+ m_1 f_0^{(1)} + 2((\lambda^+ m_1)^2/\lambda^-)(1 - B_0^*(\lambda^-))]$$

$$+ T_3[\lambda^+ m_1(1 - B_0^*(\lambda^-))] \ / \ 2\lambda^- T_2^2 \qquad (2.83)$$

- The probability that the server is busy in providing optional services and the corresponding mean number of customers in the orbit are given by

$$P = \lim_{z\to 1} P(z)$$

$$= \frac{\alpha \lambda^+ m_1 A^*(\lambda^+) B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))}{\lambda^- T_4} \qquad (2.84)$$

$$L_P = \lim_{z\to 1} \frac{d}{dz} P(z)$$

$$= I_0 \alpha A^*(\lambda^+)\{T_2[\lambda^+ m_2 h_1 B_0^*(\lambda^-) + 2\lambda^+ m_1 [h_1 f_0^{(1)} + B_0^*(\lambda^-)(h_2 - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)$$
$$f_{i,j_i}^{(1)})] + 2((\lambda^+ m_1)^2/\lambda^-) h_1 B_0^*(\lambda^-)] + T_3[\lambda^+ m_1 h_1 B_0^*(\lambda^-)]\}/2\lambda^- T_2^2 \tag{2.85}$$

- The probability that the server is under repair due to starting failure is

$$S = \lim_{z\to 1} S(z)$$
$$= \frac{\bar{\alpha}\lambda^+ m_1 A^*(\lambda^+)\gamma_1}{T_4} \tag{2.86}$$

- The probability that the server is under repair due to negative arrival is

$$R = \lim_{z\to 1} \left(R_0(z) + R(z)\right)$$
$$= \frac{\alpha\lambda^+ m_1 A^*(\lambda^+)\{(1-B_0^*(\lambda^-))\beta_0^{(1)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(1-B_{i,j_i}^*(\lambda^-))\Lambda_{i-1}^*(\lambda^-)\beta_{i,j_i}^{(1)}\}}{T_4} \tag{2.87}$$

- Mean number of customers in the orbit when the server is in failure state is

$$L_R = \lim_{z\to 1} \frac{d}{dz}[R_0(z) + R(z) + S(z)]$$

$$= I_0 A^*(\lambda^+)\{T_2[\alpha[(\lambda^+)^2 m_1^2 T_7 + \lambda^+ m_2 T_6 - 2\lambda^+ m_1 [f_0^{(1)}\beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)$$
$$(1-B_{i,j_i}^*(\lambda^-))f_0^{(1)}\beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)$$
$$B_0^*(\lambda^-))f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)}] + 2((\lambda^+ m_1)^2/\lambda^-)T_6] + \bar{\alpha}[(\lambda^+)^2 m_1^2 \gamma_1 + \lambda^+ m_2 \gamma_2$$
$$-2\lambda^+ m_1 \gamma_1 (T_1'(1) - (1/\alpha) m_1 (1 - A^*(\lambda^+)) + (1/\alpha) T_2 - (\bar{\alpha}/\alpha)\lambda^+ m_1 \gamma_1)]$$
$$+ T_3[\alpha\lambda^+ m_1 T_6 + \bar{\alpha}\lambda^+ m_1 \gamma_1]\}/2T_2^2 \tag{2.88}$$

- Expected orbit size is given by

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z)$$

$$= I_0 \alpha A^*(\lambda^+)\{T_2[\sum_{j_1=1}^{k_1} p_{j_1} f_0^{(2)} - 2\sum_{i=1}^{M} \delta_i M_i^{(1)} B_0^*(\lambda^-) - 2\sum_{i=0}^{M} \delta_i \Lambda_i^*(\lambda^-) f_0^{(1)}$$

$$-\sum_{i=1}^{M} (\delta_i + q_i)(M_i^{(2)} B_0^*(\lambda^-) + 2M_i^{(1)} f_0^{(1)} + \Lambda_i^*(\lambda^-) f_0^{(2)}) - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(2)} B_0^*(\lambda^-)$$

$$(1 - B_{i,j_i}^*(\lambda^-)) + 2h_2 f_0^{(1)} + h_1 f_0^{(2)} + 2\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)}) f_{i,j_i}^{(1)}$$

$$+\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-)) f_{i,j_i}^{(2)}] + T_3 T_4\} / 2 T_2^2 \qquad (2.89)$$

- Expected system size is given by

$$L_s = L_q + P_0 + P \qquad (2.90)$$

2.5 Reliability Indices

The system availability $\mathcal{A}$ (t) at time t is the probability that the server is either working or in an idle period.

