CHAPTER - 5

SECOND ORDER BIPOLAR FUZZY GRADATION OF OPENNESS AND SECOND ORDER BIPOLAR FUZZY COMPACTNESS

In this chapter, second order bipolar fuzzy gradation of openness is introduced.

In 1992, Hazra et al. introduced the concept of gradation of openness and gave a new definition of fuzzy topology. Using this concept, Kalaichelvi (2000) introduced the definition of second order gradation of openness and gave results related to the second order fuzzy topologies induced by second order gradation of openness.

In the first section, first order bipolar fuzzy gradation of openness is introduced and a new definition of first order bipolar fuzzy topology induced by first order bipolar fuzzy gradation of openness is given.

In the second section, second order bipolar fuzzy gradation of openness is introduced. Five interesting connections between first order bipolar fuzzy gradation of openness, second order bipolar fuzzy gradation of openness and first order gradation of openness are established. These connections are denoted as D_1, D_2, D_3, D_4 and D_5. It is proved that the associations D_1, D_3 and D_5 are functorial with respect to gradation preserving maps. Results related to the second order bipolar fuzzy topologies induced by the second order bipolar fuzzy gradation of openness are obtained. Further it is proved that if $\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{\mathrm{bp}})$ is the second order bipolar fuzzy topology induced by the second order bipolar fuzzy gradation of openness $\hat{\mathcal{G}}_{\mathrm{bp}}$, then $\hat{\mathcal{G}}_{\mathrm{bp}} \rightarrow \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{\mathrm{bp}})$ is functorial.

Third section is devoted to the study of compactness in the second order bipolar fuzzy topological spaces. Five different types of compactness, namely, second order bipolar fuzzy 1-compact, second order bipolar fuzzy 1*- compact, second order bipolar fuzzy 2-compact, second order bipolar fuzzy 2*- compact, second order bipolar fuzzy 3-compact are introduced in the second order case by extending the definition of bipolar fuzzy compactness. The relation between these concepts are analysed with special reference to R_1, R_2, R_3, R_4 and R_5. It is proved that all the five concepts are preserved under second order bipolar fuzzy continuous function.

SECTION 5.1

FIRST ORDER BIPOLAR FUZZY GRADATION OF OPENNESS

Definition:5.1.1

Let X be a non-empty set. A mapping $\mathcal{G}_{\mathrm{bp}}: \mathrm{BPF(X)} \rightarrow \mathrm{I}$ is said to be a **first order bipolar fuzzy gradation of openness** on X, if and only if the following axioms are satisfied:

(BPFGO1) $\mathcal{G}_{\mathrm{bp}}(0_{\mathrm{bp}}) = \mathcal{G}_{\mathrm{bp}}(1_{\mathrm{bp}}) = 1$

(BPFGO2) $\mathcal{G}_{\mathrm{bp}}\left((\mathrm{A}_{\mathrm{bp}})_{\mathrm{i}}\right) > 0$, for i = 1 to m $\Rightarrow \mathcal{G}_{\mathrm{bp}}\left(\cap_{\mathrm{i=1}}^{\mathrm{m}}(\mathrm{A}_{\mathrm{bp}})_{\mathrm{i}}\right) > 0.$

(BPFGO3) $\mathcal{G}_{\mathrm{bp}}\left((\mathrm{A}_{\mathrm{bp}})_{\lambda}\right) > 0$, for all $\lambda \in \Lambda \Rightarrow \mathcal{G}_{\mathrm{bp}}\left(\cup_{\lambda\in\Lambda}(\mathrm{A}_{\mathrm{bp}})_{\lambda}\right) > 0.$

Therefore $(\mathrm{X}, \mathcal{G}_{\mathrm{bp}})$ is called a **first order bipolar fuzzy gradation space**.

Example:5.1.2

Let X be a non-empty set. A mapping $\mathcal{G}_{\mathrm{bp}}: \mathrm{BPF(X)} \rightarrow \mathrm{I}$ is defined as $\mathcal{G}_{\mathrm{bp}}(0_{\mathrm{bp}}) = 1$, $\mathcal{G}_{\mathrm{bp}}(1_{\mathrm{bp}}) = 1$ and for other first order bipolar fuzzy sets $(\mathrm{A}_{\mathrm{bp}})_{\lambda} \in$ BPF(X), $\mathcal{G}_{\mathrm{bp}}\left((\mathrm{A}_{\mathrm{bp}})_{\lambda}\right) = \alpha_{\lambda}$ where $\alpha_{\lambda} \in (0,1)$. Then $\mathcal{G}_{\mathrm{bp}}$ is a first order bipolar fuzzy gradation of openness on X.

Remark:5.1.3

The family of bipolar fuzzy sets $\tau_{\mathfrak{B}}(\mathcal{G}_{\mathrm{bp}}) = \{ \mathrm{A}_{\mathrm{bp}} \in \mathrm{BPF(X)} \,/\, \mathcal{G}_{\mathrm{bp}}(\mathrm{A}_{\mathrm{bp}}) > 0\}$ forms a bipolar fuzzy topology on X.

Definition:5.1.4

Let $(\mathrm{X}, \mathcal{G}_{\mathrm{bp}})$ be a bipolar fuzzy gradation space. Then $\tau_{\mathfrak{B}}(\mathcal{G}_{\mathrm{bp}})$ is called the bipolar fuzzy topology on X induced by $\mathcal{G}_{\mathrm{bp}}$.

Definition:5.1.5

Let $(\mathcal{G}_{\mathrm{bp}})_1$ and $(\mathcal{G}_{\mathrm{bp}})_2$ be two bipolar fuzzy gradations of openness on X. If $(\mathcal{G}_{\mathrm{bp}})_1(\mathrm{A}_{\mathrm{bp}}) \geq (\mathcal{G}_{\mathrm{bp}})_2(\mathrm{A}_{\mathrm{bp}})$, for every $\mathrm{A}_{\mathrm{bp}} \in$ BPF(X), then $(\mathcal{G}_{\mathrm{bp}})_1 \geq (\mathcal{G}_{\mathrm{bp}})_2$.

Definition:5.1.6

Let $\left(X, (\mathcal{G}_{bp})_1\right)$, $\left(Y, (\mathcal{G}_{bp})_2\right)$ be two bipolar fuzzy gradation spaces. Then a map $\theta: X \to Y$ is called

(i) a **bipolar fuzzy gradation preserving map**,

if $(\mathcal{G}_{bp})_2(A_{bp}) \leq (\mathcal{G}_{bp})_1\left(\theta^{-1}(A_{bp})\right)$, for each $A_{bp} \in BPF(Y)$.

(ii) a **bipolar fuzzy strongly gradation preserving map**,

if $(\mathcal{G}_{bp})_2(A_{bp}) = (\mathcal{G}_{bp})_1\left(\theta^{-1}(A_{bp})\right)$, for each $A_{bp} \in BPF(Y)$.

(iii) a **bipolar fuzzy weakly gradation preserving map**,

if $(\mathcal{G}_{bp})_2(A_{bp}) > 0 \Rightarrow (\mathcal{G}_{bp})_1\left(\theta^{-1}(A_{bp})\right) > 0$, for each $A_{bp} \in BPF(Y)$.

Theorem:5.1.7

Let $\theta: X \to Y$ be a map and let $(\mathcal{G}_{bp})_1$ be a bipolar fuzzy gradation of openness on X. Then the largest bipolar fuzzy gradation of openness $(\mathcal{G}_{bp})_2$ on Y which makes $\theta: \left(X, (\mathcal{G}_{bp})_1\right) \to \left(Y, (\mathcal{G}_{bp})_2\right)$ a bipolar fuzzy gradation preserving map is given by $\left((\mathcal{G}_{bp})_2\right)(A_{bp}) = \left((\mathcal{G}_{bp})_1\right)\left(\theta^{-1}(A_{bp})\right)$ for each $A_{bp} \in BPF(Y)$.

Proof:

For $A_{bp} \in BPF(Y)$, define $(\mathcal{G}_{bp})_2(A_{bp}) = (\mathcal{G}_{bp})_1\left(\theta^{-1}(A_{bp})\right)$

Claim 1:

$(\mathcal{G}_{bp})_2$ is a bipolar fuzzy gradation of openness on Y.

(BPFG1) $(\mathcal{G}_{bp})_2(0_{bp}) = (\mathcal{G}_{bp})_1\left(\theta^{-1}(0_{bp})\right) = (\mathcal{G}_{bp})_1(0_{bp}) = 1.$

$(\mathcal{G}_{bp})_2(1_{bp}) = (\mathcal{G}_{bp})_1\left(\theta^{-1}(1_{bp})\right) = (\mathcal{G}_{bp})_1(1_{bp}) = 1.$

(BPFG2) $(\mathcal{G}_{bp})_2\left((A_{bp})_i\right) > 0$, for i = 1 to m implies $(\mathcal{G}_{bp})_1\left(\theta^{-1}(A_{bp})_i\right) > 0$, for i =1 to m

implies $(\mathcal{G}_{bp})_1\left(\cap_{i=1}^{m}\left(\theta^{-1}(A_{bp})_i\right)\right) > 0$

implies $(\mathcal{G}_{bp})_1\left(\theta^{-1}\left(\cap_{i=1}^{m}(A_{bp})_i\right)\right) > 0$

implies $(\mathcal{G}_{bp})_2\left(\cap_{i=1}^{m}(A_{bp})_i\right) > 0.$

(BPFG3) $(\mathcal{G}_{\mathrm{bp}})_2\left((\mathrm{A}_{\mathrm{bp}})_\lambda\right) > 0$, for $\lambda \in \Lambda$ implies $(\mathcal{G}_{\mathrm{bp}})_1\left(\theta^{-1}\left((\mathrm{A}_{\mathrm{bp}})_\lambda\right)\right) > 0$, for $\lambda \in \Lambda$

implies $(\mathcal{G}_{\mathrm{bp}})_1\left(\cup_{\lambda\in\Lambda}\left(\theta^{-1}\left((\mathrm{A}_{\mathrm{bp}})_\lambda\right)\right)\right) > 0$

implies $(\mathcal{G}_{\mathrm{bp}})_1\left(\theta^{-1}\left(\cup_{\lambda\in\Lambda}(\mathrm{A}_{\mathrm{bp}})_\lambda\right)\right) > 0$

implies $(\mathcal{G}_{\mathrm{bp}})_2\left(\cup_{\lambda\in\Lambda}(\mathrm{A}_{\mathrm{bp}})_\lambda\right) > 0$.

Claim 2:

$(\mathcal{G}_{\mathrm{bp}})_2$ is the largest bipolar fuzzy gradation of openness on Y such that $\theta\colon\left(\mathrm{X},(\mathcal{G}_{\mathrm{bp}})_1\right) \to \left(\mathrm{Y},(\mathcal{G}_{\mathrm{bp}})_2\right)$ is a bipolar fuzzy gradation preserving map, then $(\mathcal{G}_{\mathrm{bp}})_1\left(\theta^{-1}(\mathrm{A}_{\mathrm{bp}})\right) \leq (\mathcal{G}_{\mathrm{bp}})_2(\mathrm{A}_{\mathrm{bp}})$. (1)

Suppose $\left((\mathcal{G}_{\mathrm{bp}})_2\right)'$ is the bipolar fuzzy gradation of openness on Y such that $\theta\colon\left(\mathrm{X},(\mathcal{G}_{\mathrm{bp}})_1\right) \to \left(\mathrm{Y},\left((\mathcal{G}_{\mathrm{bp}})_2\right)'\right)$ is a bipolar fuzzy gradation preserving map, then $\left((\mathcal{G}_{\mathrm{bp}})_2\right)'(\mathrm{A}_{\mathrm{bp}}) \leq (\mathcal{G}_{\mathrm{bp}})_1\left(\theta^{-1}(\mathrm{A}_{\mathrm{bp}})\right)$, for each $\mathrm{A}_{\mathrm{bp}} \in \mathrm{BPF(Y)}$.

Therefore $\left((\mathcal{G}_{\mathrm{bp}})_2\right)'(\mathrm{A}_{\mathrm{bp}}) \leq (\mathcal{G}_{\mathrm{bp}})_2(\mathrm{A}_{\mathrm{bp}})$ by (1). Hence $(\mathcal{G}_{\mathrm{bp}})_2$ is the largest bipolar fuzzy gradation of openness on Y such that $\theta\colon\left(\mathrm{X},(\mathcal{G}_{\mathrm{bp}})_1\right) \to \left(\mathrm{Y},(\mathcal{G}_{\mathrm{bp}})_2\right)$ is a bipolar fuzzy gradation preserving map.

