CHAPTER - I

CHAPTER-I

SOFT SETS AND SOFT TOPOLOGICAL SPACES

SECTION 1.1

SOFT SETS

Definition 1.1.1

Let U be an initial universe set and E be the set of parameters. Let P(U) denote the power set of U and let $A \subseteq E$. A pair (F, A) is called a **soft set** over U, where F is a mapping given by $F: A \rightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

The soft set (F, A) is also denoted by F_A.

Definition 1.1.2

Let (F, A) and (G, B) be soft sets over a common universe U, we say that (F, A) is a **soft subset** of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$, if $A \subseteq B$ and for all $e \in A$, $F(e) \subseteq G(e)$.

Definition 1.1.3

Two soft sets (F, A) and (G, B) over a common universe U are said to be **soft equal** if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Notation 1.1.4

Let U be the initial universe. Let E be the set of parameters. The set of all soft sets over U is denoted as S(U, E) or S(U).

Definition 1.1.5

A soft set (F, A) over U is said to be a **null soft set** denoted by Φ if for all $e \in A$, $F(e) = \phi$.

Definition 1.1.6

A soft set (F, E) over U is said to be an **absolute soft set** denoted by $\widetilde{A}$ if for all $e \in A$, $F(e) = U$.

Definition 1.1.7

The **intersection of two soft sets** (F, A) and (G, B) over U is the soft set (H, C), where $C = A \cap B$ and $e \in C$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Definition 1.1.8

The **union of two soft sets** (F, A) and (G, B) over U is a soft set, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), \text{ if } \varepsilon \in A \backslash B, \\ G(\varepsilon) \text{ if } \varepsilon \in B \backslash A \text{ and} \\ F(\varepsilon) \cup G(\varepsilon) \text{ if } \varepsilon \in A \cap B. \end{cases}$$

This relationship is denoted by $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

Definition 1.1.9

The **complement of a Soft set** (F, E) is denoted by $(F, E)^C$ and is defined by $(F, E)^C = (F^C, E)$ where $F^C: E \rightarrow P(U)$ is a mapping given by $F^C(\alpha) = U - F(\alpha)$ for all $\alpha \in E$.

Definition 1.1.10

The **difference** (H, E) of two soft sets (F, E) and (G, E) over U, denoted by $(F, E) \backslash (G, E)$, is defined as $H(e) = F(e) \backslash G(e)$, for each $e \in E$.

SECTION 1.2

SOFT TOPOLOGICAL SPACES

Definition 1.2.1

Let τ be the collection of soft sets over U. Then τ is said to be a **soft topology** on U if

i. $\Phi, \widetilde{E}$ belongs to τ

ii. The union of any number of soft sets in τ belongs to τ.

iii. The intersection of any two soft sets in τ belongs to τ.

The triplet (U, τ, E) is called a **soft topological space** over U.

Definition 1.2.2

Let (U, τ, E) be a soft topological space over U, then members of τ are called **soft open** sets in U.

Definition 1.2.3

Let (U, τ, E) be a soft topological space over U. A soft set (F, E) over U is said to be a **soft closed** in U, if its complement $(F, E)^C$ belongs to τ.

Definition 1.2.4

Let (U, τ, E) be a soft topological space over U. A sub collection $\mathfrak{B}$ of τ is said to be a **base** for τ if every member of τ can be expressed as a union of members of $\mathfrak{B}$.

Definition 1.2.5

Let (U, τ, E) be a soft topological space over U. A sub collection $\mathfrak{B}$ of τ is said to a **subbase** for τ if the family of all finite intersections members $\mathfrak{B}$ forms a base for τ.

Definition 1.2.6

Let (U, τ, E) be a soft topological space over U and let (F, E) be a soft set over U. Then the **soft closure** of (F, E) over U, is denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E).

Definition 1.2.7

Let (U, τ, E) be a soft topological space over U and let (F, E) be a soft set over U. Then the **soft interior** of (F, E) over U, is denoted by $(F, E)^{o}$ and is defined as the union of all soft open sets contained in (F, E). Thus (F, E) is the largest fuzzy soft open set contained in (F, E).

Definition 1.2.8

Let (F, E) be a soft set over U. The soft set (F, E) is called a **soft point**, denoted by (x_e, E), if for the element $e \in E$, $F(e) = \{x\}$ and $F(e_1) = \phi$ for all $e_1 \in E - \{e\}$.