The steady state availability of the server is

$$\mathcal{A} = I_0 + \lim_{z\to 1} [\int_0^\infty I(x,z)\,dx + \int_0^\infty P_0(x,z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i}(x,z)\,dx]$$

$$= I_0 + \lim_{z\to 1} [I(z) + P_0(z) + P(z)]$$

$$= I_0 + I + P_0 + P$$

$$= I_0 A^*(\lambda^+)\{\alpha - \bar{\alpha}\lambda^+ m_1 \gamma_1 - \alpha T_1'(1) + \alpha(\lambda^+ m_1/\lambda^-)[1 - B_0^*(\lambda^-)$$

$$+ B_0^*(\lambda^-)\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))]\} / T_2 \qquad (2.91)$$

The steady state failure frequency of the server is

$$\mathcal{F} = \lambda^- \lim_{z\to 1} [\int_0^\infty P_0(x,z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i}(x,z)\,dx]$$

$$= \lambda^- \lim_{z\to 1} [P_0(z) + P(z)]$$

$$= \lambda^- (P_0 + P)$$

$$= I_0 A^*(\lambda^+)\, \alpha\, \lambda^+ m_1 [1 - B_0^*(\lambda^-) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))] / T_2$$

$$(2.92)$$

2.6 Stochastic Decomposition

In this section stochastic decomposition law for the model under consideration is verified.

Theorem 2.1

The number of customers in the system in steady state at a random time is distributed as the sum of two independent random variables, one of which is the number of customers in the classical $M^X/G/1$ G-queue with multistage and multi-optional services and feedback at a random point of time and the other is the mean number of customers in the orbit when the server is under repair due to starting failure or idle.

Proof.

Let $\phi(z)$ be the probability generating function of the number of customers in the classical batch arrival G-queue with multistage and multi-optional services and feedback.

$$\phi(z) = \lim_{A^*(\lambda^+)\to 1} P_s(z)$$

$$= \frac{\alpha I_0 \{ z - \sum_{i=0}^{M} (\delta_i z + q_i) \Lambda_i^*(g(z)) B_0^*(g(z)) + (z\lambda^+ (C(z)-1) - \lambda^-) [((1-B_0^*(g(z))/g(z)) + B_0^*(g(z)) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z))/g(z))))] \}}{(z - \bar{\alpha} z S^*(h(z)) - \alpha T_1(z))} \quad (2.93)$$

Let $\psi(z)$ be the probability generating function of the number of customers in the vacation system at a random point of time, given that the server is on vacation. In this context, server vacation represents the server is either under repair due to starting failure or idle.

$$\psi(z) = \frac{I_0 + I(z) + S(z)}{I_0 + I(1) + S(1)}$$

$$= \frac{[z - T_1(z)]\, T_2}{D(z)\, [1 - T_1'(1)]} \quad (2.94)$$

The equations (2.79), (2.93) and (2.94) imply

$$P_s(z) = \phi(z)\ \psi(z)$$

$$L_s = \lim_{z\to 1} \frac{d}{dz} P_s(z)$$

$$= \phi'(1) + \psi'(1)$$

$$= L_C + L_{Id}$$

Thus the mean number of customers in the system (L_s) is the sum of the mean number of customers in the batch arrival G-queue with multistage and multi-optional services and feedback (L_C) and the mean number of customers in the orbit when the server is under repair due to starting failure or idle (L_{Id}).

2.7 Special Cases

Case (i) : If $\lambda^- = 0$, M = 0, δ_0 = 0 and C(z) = z, then the system reduces to M/G/1 retrial queue with starting failure and the corresponding probability generating function of the system size and the probability that the server is idle in the empty system are given by

$$P_s(z) = \frac{\alpha I_0 A^*(\lambda^+)\ (z-1)\ B_0^*(\lambda^+(1-z))}{z-[z+(1-z)A^*(\lambda^+)][\ \alpha B_0^*(\lambda^+(1-z))+\overline{\alpha}\, z S^*(\lambda^+(1-z))]}$$

$$I_0 = \frac{A^*(\lambda^+)-\overline{\alpha}\,(1+\lambda^+\gamma_1)-\alpha\lambda^+\mu_0^{(1)}}{\alpha A^*(\lambda^+)[1-\lambda^+\mu_0^{(1)}]}$$

The above results agree with the results of Yang and Li (1994) and Mokaddis et al. (2007) with no vacation and by replacing δ as α.