Theorem:5.1.8

$\theta\colon\left(\mathrm{X},(\mathcal{G}_{\mathrm{bp}})_1\right)\to\left(\mathrm{Y},(\mathcal{G}_{\mathrm{bp}})_2\right)$ is a bipolar fuzzy weakly gradation preserving map iff $\theta\colon\left(\mathrm{X},\tau_{\mathfrak{B}}\left((\mathcal{G}_{\mathrm{bp}})_1\right)\right) \to \left(\mathrm{Y},\tau_{\mathfrak{B}}\left((\mathcal{G}_{\mathrm{bp}})_2\right)\right)$ is bipolar fuzzy continuous.

Proof:

Assume θ is a bipolar fuzzy weakly gradation preserving map. Let $\mathrm{A}_{\mathrm{bp}} \in \tau_{\mathfrak{B}}\left((\mathcal{G}_{\mathrm{bp}})_2\right)$. Then $(\mathcal{G}_{\mathrm{bp}})_2(\mathrm{A}_{\mathrm{bp}}) > 0$ implies $(\mathcal{G}_{\mathrm{bp}})_1\left(\theta^{-1}(\mathrm{A}_{\mathrm{bp}})\right) > 0$ implies $\theta^{-1}(\mathrm{A}_{\mathrm{bp}}) \in \tau_{\mathfrak{B}}\left((\mathcal{G}_{\mathrm{bp}})_1\right)$.

Therefore θ is bipolar fuzzy continuous.

Conversely, Assume $\theta:\left(X, \tau_{\mathfrak{B}}\left(\left(\mathcal{G}_{bp}\right)_1\right)\right) \rightarrow \left(Y, \tau_{\mathfrak{B}}\left(\left(\mathcal{G}_{bp}\right)_2\right)\right)$ is bipolar fuzzy continuous.

Let $\left(\mathcal{G}_{bp}\right)_2(A_{bp}) > 0$ implies $A_{bp} \in \tau_{\mathfrak{B}}\left(\left(\mathcal{G}_{bp}\right)_2\right)$ implies $\theta^{-1}\left(A_{bp}\right) \in \tau_{\mathfrak{B}}\left(\left(\mathcal{G}_{bp}\right)_1\right)$ implies $\left(\mathcal{G}_{bp}\right)_1\left(\theta^{-1}\left(A_{bp}\right)\right) > 0$.

Hence θ is a bipolar fuzzy weakly gradation preserving map.

SECTION 5.2

SECOND ORDER BIPOLAR FUZZY GRADATION OF OPENNESS

Definition:5.2.1

Let X be a non-empty set. A mapping $\hat{\mathcal{G}}_{\mathrm{bp}}$: SBPF(X) → I is said to be a **second order bipolar fuzzy gradation of openness** on X iff the following conditions are satisfied:

(SBPFG1) $\hat{\mathcal{G}}_{\mathrm{bp}}(\hat{0}_{\mathrm{bp}}) = \hat{\mathcal{G}}_{\mathrm{bp}}(\hat{1}_{\mathrm{bp}}) = 1$.

(SBPFG2) $\hat{\mathcal{G}}_{\mathrm{bp}}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right) > 0$, for i = 1 to m ⇒ $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right) > 0$.

(SBPFG3) $\hat{\mathcal{G}}_{\mathrm{bp}}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) > 0$, for all $\lambda \in \Lambda \Rightarrow \hat{\mathcal{G}}_{\mathrm{bp}}\left(\cup_{\lambda\in\Lambda}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) > 0$.

Therefore $(\mathrm{X}, \hat{\mathcal{G}}_{\mathrm{bp}})$ is called a **second order bipolar fuzzy gradation space**.

Example:5.2.2

Let X be a non empty set. A mapping $\hat{\mathcal{G}}_{\mathrm{bp}}$: SBPF(X) → I is defined as $\hat{\mathcal{G}}_{\mathrm{bp}}(\hat{0}_{\mathrm{bp}}) = 1, \hat{\mathcal{G}}_{\mathrm{bp}}(\hat{1}_{\mathrm{bp}}) = 1$ and for other second order bipolar fuzzy sets $(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda} \in$ SBPF(X), $\hat{\mathcal{G}}_{\mathrm{bp}}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) = \alpha_{\lambda}$ where $\alpha_{\lambda} \in (0,1)$. Then $\hat{\mathcal{G}}_{\mathrm{bp}}$ is a second order bipolar fuzzy gradation of openness on X.

Theorem:5.2.3

If $\mathrm{A}_{\mathrm{bp}} \in$ BPF(X), then $\widehat{\mathrm{A}}_{\mathrm{bp}} \in$ SBPF(X) is given by ${\widehat{\mathrm{A}}_{\mathrm{bp}}}^{+}(\mathrm{x})(\alpha) = {\mathrm{A}_{\mathrm{bp}}}^{+}(\mathrm{x}), {\widehat{\mathrm{A}}_{\mathrm{bp}}}^{-}(\mathrm{x})(\alpha) = {\mathrm{A}_{\mathrm{bp}}}^{-}(\mathrm{x})$, for every x ∈ X and for every α ∈ I.

Let $\hat{\mathcal{G}}_{\mathrm{bp}}$: SBPF(X) → I be a second order bipolar fuzzy gradation of openness on X. Define $\mathcal{G}_{\mathrm{bp}}$: BPF(X) → I as $\mathcal{G}_{\mathrm{bp}}(\mathrm{A}_{\mathrm{bp}}) = \hat{\mathcal{G}}_{\mathrm{bp}}(\widehat{\mathrm{A}}_{\mathrm{bp}})$. Then $\mathcal{G}_{\mathrm{bp}}$ is a first order bipolar fuzzy gradation of openness on X. The correspondence $\hat{\mathcal{G}}_{\mathrm{bp}} \to \mathcal{G}_{\mathrm{bp}}$ is denoted as $\mathbb{D}_1$.

Proof:

Let $\hat{\mathcal{G}}_{\mathrm{bp}}$: SBPF(X) → I be a second order bipolar fuzzy gradation of openness on X. Let $\mathcal{G}_{\mathrm{bp}}$: BPF(X) → I be defined as $\mathcal{G}_{\mathrm{bp}}(\mathrm{A}_{\mathrm{bp}}) = \hat{\mathcal{G}}_{\mathrm{bp}}(\widehat{\mathrm{A}}_{\mathrm{bp}})$.

Then $\mathcal{G}_{\mathrm{bp}}(0_{\mathrm{bp}}) = \hat{\mathcal{G}}_{\mathrm{bp}}(\hat{0}_{\mathrm{bp}}) = 1$.

Similarly $\mathcal{G}_{\mathrm{bp}}(1_{\mathrm{bp}}) = \hat{\mathcal{G}}_{\mathrm{bp}}(\hat{1}_{\mathrm{bp}}) = 1$.

Therefore (BPFG1) is satisfied.

Let $\mathcal{G}_{bp}\left(\left(A_{bp}\right)_i\right) > 0$, for i = 1 to m implies $\hat{\mathcal{G}}_{bp}\left(\left(\widehat{A}_{bp}\right)_i\right) > 0$, for i = 1 to m

implies $\hat{\mathcal{G}}_{bp}\left(\cap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i\right) > 0$ implies $\hat{\mathcal{G}}_{bp}\left(\widehat{\cap_{i=1}^{m}\left(A_{bp}\right)_i}\right) > 0$ implies $\mathcal{G}_{bp}\left(\cap_{i=1}^{m}\left(A_{bp}\right)_i\right) > 0$.

Therefore (BPFG2) is satisfied.

Let $\mathcal{G}_{bp}\left(\left(A_{bp}\right)_\lambda\right) > 0$, for $\lambda \in \Lambda$ implies $\hat{\mathcal{G}}_{bp}\left(\left(\widehat{A}_{bp}\right)_\lambda\right) > 0$, for $\lambda \in \Lambda$

implies $\hat{\mathcal{G}}_{bp}\left(\cup_{\lambda\in\Lambda}\left(\widehat{A}_{bp}\right)_\lambda\right) > 0$ implies $\hat{\mathcal{G}}_{bp}\left(\widehat{\cup_{\lambda\in\Lambda}\left(A_{bp}\right)_\lambda}\right) > 0$

implies $\mathcal{G}_{bp}\left(\cup_{\lambda\in\Lambda}\left(A_{bp}\right)_\lambda\right) > 0$.

Therefore (BPFG3) is satisfied.

Example:5.2.4

Let a second order bipolar fuzzy gradation of openness $\hat{\mathcal{G}}_{bp}: SBPF(X) \to I$ on X be defined by $\hat{\mathcal{G}}_{bp}\left(\hat{0}_{bp}\right) = 1, \hat{\mathcal{G}}_{bp}\left(\hat{1}_{bp}\right) = 1$ and for other second order bipolar fuzzy sets $\left(\widehat{A}_{bp}\right)_\lambda \in SBPF(X), \hat{\mathcal{G}}_{bp}\left(\left(\widehat{A}_{bp}\right)_\lambda\right) = \beta_\lambda$ where $\beta_\lambda \in (0,1)$. If $A_{bp} \in BPF(X)$ then $\widehat{A}_{bp} \in SBPF(X)$ is defined by $\widehat{A}_{bp}{}^{+}(x)(\alpha) = A_{bp}{}^{+}(x)$, $\widehat{A}_{bp}{}^{-}(x)(\alpha) = A_{bp}{}^{-}(x)$, for every $x \in X$ and for every $\alpha \in I$. Define $\mathcal{G}_{bp}: BPF(X) \to I$ as $\mathcal{G}_{bp}\left(A_{bp}\right) = \hat{\mathcal{G}}_{bp}\left(\widehat{A}_{bp}\right)$. Then $\mathcal{G}_{bp}$ is a first order bipolar fuzzy gradation of openness on X.

Theorem:5.2.5

If $A_{bp} \in BPF(I)$, then $\left(\widehat{A}_{bp}\right)_I \in SBPF(X)$ is given by $\left(\widehat{A}_{bp}{}^{+}\right)_I(x) = A_{bp}{}^{+}$, $\left(\widehat{A}_{bp}{}^{-}\right)_I(x) = A_{bp}{}^{-}$, for every $x \in X$. Let $\hat{\mathcal{G}}_{bp}: SBPF(X) \to I$ be a second order bipolar fuzzy gradation of openness on X. Define $\left(\hat{\mathcal{G}}_{bp}\right)_I: BPF(X) \to I$ as $\left(\hat{\mathcal{G}}_{bp}\right)_I\left(A_{bp}\right) = \hat{\mathcal{G}}_{bp}\left(\left(\widehat{A}_{bp}\right)_I\right)$. Then $\left(\hat{\mathcal{G}}_{bp}\right)_I$ is a first order bipolar fuzzy gradation of openness on I. The correspondence $\hat{\mathcal{G}}_{bp} \to \left(\hat{\mathcal{G}}_{bp}\right)_I$ is denoted as $\mathbb{D}_2$.

Proof:

Let $\hat{\mathcal{G}}_{bp}: SBPF(X) \to I$ be a second order bipolar fuzzy gradation of openness on X. Let $\left(\hat{\mathcal{G}}_{bp}\right)_I$ be defined as $\left(\hat{\mathcal{G}}_{bp}\right)_I\left(A_{bp}\right) = \hat{\mathcal{G}}_{bp}\left(\left(\widehat{A}_{bp}\right)_I\right)$.

Then $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(0_{\mathrm{bp}}\right)=\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\hat{0}_{\mathrm{bp}}\right)_{\mathrm{I}}\right)=1$ as $\left(\hat{0}_{\mathrm{bp}}\right)_{\mathrm{I}}=\hat{0}_{\mathrm{bp}}$.

Similarly, $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(1_{\mathrm{bp}}\right)=\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\hat{1}_{\mathrm{bp}}\right)_{\mathrm{I}}\right)=1$ as $\left(\hat{1}_{\mathrm{bp}}\right)_{\mathrm{I}}=\hat{1}_{\mathrm{bp}}$.