Definition 1.2.9

A soft set (F, E) in a soft topological space (U, τ, A) is called a **soft neighborhood of the soft point** $(x_e, E) \tilde{\in} (F, E)$, if there exists a soft open set (G, E) such that $(x_e, E) \tilde{\in} (G, E) \tilde{\subseteq} (F, E)$.

Definition 1.2.10

A soft set (G, E) in a soft topological space (U, τ, E) is called a **soft neighborhood** of the soft set (F, E) if there exists a soft open set (H, E) such that $(F, E) \tilde{\subseteq} (H, E) \tilde{\subseteq} (G, E)$.

SECTION 1.3

SOFT MAPPINGS

Definition 1.3.1

Let (U,τ,E) and (U',τ',E) be two soft topological spaces, $f:(U,\tau,E)\rightarrow(U',\tau',E)$ be a mapping. For each soft neighborhood (H,E) of $(f(x)_e,E)$, if there exists a soft neighborhood (F,E) of (x_e,E) such that $f((F,E))\subset(H,E)$, then f is said to be soft continuous mapping at (x_e,E).

If f is soft continuous mapping for all (x_e,E), then f is called **soft continuous mapping.**

Example 1.3.2

Let $U=\{h_1,h_2,h_3\}$, $E=\{e_1,e_2\}$, and $\tau=\{\Phi,\tilde{E},(F_1,E),(F_2,E)\}$, $\tau'=\{\Phi,\tilde{E},(G_1,E),(G_2,E)\}$ be two soft topologies defined on U, where $(F_1,E),(F_2,E),(G_1,E)$ and (G_2,E) are soft sets over U defined as follows:

$$F_1(e_1)=\{h_1,h_2\},F_1(e_2)=\{h_3\},F_2(e_1)=U,F_2(e_2)=\{h_3\},$$

and

$$G_1(e_1)=\{h_1\},G_1(e_2)=\{h_3\},G_2(e_1)=\{h_1,h_3\},G_2(e_2)=\{h_2,h_3\},$$

If we get the mapping $f:U\rightarrow U$ defined as

$$f(h_1)=f(h_2)=h_1,f(h_3)=h_3$$

Then since $f^{-1}(G_1,E)=(F_1,E)$ and $f^{-1}(G_2,E)=(F_2,E)$, f is a soft continuous mapping.

Theorem 1.3.3

Let (U,τ,E) and (U',τ',E) be two soft topological spaces, $f:(U,\tau,E)\rightarrow(U',\tau',E)$ be a mapping. Then the following conditions are equivalent:

i. $f:(U,\tau,E)\rightarrow(U',\tau',E)$ is a soft continuous mapping,

ii. For each soft open set (G,E) over U', $f^{-1}((G,E))$ is a soft open set over U,

iii. For each soft closed set (H,E) over U', $f^{-1}((G,E))$ is a soft closed set over U,

iv. For each soft set (F,E) over U, $\overline{f((F,E))} \subset \overline{(f(F,E))}$,

v. For each soft set (G,E) over U', $\overline{(f^{-1}(G,E))} \subset f^{-1}\overline{((G,E))}$,

vi. For each soft set (G,E) over U', $f^{-1}((G,E)^{\circ}) \subset \left(f^{-1}(G,E)\right)^{\circ}$.

Proof

(i)⇒(ii) Let (G,E) be a soft open set over U' and $(x_e,E) \in f^{-1}\big((G,E)\big)$ be an arbitrary soft point. Then $f(x_e,E) = (f(x)_e,E) \in (G,E)$. Since f is soft continuous mapping, there exists $(x_e,E) \in (F,E) \in \tau$ such that $f(F,E) \subset (G,E)$. This implies that $(x_e,E) \in (F,E) \subset f^{-1}\big((G,E)\big)$, $f^{-1}\big((G,E)\big)$ is a soft open set over U.

(ii) ⇒(i) Let (x_e,E) be a soft point and $(f(x)_e,E) \in (G,E)$ be an arbitrary soft neighborhood. Then $(x_e,E) \in f^{-1}\big((G,E)\big)$ is a soft neighborhood and $f(f^{-1}(G,E)) \subset (G,E)$.