Case (ii) : If $\lambda^- = 0$, M = 0 and δ_0 = 0, then our model can be reduced to a bulk arrival retrial queue with starting failures. Then the probability generating function of the system size becomes

$$P_s(z) = \frac{\alpha I_0 A^*(\lambda^+)\ (z-1)\ B_0^*(\lambda^+(1-C(z)))}{z-[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))][\ \alpha\ B_0^*(\lambda^+(1-C(z)))+\overline{\alpha}\, z S^*(\lambda^+(1-C(z)))]}$$

which agrees with the result of Murugan and Vijaykrishnaraj (2020) with no vacation.

2.8 Practical Justification of the Model

In call center scenario, group of callers (positive customers) who find the server line busy may join a virtual queue called orbit to try their request after some random time (retrial). If the service line is free, then one caller from the group can connect to IVR system (Interactive Voice Response) and others again try their request. IVR facility is considered as the first phase essential service in our model. All the subscribers can get the service of IVR system. After the completion of first phase essential service, the caller may satisfy with the response from IVR system and disconnects the call. Otherwise, the caller is connected to user agent services based on their queries. There are several backend support system for some complicated issues. This includes multi-layered services, where a layer represents say a level of expertise and customers could potentially be transferred through several layers until being served to satisfaction. This is considered as the multistage and multi-optional services.

If the caller is not satisfied with the response from IVR or user agent, then he will try again for the same service like the new customer. Those customers are called as feedback customers.

The server could fail on start up while receiving a call. During the time, the repairing works are carried out. This situation is named as starting failure in queueing theory.

Also, the system may be affected by a virus (negative customers) during the service period. This causes electronic failure (breakdown) in the system and the call at that moment is disconnected (which means it removes the positive customer being in service).

2.9 Numerical Results

In this section, numerical results are presented in order to study the effect of various parameters on the system performance. Assume that the retrial time, first phase service time, second phase service time, repair time due to starting failure, repair time at first phase due to negative arrival and repair time at second phase due to negative arrival are exponentially distributed with respective parameters η, μ_0 , μ_{i,j_i} , r, β_0, β_{i,j_i} . Set the default parameters as $\lambda^- = 0.1$, $\eta = 50$, $M = 3$, $k_1 = 2$, $k_2 = 3$,

$k_3 = 2$, $p_1 = [0.4, 0.3]$, $p_2 = [0.2, 0.3, 0.1]$, $p_3 = [0.4, 0.2]$, $\delta_0 = 0.2$, $q_0 = 0.1$, $\delta = [0.2\ 0.3\ 0.6]$, $q = [0.2\ 0.1\ 0.4]$, $\alpha = 0.8$, $\beta_0 = 2$, $r = 20$, $\mu_1 = [70\ 40]$, $\mu_2 = [62\ 42\ 52]$, $\mu_3 = [85\ 73]$, $\beta_1 = [1\ 3]$, $\beta_2 = [5\ 7\ 10]$, $\beta_3 = [12\ 14]$.

Table 2.1 presents the combined effect of λ^+ and μ_0 on the performance measures I_0 – the probability that the server is idle in the empty system, I – the probability that the server is idle in the non-empty system , P_0 – the probability that the server is busy in first phase , P – the probability that the server is busy in second phase, S – the probability that the server is under repair due to starting failure, R – the probability that the server is under repair due to negative arrival, L_q – expected orbit size and L_s – expected system size.

It shows that

- I_0 is a decreasing function of λ^+ and increasing functions of μ_0.
- I , P_0 , R , L_q and L_s are increasing functions of λ^+ and decreasing functions of μ_0.
- P and S are increasing functions of both λ^+ and μ_0.

Table 2.2 depicts the influence of β_0 and η on the performance measures L_I – the expected number of customers in the orbit when the server is idle in the non-empty system , L_{P_0} – the expected number of customers in the orbit when the server is busy in first phase , L_P – the expected number of customers in the orbit when the server is busy in second phase, L_R – the expected number of customers in the orbit when the server is under repair and L_s – the expected system size.

It is observed in Table 2.2 that increase in β_0 or η decreases L_I , L_{P_0} , L_P , L_R and L_s .

Fig. 2.2 to 2.5 display the effects of η and α on I_0 , I , S and L_s. It is observed that increase in η and α increases I_0 and decreases I, S and L_s .