Therefore (BPFG1) is satisfied.

Let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(\left(\mathrm{A}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)>0$, for i = 1 to m implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)_{\mathrm{I}}\right)>0$, for i = 1 to m implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)_{\mathrm{I}}\right)>0$ implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left(\mathrm{A}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)>0$.

Therefore (BPFG2) is satisfied.

Let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(\left(\mathrm{A}_{\mathrm{bp}}\right)_{\lambda}\right)>0$, for $\lambda\in\Lambda$ implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right)_{\mathrm{I}}\right)>0$, for $\lambda\in\Lambda$ implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\cup_{\lambda\in\Lambda}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right)_{\mathrm{I}}\right)>0$ implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(\cup_{\lambda\in\Lambda}\left(\mathrm{A}_{\mathrm{bp}}\right)_{\lambda}\right)>0$.

Therefore (BPFG3) is satisfied.

Hence $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}$ is a first order bipolar fuzzy gradation of openness on I.

Example:5.2.6

As in example (5.2.4), $\hat{\mathcal{G}}_{\mathrm{bp}}$ is a second order bipolar fuzzy gradation of openness on X. If $\mathrm{A}_{\mathrm{bp}}\in\mathrm{BPF(I)}$, then $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{I}}\in\mathrm{SBPF(X)}$ is defined by $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\mathrm{I}}(\mathrm{x})=\mathrm{A}_{\mathrm{bp}}{}^{+}$, $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{I}}(\mathrm{x})=\mathrm{A}_{\mathrm{bp}}{}^{-}$, for every $\mathrm{x}\in\mathrm{X}$. Define $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}:\mathrm{BPF(I)}\rightarrow\mathrm{I}$ as $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}\left(\mathrm{A}_{\mathrm{bp}}\right)=\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{I}}\right)$. Then $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{I}}$ is a first order bipolar fuzzy gradation of openness on I.

Theorem:5.2.7

If $\widehat{\mathrm{A}}_{\mathrm{bp}}\in\mathrm{SBPF(X)}$ then for $\alpha\in\mathrm{I}$, $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\alpha}\in\mathrm{BPF(X)}$ is given by $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\alpha}(\mathrm{x})=\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(\alpha)$, $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\alpha}(\mathrm{x})=\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(\alpha)$, for every $\mathrm{x}\in\mathrm{X}$. Let $\mathcal{G}_{\mathrm{bp}}:\mathrm{BPF(X)}\rightarrow\mathrm{I}$ be a first order bipolar fuzzy gradation of openness on X. For $\alpha\in\mathrm{I}$, define $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}:\mathrm{SBPF(X)}\rightarrow\mathrm{I}$ as $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)=\mathcal{G}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\alpha}\right)$. Then $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}$ is a second order bipolar fuzzy gradation of openness on X. The correspondence $\mathcal{G}_{\mathrm{bp}}\rightarrow\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}$ is denoted as $\mathbb{D}_3$.

Proof:

Let $\mathcal{G}_{\mathrm{bp}}\colon \mathrm{BPF(X)} \rightarrow \mathrm{I}$ be a first order bipolar fuzzy gradation of openness on X.

For $\alpha \in \mathrm{I}$, $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}\colon \mathrm{SBPF(X)} \rightarrow \mathrm{I}$ is defined as $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}(\widehat{\mathrm{A}}_{\mathrm{bp}}) = \mathcal{G}_{\mathrm{bp}}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\alpha}\right)$.

Then $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}(\hat{0}_{\mathrm{bp}}) = \mathcal{G}_{\mathrm{bp}}\left((\hat{0}_{\mathrm{bp}})_{\alpha}\right) = 1$, where $(\hat{0}_{\mathrm{bp}})_{\alpha} = 0_{\mathrm{bp}}$.

Similarly, $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}(\hat{1}_{\mathrm{bp}}) = \mathcal{G}_{\mathrm{bp}}\left((\hat{1}_{\mathrm{bp}})_{\alpha}\right) = 1$, where $(\hat{1}_{\mathrm{bp}})_{\alpha} = 1_{\mathrm{bp}}$.

Therefore (SBPFG1) is satisfied.

Let $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right) > 0$, for i = 1 to m implies $\mathcal{G}_{\mathrm{bp}}\left(\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right)_{\alpha}\right) > 0$, for i = 1 to m

implies $\mathcal{G}_{\mathrm{bp}}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right)_{\alpha}\right) > 0$ implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\mathrm{i}}\right) > 0$.

Therefore (SBPFG2) is satisfied.

Let $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) > 0$, for $\lambda \in \Lambda$ implies $\mathcal{G}_{\mathrm{bp}}\left(\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right)_{\alpha}\right) > 0$, for $\lambda \in \Lambda$

implies $\mathcal{G}_{\mathrm{bp}}\left(\cup_{\lambda\in\Lambda}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right)_{\alpha}\right) > 0$ implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}\left(\cup_{\lambda\in\Lambda}(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\lambda}\right) > 0$.

Therefore (SBPFG3) is satisfied.

Hence $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}$ is a second order bipolar fuzzy gradation of openness on X.

Example:5.2.8

Let a first order bipolar fuzzy gradation of openness $\mathcal{G}_{\mathrm{bp}}\colon \mathrm{BPF(X)} \rightarrow \mathrm{I}$ on X be defined by $\mathcal{G}_{\mathrm{bp}}(0_{\mathrm{bp}}) = 1$, $\mathcal{G}_{\mathrm{bp}}(1_{\mathrm{bp}}) = 1$ and for other first order bipolar fuzzy sets $(\mathrm{A}_{\mathrm{bp}})_{\lambda} \in \mathrm{BPF(X)}$, $\mathcal{G}_{\mathrm{bp}}\left((\mathrm{A}_{\mathrm{bp}})_{\lambda}\right) = \beta_{\lambda}$ where $\beta_{\lambda} \in (0,1)$. If $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \mathrm{SBPF(X)}$ then for $\alpha \in \mathrm{I}$, $(\widehat{\mathrm{A}}_{\mathrm{bp}})_{\alpha} \in \mathrm{BPF(X)}$ is defined by $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\alpha}(\mathrm{x}) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(\alpha)$, $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\alpha}(\mathrm{x}) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(\alpha)$, for every $\mathrm{x} \in \mathrm{X}$. For $\alpha \in \mathrm{I}$, define $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}\colon \mathrm{SBPF(X)} \rightarrow \mathrm{I}$ as $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}(\widehat{\mathrm{A}}_{\mathrm{bp}}) = \mathcal{G}_{\mathrm{bp}}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_{\alpha}\right)$. Then $(\hat{\mathcal{G}}_{\mathrm{bp}})_{\alpha}$ is a second order bipolar fuzzy gradation of openness on X.

Theorem : 5.2.9

Let $\mathcal{G}$ be a first order gradation of openness on I. For $x \in X$, define $\left(\hat{\mathcal{G}}_{bp}\right)_x : SBPF(X) \to I$ such that $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\widehat{A}_{bp}\right) = \mathcal{G}\left(\left(\widehat{A}_{bp}\right)_x\right)$. Then $\left(\hat{\mathcal{G}}_{bp}\right)_x$ is a second order bipolar fuzzy gradation of openness on X. The correspondence $\mathcal{G} \to \left(\hat{\mathcal{G}}_{bp}\right)_x$ is denoted as $\mathbb{D}_4$.

Proof:

Let $\mathcal{G}: I^I \to I$ be a first order gradation of openness on I.

For $x \in X$, $\left(\hat{\mathcal{G}}_{bp}\right)_x : SBPF(X) \to I$ be defined as $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\widehat{A}_{bp}\right) = \mathcal{G}\left(\left(\widehat{A}_{bp}\right)_x\right)$.

Then $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\hat{0}_{bp}\right) = \mathcal{G}\left(\left(\hat{0}_{bp}\right)_x\right) = 1$ as $\left(\hat{0}_{bp}\right)_x = 0_{bp}$.

Similarly $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\hat{1}_{bp}\right) = \mathcal{G}\left(\left(\hat{1}_{bp}\right)_x\right) = 1$ as $\left(\hat{1}_{bp}\right)_x = 1_{bp}$.

Therefore (BPFG1) is satisfied.

Let $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\left(\widehat{A}_{bp}\right)_i\right) > 0$, for i = 1 to m implies $\mathcal{G}\left(\left(\left(\widehat{A}_{bp}\right)_i\right)_x\right) > 0$, for i = 1 to m implies $\mathcal{G}\left(\bigcap_{i=1}^{m}\left(\left(\widehat{A}_{bp}\right)_i\right)_x\right) > 0$ implies $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\bigcap_{i=1}^{m}\left(\widehat{A}_{bp}\right)_i\right) > 0$.

Therefore (BPFG2) is satisfied.

Let $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\left(\widehat{A}_{bp}\right)_\lambda\right) > 0$, for $\lambda \in \Lambda$ implies $\mathcal{G}\left(\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_x\right) > 0$, for $\lambda \in \Lambda$ implies $\mathcal{G}\left(\bigcup_{\lambda\in\Lambda}\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_x\right) > 0$ implies $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\bigcup_{\lambda\in\Lambda}\left(\widehat{A}_{bp}\right)_\lambda\right) > 0$

Therefore (BPFG3) is satisfied.

Hence $\left(\hat{\mathcal{G}}_{bp}\right)_x$ is a second order bipolar fuzzy gradation of openness on X.

Example:5.2.10

Let $\mathcal{G}: I^I \to I$ be a first order gradation of openness on I defined by $\mathcal{G}(\mathbf{0}) = 1$, $\mathcal{G}(\mathbf{1}) = 1$ and for other first order fuzzy sets $A_\lambda \in I^I, \mathcal{G}(A_\lambda) = \beta_\lambda$ where $\beta_\lambda \in (0,1)$. If for $x \in X$, $\left(\hat{\mathcal{G}}_{bp}\right)_x : SBPF(X) \to I$ is defined as $\left(\hat{\mathcal{G}}_{bp}\right)_x\left(\widehat{A}_{bp}\right) = \mathcal{G}\left(\left(\widehat{A}_{bp}\right)_x\right)$. Then $\left(\hat{\mathcal{G}}_{bp}\right)_x$ is a second order bipolar fuzzy gradation of openness on X.

Theorem:5.2.11

Let $\hat{\mathcal{G}}_{\mathrm{bp}}$: SBPF(X) → I be a second order bipolar fuzzy gradation of openness on X. Let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$: SBPF(X) → I be defined as follows: $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$ where $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\mathrm{c}}(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(1-\alpha)$, $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\mathrm{c}}(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(1-\alpha)$, for every $\mathrm{x} \in \mathrm{X}$ and for every $\alpha \in \mathrm{I}$. Then $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$ is a second order bipolar fuzzy gradation of openness on X. The correspondence $\hat{\mathcal{G}}_{\mathrm{bp}} \rightarrow \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$ is denoted as $\mathbb{D}_5$.

Proof:

Let $\hat{\mathcal{G}}_{\mathrm{bp}}$ be a second order bipolar fuzzy gradation of openness on X.

Let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$: SBPF(X) → I be defined as $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$.

Then $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\hat{0}_{\mathrm{bp}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\hat{0}_{\mathrm{bp}}\right)_{\mathrm{c}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\hat{1}_{\mathrm{bp}}\right) = 1$.

Similarly, $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\hat{1}_{\mathrm{bp}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\hat{1}_{\mathrm{bp}}\right)_{\mathrm{c}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\hat{0}_{\mathrm{bp}}\right) = 1$.

Therefore (SBPFG1) is satisfied.

Let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right) > 0$, for i = 1 to m implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)_{\mathrm{c}}\right) > 0$, for i = 1 to m

implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)_{\mathrm{c}}\right) > 0$ implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right) > 0$.

Therefore (SBPFG2) is satisfied.

Let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right) > 0$, for $\lambda \in \Lambda$ implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right)_{\mathrm{c}}\right) > 0$, for $\lambda \in \Lambda$

implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\cup_{\lambda\in\Lambda}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right)_{\mathrm{c}}\right) > 0$ implies $\hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\cup_{\lambda\in\Lambda}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right)_{\mathrm{c}}\right) > 0$.

implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\cup_{\lambda\in\Lambda}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda}\right) > 0$.

Therefore (SBPFG3) is satisfied.