(iii) ⇒(iv) Let (F,E) be a soft set over U. Since $(F,E) \subset f^{-1}\big((F,E)\big)$ and $f(F,E) \subset \overline{f(F,E)}$, we have $(F,E) \subset f^{-1}\big(f(F,E)\big) \subset f^{-1}\overline{f(F,E)}$. Since $f^{-1}\overline{f(F,E)}$ is a soft closed set over U, $\overline{(F,E)} \subset f^{-1}\overline{f(F,E)}$. Thus $f\overline{(F,E)} \subset f(f^{-1}\overline{f(F,E)}) \subset \overline{f(F,E)}$.

(iv) ⇒(v) Let (G,E) be a soft set over U' and $f^{-1}\big((G,E)\big) = (F,E)$. Then $f\big(\overline{(F,E)}\big) = f\overline{(f^{-1}(G,E))} \subset \overline{f(f^{-1}(G,E))} \subset \overline{(G,E)}$. Then $\overline{f^{-1}(G,E)} = \overline{(F,E)} \subset f^{-1}(f\overline{(F,E)}) \subset f^{-1}\big(\overline{(G,E)}\big)$.

(v) ⇒(vi) Let (G,E) be a soft set over U'. Substituting $(G,E)^C$ for condition (v). Then $f^{-1}((G,E)^C) \subset f^{-1}\overline{(G,E)^C}$. Since $(G,E)^{o} = (\overline{(G,E)^C})^C$ then we have, $f^{-1}((G,E)^{o}) = f^{-1}\left(\left(\overline{(G,E)^C}\right)^C\right) = (f^{-1}\overline{(G,E)^C})^C \subset \overline{(f^{-1}((G,E)^C))}^C = \overline{(f^{-1}((G,E)^C))}^C = (f^{-1}(G,E))^{o}$.

(vi) ⇒ (ii) Let (G,E) be a soft open set over U'. Then since $(f^{-1}(G,E))^{o} \subset f^{-1}(G,E) = (f^{-1}(G,E)^{o}) \subset (f^{-1}(G,E))^{o}$, $(f^{-1}(G,E))^{o} = f^{-1}(G,E)$. This implies that $f^{-1}(G,E)$ is a soft open set over U.

Definition 1.3.4

Let (U, τ, E) and (U', τ', E) be two soft topological spaces $f: U \to U'$ be a mapping.

i. If the image $f((F, E))$ of each soft open set (F, E) over U is a soft open set in U', then f is said to be a **soft open mapping.**

ii. If the image $f((H, E))$ of each soft closed set (H, E) over U is a soft closed set in U', then f is said to be a **soft closed mapping**.

Theorem 1.3.5

Let (U, τ, E) and (V, τ', E) be two soft topological spaces, $f: U \to V$ be a mapping.

i. f is a soft open mapping if and only if for each soft set (F, E) over U, $f((F, E)^{\circ}) \subset (f(F, E))^{\circ}$ is satisfied.

ii. f is a soft closed mapping if and only if for each soft set (F, E) over U, $\overline{f(F, E))} \subset \overline{(f(F, E))}$ is satisfied.

Proof

(i) Let f be a soft open mapping and (F, E) be a soft set over U. $(F, E)^{\circ}$ is a soft open set and $(F, E)^{\circ} \subset (F, E)$. Since f is a soft open mapping, $f((F, E)^{\circ})$ is a soft open set in V and $f((F, E)^{\circ}) \subset f((F, E))$. Thus $f((F, E)^{\circ}) \subset (f(F, E))^{\circ}$.

Conversely, let (F, E) be any soft open set over U. Then $(F, E) = (F, E)^{\circ}$. From the condition of theorem, we have $f((F, E)^{\circ}) \subset (f(F, E))^{\circ}$. Then $f((F, E)) = (F, E)^{\circ} \subset (f(F, E))^{\circ} \subset f(F, E)$. This implies that $f((F, E)) = (f(F, E))^{\circ}$. This completes the proof.

(ii) Let f be a soft closed mapping and (F, E) be any soft set over U. Since f is a soft closed mapping, $\overline{(f(F, E))}$ is a soft closed set over V and $f((F, E)) \subset \overline{(f(F, E))}$. Thus $\overline{f(F, E))} \subset \overline{(f(F, E))}$.

Conversely, let (F, E) be any soft closed set over U. From the condition of theorem, $\overline{(f(F, E))} \subset \overline{f(F, E))} = f((F, E)) \subset \overline{(f(F, E))}$. This means that $\overline{(f(F, E))} \subset \overline{f(F, E))}$. This completes the proof.