Table 2.1 Effect of λ^+ and μ_0 on I_0 , I , P_0 , P , S , R , L_q , L_s

λ^+	μ_0	I_0	I	P_0	P	S	R	L_q	L_s
1.2	**15**	0.3339	0.0898	0.2647	0.2375	0.0500	0.0241	1.5773	2.0795
	20	0.4032	0.0883	0.1992	0.2383	0.0501	0.0209	1.1217	1.5593
	25	0.4450	0.0874	0.1597	0.2388	0.0501	0.0190	0.9257	1.3243
	30	0.4730	0.0869	0.1333	0.2391	0.0501	0.0177	0.8180	1.1904
1.4	**15**	0.2383	0.1044	0.3019	0.2709	0.0570	0.0275	2.4986	3.0714
	20	0.3177	0.1024	0.2272	0.2718	0.0571	0.0239	1.5883	2.0873
	25	0.3656	0.1012	0.1821	0.2723	0.0571	0.0216	1.2467	1.7012
	30	0.3977	0.1004	0.152	0.2727	0.0572	0.0201	1.0705	1.4951
1.6	**15**	0.1464	0.1189	0.3375	0.3028	0.0637	0.0308	4.4476	5.0879
	20	0.2355	0.1163	0.2539	0.3038	0.0638	0.0266	2.3087	2.8664
	25	0.2893	0.1148	0.2035	0.3044	0.0639	0.0242	1.6822	2.1901
	30	0.3253	0.1138	0.1698	0.3047	0.0639	0.0225	1.3887	1.8632
1.8	**15**	0.0578	0.1334	0.3715	0.3333	0.0701	0.0339	11.9609	12.6657
	20	0.1564	0.1301	0.2795	0.3344	0.0702	0.0293	3.6321	4.2460
	25	0.2159	0.1282	0.2240	0.3350	0.0703	0.0266	2.3309	2.8899
	30	0.2557	0.1269	0.1869	0.3354	0.0703	0.0247	1.8138	2.3361

Table 2.2 Effect of β_0 and η on L_I , L_{P_0} , L_P , L_R and L_s

β_0	η	L_I	L_{P_0}	L_P	L_R	L_s
2	15	0.5629	0.7253	1.3173	0.4490	4.8831
	20	0.3695	0.4293	0.7862	0.2993	2.9354
	25	0.2832	0.3317	0.6111	0.2503	2.2931
	30	0.2314	0.2831	0.5238	0.2259	1.9733
	35	0.1962	0.2540	0.4716	0.2114	1.7818
6	15	0.5552	0.6939	1.2610	0.4029	4.6764
	20	0.3667	0.4152	0.7609	0.2713	2.8426
	25	0.2816	0.3219	0.5934	0.2275	2.2284
	30	0.2302	0.2752	0.5095	0.2057	1.9208
	35	0.1952	0.2471	0.4591	0.1925	1.7360
10	15	0.5540	0.6887	1.2516	0.3953	4.6422
	20	0.3662	0.4130	0.7569	0.2669	2.8281
	25	0.2813	0.3204	0.5908	0.2241	2.2187
	30	0.2300	0.2740	0.5074	0.2028	1.9131
	35	0.1951	0.2461	0.4573	0.1899	1.7295
14	15	0.5534	0.6866	1.2478	0.3922	4.6282
	20	0.3661	0.4122	0.7553	0.2652	2.8223
	25	0.2812	0.3198	0.5897	0.2227	2.2148
	30	0.2300	0.2735	0.5066	0.2015	1.9101
	35	0.1950	0.2457	0.4567	0.1888	1.7269