Hence $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$ is a second order bipolar fuzzy gradation of openness on X.

Example:5.2.12

As in example (5.2.4), $\hat{\mathcal{G}}_{\mathrm{bp}}$ is a second order bipolar fuzzy gradation of openness on X. If $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$: SBPF(X) → I is defined as $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$ where

$\left(\widehat{A}_{bp}{}^{+}\right)_c(x)(\alpha) = \widehat{A}_{bp}{}^{+}(x)(1-\alpha)$, $\left(\widehat{A}_{bp}{}^{-}\right)_c(x)(\alpha) = \widehat{A}_{bp}{}^{-}(x)(1-\alpha)$, for every $x \in X$ and for every $\alpha \in I$, then $\left(\hat{\mathcal{G}}_{bp}\right)_c$ is a second order bipolar fuzzy gradation of openness on X.

Definition:5.2.13

Let $\left(X, \left(\hat{\mathcal{G}}_{bp}\right)_1\right), \left(Y, \left(\hat{\mathcal{G}}_{bp}\right)_2\right)$ be two second order bipolar fuzzy gradation spaces. Then a map $\theta: X \to Y$ is called

(i) a **second order bipolar fuzzy gradation preserving map**, if $\left(\hat{\mathcal{G}}_{bp}\right)_2\left(\widehat{A}_{bp}\right) \leq \left(\hat{\mathcal{G}}_{bp}\right)_1\left(\theta^{-1}\left(\widehat{A}_{bp}\right)\right)$, for each $\widehat{A}_{bp} \in SBPF(Y)$.

(ii) a **second order bipolar fuzzy strongly gradation preserving map**, if $\left(\hat{\mathcal{G}}_{bp}\right)_2\left(\widehat{A}_{bp}\right) = \left(\hat{\mathcal{G}}_{bp}\right)_1\left(\theta^{-1}\left(\widehat{A}_{bp}\right)\right)$, for each $\widehat{A}_{bp} \in SBPF(Y)$.

(iii) a **second order bipolar fuzzy weakly gradation preserving map**, if $\left(\hat{\mathcal{G}}_{bp}\right)_2\left(\widehat{A}_{bp}\right) > 0 \Rightarrow \left(\hat{\mathcal{G}}_{bp}\right)_1\left(\theta^{-1}\left(\widehat{A}_{bp}\right)\right) > 0$, for each $\widehat{A}_{bp} \in SBPF(Y)$.

Theorem : 5.2.14

The association $\mathbb{D}_1$, $\mathbb{D}_3$ and $\mathbb{D}_5$ are functorial with respect to the above definition (5.2.13)

Proof:

To prove $\mathbb{D}_1$ is functorial, assume $\theta: \left(X, \left(\hat{\mathcal{G}}_{bp}\right)_1\right) \to \left(Y, \left(\hat{\mathcal{G}}_{bp}\right)_2\right)$ is a second order bipolar fuzzy gradation preserving map.

Consider $\theta: \left(X, \left(\mathcal{G}_{bp}\right)_1\right) \to \left(Y, \left(\mathcal{G}_{bp}\right)_2\right)$, where $\left(\mathcal{G}_{bp}\right)_1$ and $\left(\mathcal{G}_{bp}\right)_2$are the first order bipolar fuzzy gradations of openness from $\left(\hat{\mathcal{G}}_{bp}\right)_1$ and $\left(\hat{\mathcal{G}}_{bp}\right)_2$ respectively, via $\mathbb{D}_1$.

For each $A_{bp} \in BPF(Y)$, $\left(\mathcal{G}_{bp}\right)_2\left(A_{bp}\right) = \left(\hat{\mathcal{G}}_{bp}\right)_2\left(\widehat{A}_{bp}\right) \leq \left(\hat{\mathcal{G}}_{bp}\right)_1\left(\theta^{-1}\left(\widehat{A}_{bp}\right)\right)$

$$= \left(\hat{\mathcal{G}}_{bp}\right)_1\left(\widehat{\theta^{-1}\left(A_{bp}\right)}\right)$$

$$= \left(\mathcal{G}_{bp}\right)_1\left(\theta^{-1}\left(A_{bp}\right)\right).$$

Then $\left(\mathcal{G}_{bp}\right)_2\left(A_{bp}\right) \leq \left(\mathcal{G}_{bp}\right)_1\left(\theta^{-1}\left(A_{bp}\right)\right)$.

Thus $\theta: \left(X, \left(\mathcal{G}_{bp}\right)_1\right) \to \left(Y, \left(\mathcal{G}_{bp}\right)_2\right)$ is a first order bipolar fuzzy gradation preserving map.

Hence $\mathbb{D}_1$ is functorial.

To prove $\mathbb{D}_3$ is functorial, assume $\theta:\left(\mathrm{X},\left(\mathcal{G}_{\mathrm{bp}}\right)_1\right) \rightarrow \left(\mathrm{Y},\left(\mathcal{G}_{\mathrm{bp}}\right)_2\right)$ is a first order bipolar fuzzy gradation preserving map.

Consider $(\theta)_\alpha:\left(\mathrm{X},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_\alpha\right) \rightarrow \left(\mathrm{Y},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_\alpha\right)$, where $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_\alpha$ and $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_\alpha$ are the second order bipolar fuzzy gradations of openness from $\left(\mathcal{G}_{\mathrm{bp}}\right)_1$ and $\left(\mathcal{G}_{\mathrm{bp}}\right)_2$ respectively, via $\mathbb{D}_3$.

For each $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_\alpha \in \mathrm{SBPF(Y)}$, $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_\alpha\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \left(\mathcal{G}_{\mathrm{bp}}\right)_2\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_\alpha\right) \leq \left(\mathcal{G}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_\alpha\right)\right)$

$$= \left(\mathcal{G}_{\mathrm{bp}}\right)_1\left(\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)_\alpha\right)$$

$$= \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_\alpha\left(\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)\right).$$

Then $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_\alpha\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) \leq \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_\alpha\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)$.

Thus $(\theta)_\alpha:\left(\mathrm{X},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_\alpha\right) \rightarrow \left(\mathrm{Y},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_\alpha\right)$ is a second order bipolar fuzzy gradation preserving map.

Hence $\mathbb{D}_3$ is functorial.

To prove $\mathbb{D}_5$ is functorial, assume $\theta:\left(\mathrm{X},\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right) \rightarrow \left(\mathrm{Y},\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)$ is a second order bipolar fuzzy gradation preserving map.

Consider $(\theta)_c:\left(\mathrm{X},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_c\right) \rightarrow \left(\mathrm{Y},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_c\right)$, where $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_c$ and $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_c$ are the second order bipolar fuzzy gradations of openness from $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1$ and $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2$respectively, via $\mathbb{D}_5$.

For each $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_c \in \mathrm{SBPF(Y)}$, $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_c\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_c\right) \leq \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_c\right)\right)$

$$= \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)_c\right)$$

$$= \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_c\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right).$$

Then $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_c\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) \leq \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_c\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)$.

Thus $(\theta)_c:\left(\mathrm{X},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_c\right) \rightarrow \left(\mathrm{Y},\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_c\right)$ is a second order bipolar fuzzy gradation preserving map.

Hence $\mathbb{D}_5$ is functorial.

Theorem:5.2.15

Let $(\mathrm{I},\mathcal{G}_1)$ and $(\mathrm{I},\mathcal{G}_2)$ be two first order gradation spaces. If $\mathcal{G}_1 \geq \mathcal{G}_2$, then for $\mathrm{x} \in \mathrm{X}$ the map $\theta\colon \left(\mathrm{X}, \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_{\mathrm{x}}\right) \to \left(\mathrm{Y}, \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_{\theta(\mathrm{x})}\right)$ is a second order bipolar fuzzy gradation preserving map where $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_{\mathrm{x}}$ and $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_{\theta(\mathrm{x})}$ are from $\mathcal{G}_1$and $\mathcal{G}_2$ respectively, via $\mathbb{D}_4$.

Proof:

$$\text{For each } \widehat{\mathrm{A}}_{\mathrm{bp}} \in \mathrm{SBPF(Y)}, \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)_{\theta(\mathrm{x})}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \mathcal{G}_2\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\theta(\mathrm{x})}\right) \leq \mathcal{G}_1\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\theta(\mathrm{x})}\right)$$

$$= \mathcal{G}_1\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right)$$

$$= \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)_{\mathrm{x}}\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right).$$

Therefore θ is a second order bipolar fuzzy gradation preserving map.

Theorem:5.2.16

Let $\theta\colon \mathrm{X} \to \mathrm{Y}$ be a map and let $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1$ be a second order bipolar fuzzy gradation of openness on X. Then the largest second order bipolar fuzzy gradation of openness $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2$ on Y which makes $\theta : \left(\mathrm{X}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right) \to \left(\mathrm{Y}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)$ a second order bipolar fuzzy gradation preserving map is given by $\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)$, for each $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \mathrm{SBPF(Y)}$.

Proof:

$$\text{For } \widehat{\mathrm{A}}_{\mathrm{bp}} \in \mathrm{SBPF(Y)}, \text{ define } \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) = \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right)$$

Claim 1:$\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2$is a second order bipolar fuzzy gradation of openness on Y.

(SBPFG1) $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\hat{0}_{\mathrm{bp}}\right) = \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\hat{0}_{\mathrm{bp}}\right)\right) = \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\hat{0}_{\mathrm{bp}}\right) = 1.$

$$\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\hat{1}_{\mathrm{bp}}\right) = \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\hat{1}_{\mathrm{bp}}\right)\right) = \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\hat{1}_{\mathrm{bp}}\right) = 1.$$

(SBPFG2) $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right) > 0$, for i = 1 to m

implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right) > 0$, for i = 1 to m

implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)\right) > 0$

implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\cap_{\mathrm{i}=1}^{\mathrm{m}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{i}}\right)\right) > 0$

implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_2\left(\cap_{i=1}^{m}(\widehat{\mathrm{A}}_{\mathrm{bp}})_i\right) > 0.$

(SBPFG3) $(\hat{\mathcal{G}}_{\mathrm{bp}})_2\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_\lambda\right) > 0$, for $\lambda \in \Lambda$

implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_1\left(\theta^{-1}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_\lambda\right)\right) > 0$, for $\lambda \in \Lambda$

implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_1\left(\cup_{\lambda\in\Lambda}\left(\theta^{-1}\left((\widehat{\mathrm{A}}_{\mathrm{bp}})_\lambda\right)\right)\right) > 0$

implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_1\left(\theta^{-1}\left(\cup_{\lambda\in\Lambda}(\widehat{\mathrm{A}}_{\mathrm{bp}})_\lambda\right)\right) > 0$

implies $(\hat{\mathcal{G}}_{\mathrm{bp}})_2\left(\cup_{\lambda\in\Lambda}(\widehat{\mathrm{A}}_{\mathrm{bp}})_\lambda\right) > 0.$

Claim 2: $(\hat{\mathcal{G}}_{\mathrm{bp}})_2$ is the largest second order bipolar fuzzy gradation of openness on Y such that $\theta: \left(X, (\hat{\mathcal{G}}_{\mathrm{bp}})_1\right) \to \left(Y, (\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)$ is a second order bipolar fuzzy gradation preserving map.

Suppose $\left((\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)'$ is the largest first order bipolar fuzzy gradation of openness on Y such that $\theta: \left(X, (\hat{\mathcal{G}}_{\mathrm{bp}})_1\right) \to \left(Y, \left((\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)'\right)$ is a second order bipolar fuzzy gradation preserving map, then $\left((\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)'(\widehat{\mathrm{A}}_{\mathrm{bp}}) \le (\hat{\mathcal{G}}_{\mathrm{bp}})_1\left(\theta^{-1}(\widehat{\mathrm{A}}_{\mathrm{bp}})\right)$, for each $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \mathrm{SBPF}(Y)$.

Then $\left((\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)'(\widehat{\mathrm{A}}_{\mathrm{bp}}) \le (\hat{\mathcal{G}}_{\mathrm{bp}})_2(\widehat{\mathrm{A}}_{\mathrm{bp}})$, by the definition of $(\hat{\mathcal{G}}_{\mathrm{bp}})_2$.