Definition 1.3.6

Let (U, τ, E) and (U', τ', E) be two soft topological spaces, $f: U \to U'$ be a mapping. If f is a bijection, soft continuous and f^{-1} is a soft continuous mapping, then f is said to be **soft homeomorphism** from U to U'. When a homeomorphism f exists between U and U', we say that U is soft homeomorphic to U'.

Theorem 1.3.7

Let (U, τ, E) and (U', τ', E) be two soft topological spaces, $f: U \to U'$ be a bijective mapping. Then the following conditions are equivalent:

i. f is a soft homeomorphism,
ii. f is a soft continuous and soft closed mapping,
iii. f is a soft continuous and soft open mapping.

SECTION 1.4

SOFT COMPACTNESS

Definition 1.4.1

A family $\psi = \{(F_i, E)\}_{i \in I}$ of soft sets is a cover of a soft set (F, E) if

$$(F, E) \ \tilde{\subseteq} \ \cup_{i \in I}(F_i, E).$$

It is a **soft open cover** if each member of ψ is a soft open set. A subcover of ψ is a subfamily of ψ which is also a cover.

A soft topological space (U, τ, E) is said to be **soft compact** if each soft open cover of (U, E) has a finite subcover.

Definition 1.4.2

Let (U, τ, E) and (V, τ', E) be soft topological spaces. If $\tau \ \tilde{\subseteq} \ \tau'$, then τ' is **soft finer than**τ. If $\tau \ \tilde{\subseteq} \ \tau'$ or $\tau' \ \tilde{\subseteq} \ \tau$, then τ is **soft comparable** with τ'.

Theorem 1.4.3

Let (U, τ', E) be a soft compact space and $\tau \ \tilde{\subseteq} \ \tau'$. Then (U, τ, E) is soft compact.

Proof

Let $\{(F_i, E)\}_{i \in I}$ be a soft open cover of $\tilde{E}$ by soft open sets of (U, τ, E). Since $\tau \ \tilde{\subseteq} \ \tau'$, then $\{(F_i, E)\}_{i \in I}$ is a soft open cover of $\tilde{E}$ by soft open sets of (U, τ', E). But (U, τ', E) is soft compact. Therefore

$$(U, E) \ \tilde{\subseteq} \ (F_{i_1}, E) \cup \ldots \cup (F_{i_n}, E)$$

for some $i_1, \ldots, i_n \in I$. Hence (U, τ, E) is soft compact.

Theorem 1.4.4

Let (F, E), (G, E), (H, E) and (I, E) be soft sets in $S(U, E)$. Then the following hold.

(i) $(F, E) \ \tilde{\subseteq} \ (G, E)$ if and only if $(F, E) \cap (G, E) = (F, E)$;

(ii) $(F, E) \ \tilde{\subseteq} \ (G, E), (H, E)$ if and only if $(F, E) \ \tilde{\subseteq} \ (G, E) \cap (H, E)$;

(iii) If $(F,E) \cong (H,E)$ and $(G,E) \cong (I,E)$, then $(F,E) \cup (G,E) \cong (H,E) \cup (I,E)$;

(iv) $(F,E) \cap (F,E)^C = \Phi$;

(v) $(F,E) \cap (G,E) = \Phi$ if and only if $(F,E) \cong (G,E)^C$;

(vi) $(F,E) \cong (G,E)$ if and only if . $(G,E)^C \cong (F,E)^C$.

Proof

We prove (iii) Let $(F,E) \cup (G,E) = (J,E)$ and $(H,E) \cup (I,E) = (K,E)$. Since $(F,E) \cong (H,E)$ and $(G,E) \cong (I,E)$ then $F(e) \cong H(e)$ and $G(e) \cong I(e)$, $\forall e \in E$.

Therefore

$$J(e) = F(e) \cup G(e) \subseteq H(e) \cup I(e) = K(e).$$

Hence $(J,E) \cong (K,E)$.

The proof of (i), (ii), (iv), (v) and (vi) are similar.

Theorem 1.4.5

Let (Y, τ_Y, E) be a soft subspace of a soft space (U, τ, E). Then (Y, τ_Y, E) is soft compact if and only if every cover of $\tilde{Y}$ by soft open sets in U contains a finite subcover.

Theorem 1.4.6

Every soft compact subspace of a soft Hausdorff space is soft closed.