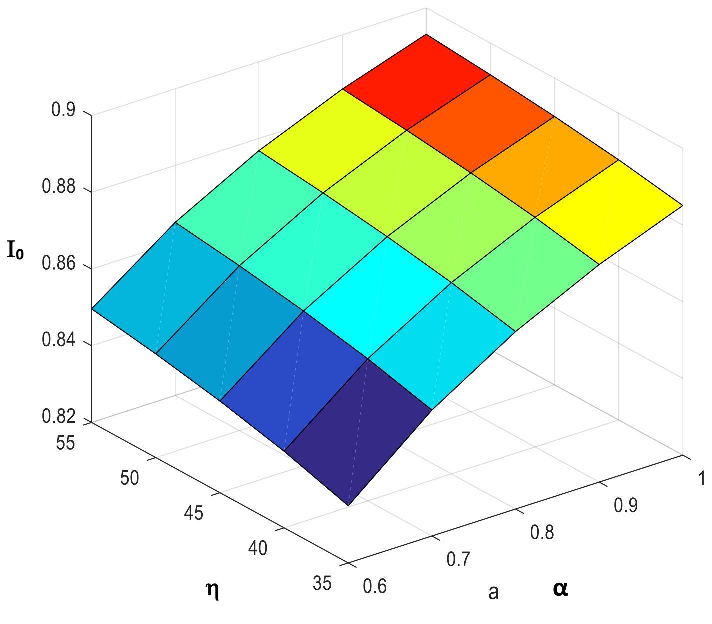

Fig. 2.2 Effect of η and α on I_0

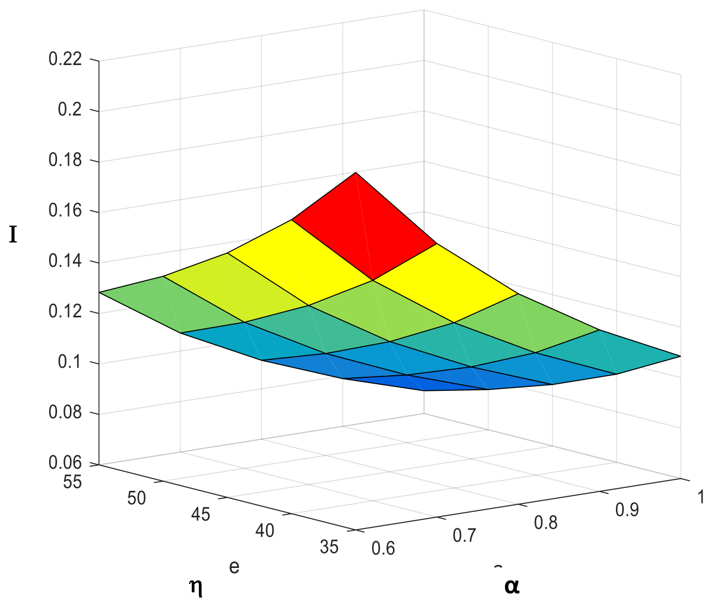

Fig. 2.3 Effect of η and α on I

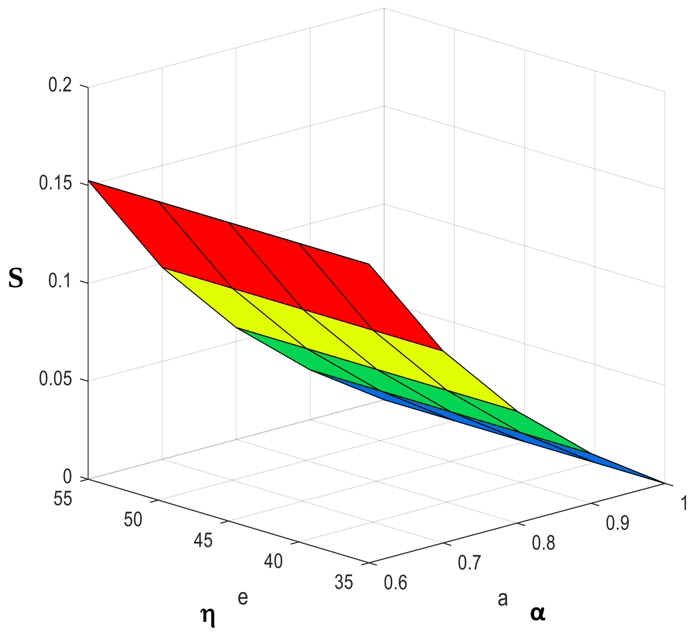

Fig. 2.4 Effect of η and α on S

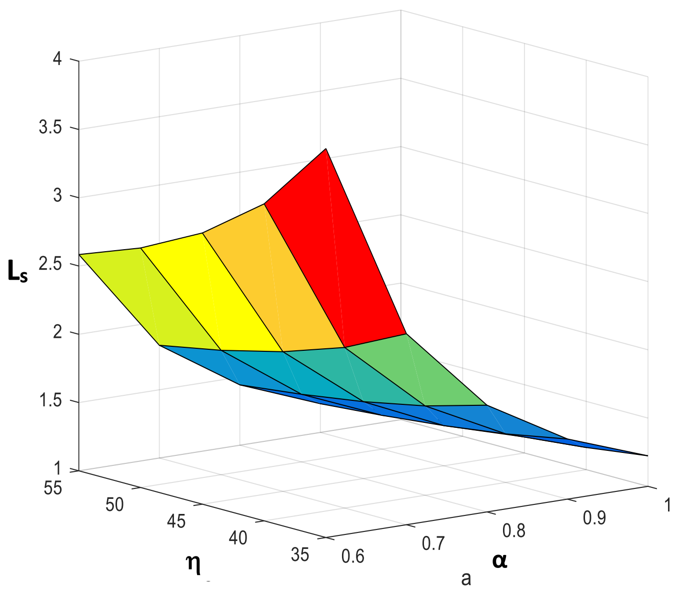

Fig. 2.5 Effect of η and α on L_s