Hence $(\hat{\mathcal{G}}_{\mathrm{bp}})_2$ is the largest second order bipolar fuzzy gradation of openness on Y such that $\theta: \left(X, (\hat{\mathcal{G}}_{\mathrm{bp}})_1\right) \to \left(Y, (\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)$ is a second order bipolar fuzzy gradation preserving map.

Note:5.2.17

The second order bipolar fuzzy gradation of openness on Y defined in the above theorem (5.2.16) is called the **second order bipolar fuzzy quotient gradation** with respect to $(\hat{\mathcal{G}}_{\mathrm{bp}})_1$ on X and it is denoted by $Q\left((\hat{\mathcal{G}}_{\mathrm{bp}})_1\right)$.

Theorem:5.2.18

Let $\theta: \mathrm{SBPF}(X) \to \mathrm{SBPF}(Y)$ be a second order bipolar fuzzy mapping. Let $(\hat{\mathcal{G}}_{\mathrm{bp}})_1$ be a second order bipolar fuzzy gradation of openness on X and $(\hat{\mathcal{G}}_{\mathrm{bp}})_2$ be the largest second order bipolar fuzzy gradation of openness on Y such that $\theta: \left(X, (\hat{\mathcal{G}}_{\mathrm{bp}})_1\right) \to \left(Y, (\hat{\mathcal{G}}_{\mathrm{bp}})_2\right)$ is a second order bipolar fuzzy gradation preserving map. Let $\left(X,(\mathcal{G}_{bp})_1\right)$ and $\left(Y,(\mathcal{G}_{bp})_2\right)$ be the first order bipolar fuzzy gradation spaces from $\left(X,(\hat{\mathcal{G}}_{bp})_1\right)$ and $\left(Y,(\hat{\mathcal{G}}_{bp})_2\right)$, respectively via $\mathbb{D}_1$. Then $(\mathcal{G}_{bp})_2$ is the largest first order bipolar fuzzy gradation of openness on Y such that $\theta\colon\left(X,(\mathcal{G}_{bp})_1\right) \to \left(Y,(\mathcal{G}_{bp})_2\right)$ is a first order bipolar fuzzy gradation preserving map.

Proof:

Let $(\hat{\mathcal{G}}_{bp})_2$ be the largest second order bipolar fuzzy gradation of openness on Y such that $\theta : \left(X,(\hat{\mathcal{G}}_{bp})_1\right) \to \left(Y,(\hat{\mathcal{G}}_{bp})_2\right)$ is a second order bipolar fuzzy gradation preserving map.

To prove $(\mathcal{G}_{bp})_2$ is the largest first order bipolar fuzzy gradation of openness on Y

Let $\left((\mathcal{G}_{bp})_2\right)'$ be the largest first order bipolar fuzzy gradation of openness on Y such that $\theta\colon\left(X,(\mathcal{G}_{bp})_1\right) \to \left(Y,\left((\mathcal{G}_{bp})_2\right)'\right)$ is a first order bipolar fuzzy gradation preserving map, then $\left((\mathcal{G}_{bp})_2\right)'(A_{bp}) = \left((\hat{\mathcal{G}}_{bp})_2\right)'(\widehat{A}_{bp}) \leq \left((\hat{\mathcal{G}}_{bp})_2\right)(\widehat{A}_{bp}) = (\mathcal{G}_{bp})_2(A_{bp})$.
Therefore $\left((\mathcal{G}_{bp})_2\right)'(A_{bp}) \leq (\mathcal{G}_{bp})_2(A_{bp})$.

Hence $(\mathcal{G}_{bp})_2$ is the largest first order bipolar fuzzy gradation of openness on Y such that $\theta : \left(X,(\mathcal{G}_{bp})_1\right) \to \left(Y,(\mathcal{G}_{bp})_2\right)$ is a first order bipolar fuzzy gradation preserving map.

Similar results can be proved for the associations $\mathbb{D}_3$ and $\mathbb{D}_5$.

Remark:5.2.19

The family of second order bipolar fuzzy sets

$\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp}) = \{\widehat{A}_{bp} \in SBPF(X) / \hat{\mathcal{G}}_{bp}(\widehat{A}_{bp}) > 0\}$ forms a second order bipolar fuzzy topology on X.

Definition:5.2.20

Let $(X, \hat{\mathcal{G}}_{bp})$ be a second order bipolar fuzzy gradation space. Then $\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$ is called the **second order bipolar fuzzy topology induced by** $\hat{\mathcal{G}}_{bp}$.

Theorem:5.2.21

Let $(X, \hat{\mathcal{G}}_{bp})$ be a second order bipolar fuzzy gradation space. Let $(X, \mathcal{G}_{bp})$ be a first order bipolar fuzzy gradation space from $(X, \hat{\mathcal{G}}_{bp})$, via $\mathbb{D}_1$. If $\hat{\mathcal{G}}_{bp}$ induces $\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$ and $\mathcal{G}_{bp}$ induces $\tau_{\mathfrak{B}}(\mathcal{G}_{bp})$, then $\left(\widehat{\tau_{\mathfrak{B}}(\mathcal{G}_{bp})}\right) \subseteq \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$ where $\left(\widehat{\tau_{\mathfrak{B}}(\mathcal{G}_{bp})}\right)$ is from $\tau_{\mathfrak{B}}(\mathcal{G}_{bp})$ via R_1.

Proof:

$\widehat{A}_{bp} \in \left(\widehat{\tau_{\mathfrak{B}}(\mathcal{G}_{bp})}\right)$ implies $A_{bp} \in \tau_{\mathfrak{B}}(\mathcal{G}_{bp})$ implies $\mathcal{G}_{bp}(A_{bp}) > 0$

implies $\hat{\mathcal{G}}_{bp}(\widehat{A}_{bp}) > 0$ implies $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$.

Therefore $\left(\widehat{\tau_{\mathfrak{B}}(\mathcal{G}_{bp})}\right) \subseteq \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$.

Theorem:5.2.22

Let $(X, \hat{\mathcal{G}}_{bp})$ be a second order bipolar fuzzy gradation space. Let $\left(I, (\mathcal{G}_{bp})_I\right)$ be a first order bipolar fuzzy gradation space from $(X, \hat{\mathcal{G}}_{bp})$, via $\mathbb{D}_2$. If $\hat{\mathcal{G}}_{bp}$ induces $\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$ on X and $(\mathcal{G}_{bp})_I$ induces $\tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)$ on I, then $\left(\widehat{\tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)}\right)_I \subseteq \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$ where $\left(\widehat{\tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)}\right)_I$ is the second order bipolar fuzzy topology on X from $\tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)$ through the association R_2.

Proof:

$(\widehat{A}_{bp})_I \in \left(\widehat{\tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)}\right)_I$ implies $A_{bp} \in \tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)$

implies $(\mathcal{G}_{bp})_I(A_{bp}) > 0$ implies $\hat{\mathcal{G}}_{bp}\left((\widehat{A}_{bp})_I\right) > 0$

implies $(\widehat{A}_{bp})_I \in \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$.

Therefore $\left(\widehat{\tau_{\mathfrak{B}}\left((\mathcal{G}_{bp})_I\right)}\right)_I \subseteq \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{bp})$.

Theorem:5.2.23

Let $(\mathrm{X}, \mathcal{G}_{\mathrm{bp}})$ be a first order bipolar fuzzy gradation space. For $\alpha \in \mathrm{I}$, let $\left(\mathrm{X}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}\right)$ be the second order bipolar fuzzy gradation space from $(\mathrm{X}, \mathcal{G}_{\mathrm{bp}})$, via $\mathbb{D}_3$. If $\mathcal{G}_{\mathrm{bp}}$ induces $\tau_{\mathfrak{B}}(\mathcal{G}_{\mathrm{bp}})$ on X and $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}$ induces $\hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}\right)$ on X, then $(\hat{\tau}_{\mathfrak{B}})_{\alpha} \subseteq \tau_{\mathfrak{B}}(\mathcal{G}_{\mathrm{bp}})$ where $(\hat{\tau}_{\mathfrak{B}})_{\alpha}$ is the first order bipolar fuzzy topology from $\hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}\right)$, via R_3.

Proof:

$\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}$ implies there exists $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}\right)$ such that $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\alpha}(\mathrm{x}) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(\alpha)$,
$\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\alpha}(\mathrm{x}) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(\alpha)$ implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\alpha}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) > 0$
implies $\mathcal{G}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\alpha}\right) > 0$ implies $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\alpha} \in \tau_{\mathfrak{B}}(\mathcal{G}_{\mathrm{bp}})$.
Therefore $(\hat{\tau}_{\mathfrak{B}})_{\alpha} \subseteq \tau_{\mathfrak{B}}(\mathcal{G}_{\mathrm{bp}})$.

Theorem:5.2.24

Let $(\mathrm{I}, \mathcal{G})$ be a first order gradation space. Let X be a non-empty set and $\mathrm{x} \in \mathrm{X}$. Let $\left(\mathrm{X}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right)$ be the second order bipolar fuzzy gradation space from $(\mathrm{I}, \mathcal{G})$, via $\mathbb{D}_4$. If $\mathcal{G}$ induces $\tau(\mathcal{G})$ on I and $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}$ induces $\hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right)$ on X, then $(\hat{\tau}_{\mathfrak{B}})_{\mathrm{x}} \subseteq \tau(\mathcal{G})$, where $(\hat{\tau}_{\mathfrak{B}})_{\mathrm{x}}$ is the first order bipolar fuzzy topology on I from $\hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right)$, via R_4.

Proof:

Given $(\hat{\tau}_{\mathfrak{B}})_{\mathrm{x}} = \left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{x}} / \ \widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right)\right\}$.
Let $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{x}} \in (\hat{\tau}_{\mathfrak{B}})_{\mathrm{x}}$ implies $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right)$
implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{x}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) > 0$ implies $\mathcal{G}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{x}}\right) > 0$ implies $\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{x}} \in \tau(\mathcal{G})$.
Therefore $(\hat{\tau}_{\mathfrak{B}})_{\mathrm{x}} \subseteq \tau(\mathcal{G})$.

Theorem:5.2.25

Let $(\mathrm{X}, \hat{\mathcal{G}}_{\mathrm{bp}})$ be a second order bipolar fuzzy gradation space. Let $\left(\mathrm{X}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{c}\right)$ be the second order bipolar fuzzy gradation space from $(\mathrm{X}, \hat{\mathcal{G}}_{\mathrm{bp}})$, via $\mathbb{D}_5$. If $\hat{\mathcal{G}}_{\mathrm{bp}}$ induces $\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}}_{\mathrm{bp}})$ and

$\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}$ induces $\hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$, then $\left(\hat{\tau}_{\mathfrak{B}}\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)\right)_{\mathrm{c}} = \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$, where $\hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$ is from $\hat{\tau}_{\mathfrak{B}}\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)$, via R_5.

Proof:

Given $\left(\hat{\tau}_{\mathfrak{B}}\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)\right)_{\mathrm{c}} = \left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}} / \ \widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)\right\}$.

Let $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right) \Leftrightarrow \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) > 0$

$\Leftrightarrow \hat{\mathcal{G}}_{\mathrm{bp}}\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right) > 0$

$\Leftrightarrow \left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}} \in \hat{\tau}_{\mathfrak{B}}\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)$

$\Leftrightarrow \left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)_{\mathrm{c}} \in \left(\hat{\tau}_{\mathfrak{B}}\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)\right)_{\mathrm{c}}$.

Therefore $\left(\hat{\tau}_{\mathfrak{B}}\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)\right)_{\mathrm{c}} \subseteq \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_{\mathrm{c}}\right)$.

Theorem:5.2.26

A function $\theta: \left(\mathrm{X}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right) \rightarrow \left(\mathrm{Y}, \left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)$ is a second order bipolar fuzzy weakly gradation preserving map if and only if $\theta: \left(\mathrm{X}, \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)\right) \rightarrow \left(\mathrm{Y}, \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)\right)$ is second order bipolar fuzzy continuous.

Proof:

Assume θ is a second order bipolar fuzzy gradation preserving map.

$\widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)$ implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) > 0$

implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right) > 0$ implies $\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)$.

Therefore θ is second order bipolar fuzzy continuous.

Conversely, assume θ is second order bipolar fuzzy continuous.

$\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) > 0$ implies $\widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_2\right)$

implies $\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) \in \hat{\tau}_{\mathfrak{B}}\left(\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\right)$ implies $\left(\hat{\mathcal{G}}_{\mathrm{bp}}\right)_1\left(\theta^{-1}\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)\right) > 0$.