Proof

Let (Y, τ_Y, E) be a soft compact subspace of soft Hausdorff space (U, τ, E). Let $x \in (U,E) - (Y,E)$. Then for all $y \in (Y,E)$, $x \neq y$. Therefore, there exist soft open sets (U_y, E) and (U_{xy}, E) containing x and y respectively such that $(U_y, E) \cap (U_{xy}, E) = \Phi$. Obviously, $\{(U_{xy}, E)\}_{y \in Y}$ is a cover of $\tilde{Y}$ by soft open sets in U. By Therefore we have $(Y,E) = (U_{xy}, E) \cup \ldots \cup (U_{xyn}, E)$ for some $y_1, \ldots, y_n \in Y$. Now, $x \in (U_y, E) \cap \ldots \cap (U_{yn}, E) = (U_x, E)$ and $(U_x, E) \cap (Y,E) = \Phi_E$. Hence $x \in (U_x, E) \cong (U,E) - (Y,E)$. Then $(U,E) - (Y,E) = \cup_{x \in X-Y}(U_x, E)$. Therefore $(U,E) - (Y,E)$ is soft open. Hence (Y,E) is soft closed.

Theorem 1.4.7

Every soft closed subset of a soft compact space is soft compact.

Proof

Let (Y, τ_Y, E) be a soft subspace of a soft compact space (U, τ, E) such that (Y, E) is a soft closed in U. Let $\{(F_i, E)\}_{i\in I}$ be a cover of $\widetilde{Y}$ by soft open sets in X. $(Y, E)^C$ is a soft open set in X. Then $\{(F_i, E)\}_{i\in I} \cup \{(Y^C, E)\}$ form a soft open cover of $\widetilde{E}$. Therefore

$$(U, E) \widetilde{\subseteq} (F_{i_1}, E) \cup \dots \cup (F_{i_n}, E) \cup (Y^C, E),$$

for some $i_1, \dots, i_n \in I$. Hence $\{(F_i, E)\}_{i\in I}$ is a subcover of $\widetilde{Y}$.

Definition 1.4.8

Let (U, τ, E) be a soft topological spaces and $\mathfrak{B} \widetilde{\subseteq} \tau$. If every element of τ can be written as a union of elements of $\mathfrak{B}$, then $\mathfrak{B}$ is called a **soft basis** for the soft topology τ. Each element of $\mathfrak{B}$ is called a soft basis element.

Theorem 1.4.9

A soft topological space (U, τ, E) is soft compact if and only if there is a soft basis $\mathfrak{B}$ for τ such that every cover of $\widetilde{E}$ by elements of $\mathfrak{B}$ has a finite subcover.

Proof

Let (U, τ, E) be soft compact. Obviousely, τ is a soft basis for τ. Therefore, every cover of $\widetilde{E}$ by elements of τ has finite subcover.

Conversely, let $\{(U_i, E)\}_{i\in I}$ be a soft open cover of $\widetilde{E}$. We can write (U_i, E) as a union of basis elements, for each $i \in I$. These elements form a soft open cover of $\widetilde{E}$ such as $\{(F_{\mathfrak{B}}, E)\}_{\mathfrak{B}\in I}$. Therefore $\widetilde{E} = (F_{\mathfrak{B}_1}, E) \cup \dots \cup (F_{\mathfrak{B}_n}, E)$, for some $\mathfrak{B}_1, \dots, \mathfrak{B}_n \in I$. Let $(F_{\mathfrak{B}_n}, E) \widetilde{\subseteq} (U_i, E)$, for each $1 \leq i \leq n$. This implies that $\{(U\_i, E)\}_{i=1\dots n}$ is a finite subcover of $\widetilde{E}$. Hence, (U, τ, E) is soft compact.

SECTION 1.5

SOFT CONNECTEDNESS

Definition 1.5.1

Let (U, τ, E) be a soft topological space over U. A **soft separation** of $\widetilde{E}$ is a pair (F, E), (G, E) of no-null soft open sets over U such that

$$\widetilde{U} = (F, E) \cup (G, E), (F, E) \cap (G, E) = \Phi$$

A soft topological space (U, τ, E) is said to be **soft connected** if there does not exist a soft separation of $\widetilde{E}$.

Theorem 1.5.2

Let (F, E) be a soft set in $S(U, E)$. Then the following hold

(i) $(F, E) \cup (F, E)^C = \widetilde{E}$;

(ii) $(F, E) \cap (F, E)^C = \Phi$;

(iii) $(F, E) \cap \widetilde{E} = (F, E)$.

Proof

We prove (ii). Let $(F, E) \cup (F, E)^C = (H,E)$. Then

$$H(e) = F(e) \cap F^C(e) = F(e) \cap (U - F(e)) = \Phi.$$

Therefore $(H, E) = \Phi$.

Using Theorem 1.5.2, we prove the following.

Theorem 1.5.3

A soft topological space (U, τ, E) is soft connected if and only if the only soft sets in $S(X, E)$ that are both soft open and soft closed over U are Φ and $\widetilde{E}$.

Proof

Let (U, τ, E) be soft connected. Suppose to the contrary that (F, E) is both soft open and soft closed in U different from Φ and $\widetilde{E}$. Clearly, $(F, E)^C$ is a soft open set in U different from Φ and $\widetilde{E}$. Therefore (F, E), $(F, E)^C$ is a soft separation of $\widetilde{E}$. This is a contradiction. Thus the only soft closed and open sets in U are Φ and $\widetilde{E}$.

Conversely, let (F, E), (G, E) be a soft separation of $\widetilde{E}$. Let $(F, E) = \widetilde{E}$. This implies that $(G, E) = \Phi$. This is a contradiction. Hence $(F, E) \neq \widetilde{E}$. Since $F(e) \cap G(e) = \Phi$ and $F(e) \cup G(e) = U$, for each $e \in E$, then we have $G^C = U - G(e) = F(e)$. Therefore $(F, E) = (G, E)^C$. This shows that (F, E) is both soft open and soft closed in U different from Φ and $\widetilde{E}$. This is a contradiction. Therefore, (U, τ, E) is soft connected.

Theorem 1.5.4

Let $S(U, A)$ and $S(V, B)$ be families of soft sets. For a function $f_{pu} : S(U, A) \rightarrow S(V, B)$ the following hold

(i) ${f^{-1}}_{pu}\left((F, B) \cup (G, B)\right) = {f^{-1}}_{pu}(F, B) \cup {f^{-1}}_{pu}(G, B)$;

(ii) ${f^{-1}}_{pu}(\widetilde{V}) = \widetilde{E}$;

(iii) $f_{pu}((F, A) \cap (G, A)) \widetilde{\subseteq} f_{pu}(F, A) \cap f_{pu}(G, A)$;

(iv) ${f^{-1}}_{pu}((F, B) \cap (G, B)) = {f^{-1}}_{pu}(F, B) \cap {f^{-1}}_{pu}(G, B)$;

(v) ${f^{-1}}_{pu}(\Phi_B) = \Phi_A$.

Proof

(i) Let $(F, B) \cup (G, B) = (H, B)$. Then ${f^{-1}}_{pu}(H, B) = ({f^{-1}}_{pu}(H), A)$, where ${f^{-1}}_{pu}(H)(x) = u^{-1}(H(p(x)))$, for each $x \in A$. On the other hand, let ${f^{-1}}_{pu}(F, B) \cup {f^{-1}}_{pu}(G, B) = (O, A)$, where

$O(x) = {f^{-1}}_{pu}(F)(x) \cup {f^{-1}}_{pu}(G)(x) = u^{-1}F\left(p(x)\right) \cup G\left(p(x)\right) = u^{-1}(H(p(x)))$,for each $x \in A$. Therefore ${f^{-1}}_{pu}(H, B) = (O, A)$.

(ii) ${f^{-1}}_{pu}\left(\widetilde{V}\right) = {f^{-1}}_{pu}(V, B) = ({f^{-1}}_{pu}(V), A)$, where ${f^{-1}}_{pu}(V)(x) = u^{-1}(V(p(x))) = u^{-1}(V) = U = U(x)$.