Therefore θ is a second order bipolar fuzzy weakly gradation preserving map.

SECTION-5.3

SECOND ORDER BIPOLAR FUZZY COMPACTNESS

Definition:5.3.1

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 1-compact at x ∈ X if the following condition is satisfied:

Given a family of second order bipolar fuzzy open sets $\left\{(\hat{A}_{bp})_{\lambda} / \lambda \in \Lambda\right\}$ with $\vee\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x) = \mathbf{1}$ and $\wedge\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x) = -\mathbf{1}$, there exists a finite subfamily $\Lambda_0(x) \subseteq \Lambda$ such that $\vee\left\{\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x) / \lambda \in \Lambda_0(x)\right\} = \mathbf{1}$ and

$$\wedge\left\{\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x) / \lambda \in \Lambda_0(x)\right\} = -\mathbf{1}$$

Then $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 1- compact in X, if it is second order bipolar fuzzy 1-compact at every x ∈ X.

Definition:5.3.2

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 1*-compact in X if the following condition is satisfied:

Given a family of second order bipolar fuzzy open sets $\left\{(\hat{A}_{bp})_{\lambda} / \lambda \in \Lambda\right\}$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that if $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda\right\} = \mathbf{1}$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x) / \lambda \in \Lambda\right\} = -\mathbf{1}$, for a given $x \in X$, then $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda_0\right\} = \mathbf{1}$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x) / \lambda \in \Lambda_0\right\} = -\mathbf{1}$.

Definition:5.3.3

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 2-compact with respect to α ∈ I if the following condition is satisfied Given a family of second order bipolar fuzzy open sets $\left\{(\hat{A}_{bp})_{\lambda} / \lambda \in \Lambda\right\}$ with $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda\right\} = 1$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda\right\} = -1$, for every $x \in X$, there exists a finite subfamily $\Lambda_0(\alpha) \subseteq \Lambda$ such that $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)\ /\lambda \in \Lambda_0(\alpha)\right\} = 1$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)\ /\lambda \in \Lambda_0(\alpha)\right\} = -1$. Then $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 2-compact in X, if it is second order bipolar fuzzy 2-compact with respect to every $\alpha \in I$.

Definition:5.3.4

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 2*-compact if the following condition is satisfied:

Given a family of second order bipolar fuzzy open sets $\left\{\left(\widehat{A}_{bp}\right)_{\lambda}\ /\lambda \in \Lambda\right\}$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that for a given $\alpha \in I$, if $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda\right\} = 1$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\ \lambda \in \Lambda\right\} = -1$, for every $x \in X$, then $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)\ /\lambda \in \Lambda_0\right\} = 1$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)\ /\lambda \in \Lambda_0\right\} = -1$, for every $x \in X$.

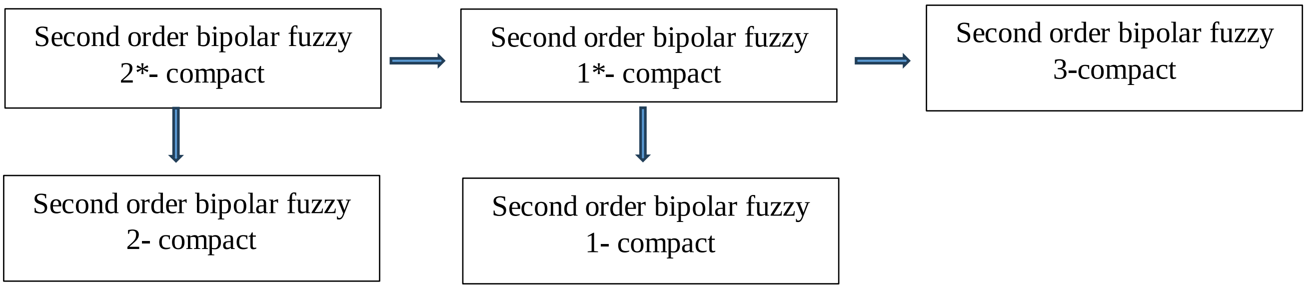
Definition:5.3.5

A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy 3-compact if the following condition is satisfied:

Given a family of second order bipolar fuzzy open sets $\left\{\left(\widehat{A}_{bp}\right)_{\lambda}\ /\lambda \in \Lambda\right\}$ with $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\ /\lambda \in \Lambda\right\} = \hat{1}_{bp}{}^{+}$and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\ /\ \lambda \in \Lambda\right\} = \hat{1}_{bp}{}^{-}$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\ /\lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{+}$and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\ /\lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{-}$.

Remark:5.3.6

All these five concepts of compactness are essentially different. The relations between these concepts are given in the following diagram



Proof :

(i) Second order bipolar fuzzy 2*-compact implies second order bipolar fuzzy 2-compact with respect to $\alpha \in I$.

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a second order bipolar fuzzy 2*-compact on X.

Given a family of second order bipolar fuzzy open sets $\left\{(\hat{A}_{bp})_{\lambda} / \lambda \in \Lambda\right\}$ with $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda\right\} = 1$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda\right\} = -1$, for every $x \in X$. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2*-compact, $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda\right\} = 1$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda\right\} = -1$, for every $x \in X$ and for every $\alpha \in I$. Then, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0\right\} = 1$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0\right\} = -1$, for every $x \in X$. Therefore for a given $\alpha \in I$, there exists a finite subfamily $\Lambda_0(\alpha) \subseteq \Lambda$ such that $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0(\alpha)\right\} = 1$ and $\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0(\alpha)\right\} = -1$.

Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is a second order bipolar fuzzy 2-compact with respect to $\alpha \in I$.

(ii) Second order bipolar fuzzy 2*-compact implies second order bipolar fuzzy 1*-compact.

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a second order bipolar fuzzy 2*-compact on X.

Given a family of second order bipolar fuzzy open sets $\left\{(\hat{A}_{bp})_{\lambda} / \lambda \in \Lambda\right\}$ with

$\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = \mathbf{1}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = -\mathbf{1}$, for every $x \in X$. Then $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda\right\} = 1$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda\right\} = -1$, for every $x \in X$ and for every $\alpha \in I$. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2*-compact, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda_0\right\} = 1$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda_0\right\} = -1$, for every $x \in X$ and for every $\alpha \in I$. That is for the subfamily $\Lambda_0 \subseteq \Lambda$, $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda \in \Lambda_0\right\} = \mathbf{1}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda \in \Lambda_0\right\} = -\mathbf{1}$, for every $x \in X$. Hence $(X, \hat{\tau}_{\mathfrak{B}})$ second order bipolar fuzzy 1*-compact.

(iii) Second order bipolar fuzzy 1*-compact implies second order bipolar fuzzy 3-compact.

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a second order bipolar fuzzy 1*-compact on X.

Given a family of second order bipolar fuzzy open sets $\left\{\left(\widehat{A}_{bp}\right)_{\lambda}/\lambda \in \Lambda\right\}$ with $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}/\lambda \in \Lambda\right\} = \hat{1}_{bp}{}^{+}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}/\lambda \in \Lambda\right\} = \hat{1}_{bp}{}^{-}$. Then $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = \mathbf{1}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = -\mathbf{1}$, for every $x \in X$. Since $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 1*-compact, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda \in \Lambda_0\right\} = \mathbf{1}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda \in \Lambda_0\right\} = -\mathbf{1}$, for every $x \in X$. That is for the subfamily $\Lambda_0 \subseteq \Lambda$, $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}/\lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{+}$and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}/\lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{-}$. Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 3-compact on X.

(iv) Second order bipolar fuzzy 1*-compact implies second order bipolar fuzzy 1-compact.

Let $(X, \hat{\tau}_{\mathfrak{B}})$ be a second order bipolar fuzzy 1*-compact on X.

Given a family of second order bipolar fuzzy open sets $\left\{\left(\widehat{A}_{bp}\right)_{\lambda}/\lambda \in \Lambda\right\}$ with $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = \mathbf{1}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = -\mathbf{1}$, at $x \in X$. Since $(X, \hat{\tau}_{\mathfrak{B}})$ be a second order bipolar fuzzy 1*-compact, $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda \in \Lambda\right\} = \mathbf{1}$

and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda\in\Lambda\right\}=-\mathbf{1}$, for a given $x\in X$. Then, there exists a finite subfamily $\Lambda_0\subseteq\Lambda$ such that $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)/\lambda\in\Lambda_0\right\}=\mathbf{1}$ and $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)/\lambda\in\Lambda_0\right\}=-\mathbf{1}$, for a given $x\in X$. Therefore, at $x\in X$, there exists a finite subfamily $\Lambda_0(x)\subseteq\Lambda$ such that $\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)/\lambda\in\Lambda_0(x)\right\}=\mathbf{1}$ and $\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)/\lambda\in\Lambda_0(x)\right\}=-\mathbf{1}$. Hence $(X,\hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 1-compact on X.

Example:5.3.7

The following example shows that second order bipolar fuzzy 2-compact does not imply second order bipolar fuzzy 2*-compact on X.

Proof:

Let $\widehat{A}_{bp},\widehat{B}_{bp}$ and $\widehat{C}_{bp}$ be second order bipolar fuzzy open sets in X. Let $\alpha_1,\alpha_2,\alpha_3\in I$. Define

$\widehat{A}_{bp}{}^{+}(x)(\alpha_1)=0,\widehat{A}_{bp}{}^{-}(x)(\alpha_1)=-1;\widehat{A}_{bp}{}^{+}(x)(\alpha_2)=0,\widehat{A}_{bp}{}^{-}(x)(\alpha_2)=0;$

$\widehat{A}_{bp}{}^{+}(x)(\alpha_3)=0,\widehat{A}_{bp}{}^{-}(x)(\alpha_3)=-1.$

$\widehat{B}_{bp}{}^{+}(x)(\alpha_1)=0,\widehat{B}_{bp}{}^{-}(x)(\alpha_1)=0;\widehat{B}_{bp}{}^{+}(x)(\alpha_2)=1,\widehat{B}_{bp}{}^{-}(x)(\alpha_2)=-1;$

$\widehat{B}_{bp}{}^{+}(x)(\alpha_3)=1,\widehat{B}_{bp}{}^{-}(x)(\alpha_3)=-1.$

$\widehat{C}_{bp}{}^{+}(x)(\alpha_1)=1,\widehat{C}_{bp}{}^{-}(x)(\alpha_1)=0;\widehat{C}_{bp}{}^{+}(x)(\alpha_2)=0,\widehat{C}_{bp}{}^{-}(x)(\alpha_2)=0;$

$\widehat{C}_{bp}{}^{+}(x)(\alpha_3)=0,\widehat{C}_{bp}{}^{-}(x)(\alpha_3)=0.$

Consider $\left(\widehat{A}_{bp}{}^{+}\vee\widehat{B}_{bp}{}^{+}\right)(x)(\alpha_1)=0,\left(\widehat{A}_{bp}{}^{-}\wedge\widehat{B}_{bp}{}^{-}\right)(x)(\alpha_1)=-1$

$\left(\widehat{A}_{bp}{}^{+}\vee\widehat{B}_{bp}{}^{+}\right)(x)(\alpha_2)=1,\left(\widehat{A}_{bp}{}^{-}\wedge\widehat{B}_{bp}{}^{-}\right)(x)(\alpha_2)=-1$

$\left(\widehat{A}_{bp}{}^{+}\vee\widehat{B}_{bp}{}^{+}\right)(x)(\alpha_3)=1,\left(\widehat{A}_{bp}{}^{-}\wedge\widehat{B}_{bp}{}^{-}\right)(x)(\alpha_3)=-1$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_1) = 1, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_1) = 0$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_2) = 1, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_2) = -1$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_3) = 1, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_3) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_1) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_1) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_2) = 0, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_2) = 0$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_3) = 0\ , \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_3) = -1.$

Let $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_1) = 1$ and $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_1) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_2) = 1$ and $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_2) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_3) = 1$ and $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_3) = -1.$

Now, for $\alpha_1 \in I$, $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_1) = 1$ and $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_1) = -1$, for every $x \in X$. Then there exists a subfamily $\{\widehat{A}_{bp}, \widehat{C}_{bp}\}$ in $\hat{\tau}_{\mathfrak{B}}$ such that $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_1) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_1) = -1.$

Similarly for $\alpha_2 \in I$, $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_2) = 1$ and $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_2) = -1$, for every $x \in X$. Then there exists a subfamily $\{\widehat{A}_{bp}, \widehat{B}_{bp}\}$ and $\{\widehat{B}_{bp}, \widehat{C}_{bp}\}$in $\hat{\tau}_{\mathfrak{B}}$ such that $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x)(\alpha_2) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x)(\alpha_2) = -1$ and

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_2) = 1, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_2) = -1.$

And for $\alpha_3 \in I$, $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_3) = 1$ and $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_3) = -1$, for every $x \in X$. Then there exists a subfamily $\{\widehat{A}_{bp}, \widehat{B}_{bp}\}$ and $\{\widehat{B}_{bp}, \widehat{C}_{bp}\}$in $\hat{\tau}_{\mathfrak{B}}$ such that $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x)(\alpha_3) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x)(\alpha_3) = -1$ and

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x)(\alpha_3) = 1, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x)(\alpha_3) = -1$

From the above it is clear that, the subfamilies $\{\widehat{A}_{bp}, \widehat{C}_{bp}\}$, $\{\{\widehat{A}_{bp}, \widehat{B}_{bp}\}, \{\widehat{B}_{bp}, \widehat{C}_{bp}\}\}$ and $\{\{\widehat{A}_{bp}, \widehat{B}_{bp}\}, \{\widehat{B}_{bp}, \widehat{C}_{bp}\}\}$ for α_1, α_2 and $\alpha_3 \in I$ respectively satisfy the condition of 2-compactness. Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2-compact with respect to $\alpha \in I$. But there exists no such subfamily for all $\alpha \in I$ for which the condition of compactness is true. Thus $(X, \hat{\tau}_{\mathfrak{B}})$ is not second order bipolar fuzzy 2*-compact.