(iii) Let $(F, A) \cap (G, A) = (H, A)$. Then $f_{pu}(H, A) = (f_{pu}(H), B)$, where

$$f_{pu}(H)(y) = \begin{cases} \bigcup_{x\in p^{-1}(y)\cap A} u(H(x)) & p^{-1}(y) \cap A \neq \phi \\ \phi & p^{-1}(y) \cap A = \phi \end{cases}$$

for each $y \in B$. On the other hand, let $f_{pu}(F, A) \cap f_{pu}(G, A) = (O, B)$, where $O(y) = f_{pu}(F)(y) \cap f_{pu}(G)(y)$, for each $y \in B$. We have

$$f_{pu}(H)(y) = \begin{cases} (\bigcup_{x\in p^{-1}(y)\cap A} u(H(x))) \cap (\bigcup_{x\in p^{-1}(y)\cap A} u(G(x))), & p^{-1}(y) \cap A \neq \phi \\ \phi & p^{-1}(y) \cap A = \phi \end{cases}$$

for each $y \in B$. Since $H(x) = F(x) \cap G(x)$, for each $x \in A$, then $f_{pu}(H)(y) \widetilde{\subseteq} O(y)$ for each $y \in B$. This implies that $f_{pu}(H, A) \widetilde{\subseteq} (O, B)$.

(iv) Let $(F, B) \cap (G, B) = (H, B)$. Then $f^{-1}{}_{pu}(H, B) = f^{-1}{}_{pu}((H), A)$, where $f^{-1}{}_{pu}(H)(x) = u^{-1}(H(p(x)))$, for each $x \in A$. On the other hand, let $f^{-1}{}_{pu}(F, B) \cap f^{-1}{}_{pu}(G, B) = (O, A)$, where

$$O(x) = f^{-1}{}_{pu}(F)(x) \cap f^{-1}{}_{pu}(G)(x) = u^{-1}\big(F(p(x))\big) \cap u^{-1}\big(G(p(x))\big)$$

$$= u^{-1}(H(p(x))),$$

for each $x \in A$. Therefore, $f^{-1}{}_{pu}(H, B) = (O, A)$.

Note 1.5.5

Let (U, τ, A) and (V, τ', B) be soft topological spaces. Let $f_{pu} : S(U, A) \to S(V, B)$ be a function. Then f_{pu} is said to be soft pu-continuous if for each $(F, B) \in \tau'$ we have $f^{-1}{}_{pu}(F, B) \in \tau$.

Theorem 1.5.6

Let (F, E), (G, E) and (H, E) be soft sets in $S(X, E)$. Then,

(i) $(F, E) \cap ((G, E) \cup (H, E)) = ((F, E) \cap (G, E)) \cup ((F, E) \cap (H, E))$;

(ii) $(F,E)^C \tilde{\subseteq} (G,E)$ if and only if $(F,E) \cap (G,E) = (F,E)$;

(iii) $(F,E)^C \tilde{\subseteq} (G,E)$ if and only if $(F,E) \cup (G,E) = (G,E)$.

Proof

(i) Let $(G,E) \cup (H,E) = (A,E)$ and $(F,E) \cap (A,E) = (B,E)$. Then,

$B(e) = F(e) \cap A(e) = F(e) \cap (G(e) \cup H(e)) = (F(e) \cap G(e)) \cup (F(e) \cap H(e))$, for each $e \in E$.

On the other hand, if $(F,E) \cap (G,E) = (C,E), (F,E) \cap (H,E) = (D,E)$ and $(C,E) \cup (D,E) = (I,E)$, then $I(e) = C(e) \cup D(e) = (F(e) \cap G(e)) \cup (F(e) \cap H(e))$ for each $e \in E$. Therefore, $(B,E) = (I,E)$.

The proof of (ii) and (iii) are similar.

Theorem 1.5.7

Let f_{pu} be a soft pu-continuous function carrying the soft connected space (U,τ,A) onto the soft space (V,τ',B). Then (V,τ',B) is soft connected.

Proof

Suppose to the contrary there exists a soft separation $(F,B), (G,B)$ of $\tilde{V}$. Then

$\tilde{U} = f^{-1}{}_{pu}\left((F,B) \cup (G,B)\right) = f^{-1}{}_{pu}(F,B) \cup f^{-1}{}_{pu}(G,B)$

$= f^{-1}{}_{pu}(F,B) \cap f^{-1}{}_{pu}(G,B)$

$= f^{-1}{}_{pu}(\Phi_B) = \Phi_A$.

Let $f^{-1}{}_{pu}(F,B) = \Phi_A$. Since f_{pu} is surjective, then we have, $(F,B) = \Phi_B$.

Therefore $f^{-1}{}_{pu}(F,B)$ and by a similar reason $f^{-1}{}_{pu}(G,B)$ are different from Φ_A. Then this shows that $f^{-1}{}_{pu}(F,B)$, $f^{-1}{}_{pu}(G,B)$ is a soft separation of $\tilde{U}$. This is a contradiction.