Example:5.3.8

The following example shows that second order bipolar fuzzy 1*-compact does not imply second order bipolar fuzzy 2*-compact.

Let $X = \{x_1, x_2\}$ and $\alpha = \{\alpha_1, \alpha_2\} \in I$. Let $\widehat{A}_{bp}, \widehat{B}_{bp}, \widehat{C}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ be second order bipolar fuzzy open sets where

$\widehat{A}_{bp}{}^{+}(x_1)(\alpha_1) = 1, \widehat{A}_{bp}{}^{-}(x_1)(\alpha_1) = -1; \widehat{A}_{bp}{}^{+}(x_1)(\alpha_2) = 0, \widehat{A}_{bp}{}^{-}(x_1)(\alpha_2) = -1;$

$\widehat{A}_{bp}{}^{+}(x_2)(\alpha_1) = 1, \widehat{A}_{bp}{}^{-}(x_2)(\alpha_1) = -1; \widehat{A}_{bp}{}^{+}(x_2)(\alpha_2) = 0, \widehat{A}_{bp}{}^{-}(x_2)(\alpha_2) = 0.$

$\widehat{B}_{bp}{}^{+}(x_1)(\alpha_1) = 0, \widehat{B}_{bp}{}^{-}(x_1)(\alpha_1) = -0.2; \widehat{B}_{bp}{}^{+}(x_1)(\alpha_2) = 0.1, \widehat{B}_{bp}{}^{-}(x_1)(\alpha_2) = -0.3;$

$\widehat{B}_{bp}{}^{+}(x_2)(\alpha_1) = 0, \widehat{B}_{bp}{}^{-}(x_2)(\alpha_1) = 0; \widehat{B}_{bp}{}^{+}(x_2)(\alpha_2) = 1, \widehat{B}_{bp}{}^{-}(x_2)(\alpha_2) = 0.$

$\widehat{C}_{bp}{}^{+}(x_1)\ (\alpha_1) = 1, \widehat{C}_{bp}{}^{-}(x_1)(\alpha_1) = -0.4; \widehat{C}_{bp}{}^{+}(x_1)(\alpha_2) = 0.3, \widehat{C}_{bp}{}^{-}(x_1)(\alpha_2) = -0.6;$

$\widehat{C}_{bp}{}^{+}(x_2)(\alpha_1) = 0.4, \widehat{C}_{bp}{}^{-}(x_2)(\alpha_1) = -0.5; \widehat{C}_{bp}{}^{+}(x_2)(\alpha_2) = 0.7, \widehat{C}_{bp}{}^{-}(x_2)(\alpha_2) = -1.$

Then, $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_1)(\alpha_1) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_1)(\alpha_1) = -1$

implies $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_1) = \mathbf{1}, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_1) = \mathbf{-1}$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_1)(\alpha_2) = 0.1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_1)(\alpha_2) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_2)(\alpha_1) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_2)(\alpha_1) = -1$

implies $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_2) = \mathbf{1}, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_2) = \mathbf{-1}$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_2)(\alpha_2) = 1$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_2)(\alpha_2) = 0$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha_1) = 1$,$\left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha_1) = -0.4$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha_2) = 0.3$, $\left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha_2) = -0.6$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha_1) = 0.4$, $\left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha_1) = -0.5$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha_2) = 1$, $\left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha_2) = -1$

implies $\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2) = \mathbf{1}$, $\left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2) = \mathbf{-1}$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha_1) = 1$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha_1) = -1$

implies $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1) = \mathbf{1}$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1) = \mathbf{-1}$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha_2) = 0.3$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha_2) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha_1) = 1$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha_1) = -1$

implies $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2) = \mathbf{1}$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2) = \mathbf{-1}$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha_2) = 0.7$, $\left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha_2) = -1$.

From the above it is clear that for every $x \in X$ there exists subfamilies $\{\widehat{A}_{bp}, \widehat{B}_{bp}\}$ and $\{\widehat{A}_{bp}, \widehat{C}_{bp}\}$ satisfy the condition of 1*-compactness. Hence $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 1*-compact. But these two subfamilies do not hold the condition of compactness for $\alpha_2 \in I$. Thus $(X, \hat{\tau}_{\mathfrak{B}})$ is not second order bipolar fuzzy 2*-compact.

Example:5.3.9

The following example shows that second order bipolar fuzzy 3-compact does not imply second order bipolar fuzzy 1*-compact.

Let $X = \{x_1, x_2\}$ and $\alpha \in I$. Let $\widehat{A}_{bp}, \widehat{B}_{bp}, \widehat{C}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ be second order bipolar fuzzy open sets where $\widehat{A}_{bp}{}^{+}(x_1)(\alpha) = 1, \widehat{A}_{bp}{}^{-}(x_1)(\alpha) = -1; \widehat{A}_{bp}{}^{+}(x_2)(\alpha) = 0, \widehat{A}_{bp}{}^{-}(x_2)(\alpha) = 0.$

$\widehat{B}_{bp}{}^{+}(x_1)(\alpha) = 0.4, \widehat{B}_{bp}{}^{-}(x_1)(\alpha) = -0.1; \widehat{B}_{bp}{}^{+}(x_2)(\alpha) = 1, \widehat{B}_{bp}{}^{-}(x_2)(\alpha) = 0.$

$\widehat{C}_{bp}{}^{+}(x_1)\,(\alpha) = 0.1, \widehat{C}_{bp}{}^{-}(x_1)(\alpha) = -0.1; \widehat{C}_{bp}{}^{+}(x_2)(\alpha) = 0, \widehat{C}_{bp}{}^{-}(x_2)(\alpha) = -1.$

Then $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_1)(\alpha) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_1)(\alpha) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+}\right)(x_2)(\alpha) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-}\right)(x_2)(\alpha) = 0$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha) = 0.4, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha) = -0.1$

$\left(\widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha) = 1, \left(\widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha) = 0.1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha) = 0, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_1)(\alpha) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_1)(\alpha) = -1$

$\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x_2)(\alpha) = 1, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x_2)(\alpha) = -1$

implies $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right) = \widehat{1}_{bp}{}^{+}, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right) = \widehat{1}_{bp}{}^{-}$

implies $\left(\widehat{A}_{bp}{}^{+} \vee \widehat{B}_{bp}{}^{+} \vee \widehat{C}_{bp}{}^{+}\right)(x) = \mathbf{1}, \left(\widehat{A}_{bp}{}^{-} \wedge \widehat{B}_{bp}{}^{-} \wedge \widehat{C}_{bp}{}^{-}\right)(x) = \mathbf{-1.}$

From the above membership functions, it is obvious that the subfamilies $\{\widehat{A}_{bp}, \widehat{B}_{bp}\}$ and $\{\widehat{B}_{bp}, \widehat{C}_{bp}\}$ satisfy the condition of 3-compactness. But these two subfamilies do not satisfy the condition of second order bipolar fuzzy 1*-compactness for $x_2 \in X$ and $x_1 \in X$ respectively. Thus $(X, \widehat{\tau}_{\mathfrak{B}})$ is not second order bipolar fuzzy 1*-compact.

Example:5.3.10

The following example shows that second order bipolar fuzzy 1-compact does not imply second order bipolar fuzzy 1*-compact.

Let $X = \{x_1, x_2\}$ and $\alpha \in I$. Let $\widehat{A}_{bp}, \widehat{B}_{bp}, \widehat{C}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ be second order bipolar fuzzy open sets where ${\widehat{A}_{bp}}^{+}(x_1)(\alpha) = 1, {\widehat{A}_{bp}}^{-}(x_1)(\alpha) = -1; {\widehat{A}_{bp}}^{+}(x_2)(\alpha) = 0, {\widehat{A}_{bp}}^{-}(x_2)(\alpha) = 0.$

${\widehat{B}_{bp}}^{+}(x_1)(\alpha) = 0.4, {\widehat{B}_{bp}}^{-}(x_1)(\alpha) = -0.1; {\widehat{B}_{bp}}^{+}(x_2)(\alpha) = 1, {\widehat{B}_{bp}}^{-}(x_2)(\alpha) = 0.$

${\widehat{C}_{bp}}^{+}(x_1)\,(\alpha) = 0.1, {\widehat{C}_{bp}}^{-}(x_1)(\alpha) = -0.1; {\widehat{C}_{bp}}^{+}(x_2)(\alpha) = 0, {\widehat{C}_{bp}}^{-}(x_2)(\alpha) = -1.$

Then, $\left({\widehat{A}_{bp}}^{+} \vee {\widehat{B}_{bp}}^{+}\right)(x_1)(\alpha) = 1, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{B}_{bp}}^{-}\right)(x_1)(\alpha) = -1$

$\left({\widehat{A}_{bp}}^{+} \vee {\widehat{B}_{bp}}^{+}\right)(x_2)(\alpha) = 1, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{B}_{bp}}^{-}\right)(x_2)(\alpha) = 0$

$\left({\widehat{B}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_1)(\alpha) = 0.4, \left({\widehat{B}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_1)(\alpha) = -0.1$

$\left({\widehat{B}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_2)(\alpha) = 1, \left({\widehat{B}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_2)(\alpha) = -1$

$\left({\widehat{A}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_1)(\alpha) = 0.1, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_1)(\alpha) = -1$

$\left({\widehat{A}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_2)(\alpha) = 0, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_2)(\alpha) = -1$

Now for $x_2, x_2 \in X$,

$\left({\widehat{A}_{bp}}^{+} \vee {\widehat{B}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_1)(\alpha) = 1, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{B}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_1)(\alpha) = -1$

$\left({\widehat{A}_{bp}}^{+} \vee {\widehat{B}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_2)(\alpha) = 1, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{B}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_2)(\alpha) = -1.$

Then for $x_1 \in X$, there exists a subfamily $\{\widehat{A}_{bp}, \widehat{B}_{bp}\}$ in $\hat{\tau}_{\mathfrak{B}}$ such that $\left({\widehat{A}_{bp}}^{+} \vee {\widehat{B}_{bp}}^{+}\right)(x_1)(\alpha) = 1, \left({\widehat{A}_{bp}}^{-} \wedge {\widehat{B}_{bp}}^{-}\right)(x_1)(\alpha) = -1.$

Similarly for $x_2 \in X$, there exists a subfamily $\{\widehat{B}_{bp}, \widehat{C}_{bp}\}$ in $\hat{\tau}_{\mathfrak{B}}$ such that $\left({\widehat{B}_{bp}}^{+} \vee {\widehat{C}_{bp}}^{+}\right)(x_2)(\alpha) = 1, \left({\widehat{B}_{bp}}^{-} \wedge {\widehat{C}_{bp}}^{-}\right)(x_2)(\alpha) = -1.$

From the above it is clear that, the subfamilies $\{\widehat{A}_{bp}, \widehat{B}_{bp}\}$ and $\{\widehat{B}_{bp}, \widehat{C}_{bp}\}$ for $x_1, x_2 \in X$ respectively satisfy the condition of second order bipolar fuzzy 1-compactness. But there is no such subfamily for all $x \in X$ for which the condition of compactness is true. Thus $(X, \hat{\tau}_{\mathfrak{B}})$ is not second order bipolar fuzzy 1*-compact.

Theorem:5.3.11

Let $(X, \tau_{\mathfrak{B}})$ be a first order bipolar fuzzy topological space. Let $(X, \hat{\tau}_{\mathfrak{B}})$ be the second order bipolar fuzzy topological space from $(X, \tau_{\mathfrak{B}})$ through the relation R_1. Then the following statements are equivalent:

a. $(X, \tau_{\mathfrak{B}})$ is bipolar fuzzy compact.

b. $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2*-compact.

c. $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2-compact.

d. $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 3-compact.

Proof:

Given $(X, \tau_{\mathfrak{B}})$, $\hat{\tau}_{\mathfrak{B}} = \{\hat{A}_{bp} / A_{bp} \in \tau_{\mathfrak{B}}\}$ where $\hat{A}_{bp}{}^{+}(x)(\alpha) = A_{bp}{}^{+}(x)$, $\hat{A}_{bp}{}^{-}(x)(\alpha) = A_{bp}{}^{-}(x)$, for every $x \in X$ and for every $\alpha \in I$.

(a)⇒(b)

Given $\alpha \in I$, $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = 1$

$\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = -1$, for every $x \in X$.

Then $\vee\left\{\left(A_{bp}{}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = 1$, $\wedge\left\{\left(A_{bp}{}^{-}\right)_{\lambda}(x) / \lambda \in \Lambda, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = -1$, for every $x \in X$.

Therefore there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that

$\vee\left\{\left(A_{bp}{}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda_0, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = 1$, $\wedge\left\{\left(A_{bp}{}^{-}\right)_{\lambda}(x) / \lambda \in \Lambda_0, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = -1$, for every $x \in X$

implies $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = 1$

$\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = -1$, for every $x \in X$.

Therefore $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2*-compact.

(d)⇒(a)

Let $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda}(x)/\lambda\in\Lambda, A_{bp}\in\tau_{\mathfrak{B}}\right\}=1,\quad \wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda}(x)/\lambda\in\Lambda, A_{bp}\in\tau_{\mathfrak{B}}\right\}=-1,$ for every $x\in X$.

Therefore

$$\vee\left\{\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}}\right\}=1$$

$\wedge\left\{\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}}\right\}=-1$, for every $x\in X$, for every $\alpha\in I$

implies $\vee\left\{\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}}\right\}={\hat{1}_{bp}}^{+}$

$\wedge\left\{\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}}\right\}={\hat{1}_{bp}}^{-}$.

Then there exists a finite subfamily $\Lambda_0\subseteq\Lambda$ such that

$\vee\left\{\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}/\lambda\in\Lambda_0\right\}={\hat{1}_{bp}}^{+}$, $\wedge\left\{\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}/\lambda\in\Lambda_0\right\}={\hat{1}_{bp}}^{-}$.

Therefore $\vee\left\{\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda_0\right\}=1$

$\wedge\left\{\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda_0\right\}=-1$, for every $\in X$, for every $\alpha\in I$.

Therefore $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda}(x)/\lambda\in\Lambda_0\right\}=1$

$\wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda}(x)/\lambda\in\Lambda_0\right\}=-1$, for every $x\in X$.

Therefore $(X,\tau_{\mathfrak{B}})$ is bipolar fuzzy compact.

(c)⇒(a)

Let $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda}(x)/\lambda\in\Lambda\right\}=1$

$\wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda}(x)/\lambda\in\Lambda\right\}=-1$, for every $x\in X$.

Therefore $\vee\left\{\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}}\right\}=1$ and

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}}\right\}=-1 \text{ for every } x\in X \text{ and for every } \alpha\in I.$$

Therefore there exists a finite subfamily $\Lambda_0(\alpha)\subseteq\Lambda$ such that

$$\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda_0(\alpha)\right\}=1 \text{ and}$$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda_0(\alpha)\right\}=-1, \text{ for every } x\in X.$$

The subfamily $\Lambda_0(\alpha)$ independent of α, because for any $\alpha,\beta\in I$.

$$\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)=\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\beta)=\left(A_{bp}{}^{+}\right)_{\lambda}(x) \text{ and}$$

$$\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)=\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\beta)=\left(A_{bp}{}^{-}\right)_{\lambda}(x).$$

Therefore denote $\Lambda_0(\alpha)$ as Λ_0

$$\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda_0\right\}=1 \text{ and}$$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda\in\Lambda_0\right\}=-1, \text{ for every } x\in X \text{ and for every } \alpha\in I.$$

implies $\vee\left\{\left(\left(A_{bp}{}^{+}\right)_{\lambda}\right)(x)\ /\lambda\in\Lambda_0\right\}=1$

$$\wedge\left\{\left(\left(A_{bp}{}^{-}\right)_{\lambda}\right)(x)\ /\lambda\in\Lambda_0\right\}=-1, \text{ for every } x\in X.$$

Therefore $(X,\tau_{\mathfrak{B}})$ is bipolar fuzzy compact.

(b) ⇒ (c) and (b) ⇒ (d) follow from remark 5.3.6

Theorem:5.3.12

Given a first order bipolar fuzzy topological space $(I,\tau_{\mathfrak{B}})$, let $(X,(\hat{\tau}_{\mathfrak{B}})_I)$ be the second order bipolar fuzzy topological space from $(I,\tau_{\mathfrak{B}})$ through the relation R_4. Then the following statements are equivalent:

a) $(I,\tau_{\mathfrak{B}})$ is bipolar fuzzy compact.

b) $(X,(\hat{\tau}_{\mathfrak{B}})_I)$ is second order bipolar fuzzy 1*-compact.

c) $(X,(\hat{\tau}_{\mathfrak{B}})_I)$ is second order bipolar fuzzy 1-compact.

d) $(X,(\hat{\tau}_{\mathfrak{B}})_I)$ is second order bipolar fuzzy 3-compact

Proof:

Given $(I,\hat{\tau}_{\mathfrak{B}})$, $(\hat{\tau}_{\mathfrak{B}})_I=\left\{\left(\widehat{A}_{bp}\right)_I/A_{bp}\in\tau_{\mathfrak{B}}\right\}$ where

$\left(\widehat{A}_{bp}{}^{+}\right)_I(x)=A_{bp}{}^{+}$, $\left(\widehat{A}_{bp}{}^{-}\right)_I(x)=A_{bp}{}^{-}$, for every $x\in X$.

(a) ⇒ (b)

For a given $x\in X$, assume

$$\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I(x)\,/\lambda\in\Lambda\,,\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I\in(\hat{\tau}_{\mathfrak{B}})_I\right\}=\mathbf{1}$$

$$\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I(x)\,/\lambda\in\Lambda,\left(\left(\widehat{A}_{bp}\right)_\lambda\right)_I\in(\hat{\tau}_{\mathfrak{B}})_I\right\}=-\mathbf{1}.$$

Therefore $\vee\left\{\left(A_{bp}{}^{+}\right)_\lambda/\lambda\in\Lambda\,,\left(A_{bp}\right)_\lambda\in\tau_{\mathfrak{B}}\right\}=\mathbf{1}$

$$\wedge\left\{\left(A_{bp}{}^{-}\right)_\lambda/\lambda\in\Lambda,\left(A_{bp}\right)_\lambda\in\tau_{\mathfrak{B}}\right\}=-\mathbf{1}$$

implies $\vee\left\{\left(A_{bp}{}^{+}\right)_\lambda(x)/\lambda\in\Lambda\,,\left(A_{bp}\right)_\lambda\in\tau_{\mathfrak{B}}\right\}=1$

$\wedge\left\{\left(A_{bp}{}^{-}\right)_\lambda(x)/\lambda\in\Lambda,\left(A_{bp}\right)_\lambda\in\tau_{\mathfrak{B}}\right\}=-1$, for every $x\in X$.

Therefore there exists a finite subfamily $\Lambda_0\subseteq\Lambda$ such that

$\vee\left\{\left(A_{bp}{}^{+}\right)_\lambda(x)\,/\lambda\in\Lambda_0\right\}=1$, $\wedge\left\{\left(A_{bp}{}^{-}\right)_\lambda(x)/\lambda\in\Lambda_0\right\}=-1$, for every $x\in X$

implies $\vee\left\{\left(A_{bp}{}^{+}\right)_\lambda/\lambda\in\Lambda_0\right\}=\mathbf{1}$, $\wedge\left\{\left(A_{bp}{}^{-}\right)_\lambda/\lambda\in\Lambda_0\right\}=-\mathbf{1}$

implies $\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_\lambda\right)_I(x)\,/\lambda\in\Lambda_0\right\}=\mathbf{1}$, $\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_\lambda\right)_I(x)\,/\lambda\in\Lambda_0\right\}=-\mathbf{1}$, for every $x\in X$.

Therefore $(X,(\hat{\tau}_{\mathfrak{B}})_I)$ is second order bipolar fuzzy 1*-compact.

(b)⇒(c) and (b)⇒(d) follow from remark 5.3.6.

(d)⇒(a)

Given a family of bipolar fuzzy open sets $\left\{\left(A_{bp}\right)_{\lambda} / \lambda \in \Lambda\right\}$.

Assume $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = 1$

$\wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda}(x) / \lambda \in \Lambda, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = -1$, for every $x \in X$

implies $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda} / \lambda \in \Lambda, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = \mathbf{1}$

$\wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda} / \lambda \in \Lambda, \left(A_{bp}\right)_{\lambda} \in \tau_{\mathfrak{B}}\right\} = \mathbf{-1.}$

Therefore $\vee\left\{\left(\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}\right)_{I}(x) / \lambda \in \Lambda, \left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{I} \in (\hat{\tau}_{\mathfrak{B}})_{I}\right\} = \mathbf{1}$

$\wedge\left\{\left(\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}\right)_{I}(x) / \lambda \in \Lambda, \left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{I} \in (\hat{\tau}_{\mathfrak{B}})_{I}\right\} = -\mathbf{1}$, for every $x \in X$

implies $\vee\left\{\left(\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}\right)_{I} / \lambda \in \Lambda\right\} = {\hat{1}_{bp}}^{+}, \wedge\left\{\left(\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}\right)_{I} / \lambda \in \Lambda\right\} = {\hat{1}_{bp}}^{-}$.

Then there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that

$\vee\left\{\left(\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}\right)_{I} / \lambda \in \Lambda_0\right\} = {\hat{1}_{bp}}^{+}, \wedge\left\{\left(\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}\right)_{I} / \lambda \in \Lambda_0\right\} = {\hat{1}_{bp}}^{-}$

implies

$\vee\left\{\left(\left({\widehat{A}_{bp}}^{+}\right)_{\lambda}\right)_{I}(x) / \lambda \in \Lambda_0\right\} = \mathbf{1}, \wedge\left\{\left(\left({\widehat{A}_{bp}}^{-}\right)_{\lambda}\right)_{I}(x) / \lambda \in \Lambda_0\right\} = -\mathbf{1}$, for every $x \in X$

implies $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda} / \lambda \in \Lambda_0\right\} = \mathbf{1}, \wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda} / \lambda \in \Lambda_0\right\} = -\mathbf{1}$

implies $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda_0\right\} = 1, \wedge\left\{\left({A_{bp}}^{-}\right)_{\lambda}(x) / \lambda \in \Lambda_0\right\} = -1$**,** for every $x \in X$.

Therefore $(I, \tau_{\mathfrak{B}})$ is bipolar fuzzy compact.

(c)⇒(a)

Given a family of bipolar fuzzy open set $\left\{\left(\widehat{A}_{bp}\right)_{\lambda} / \lambda \in \Lambda\right\}$.

Assume $\vee\left\{\left({A_{bp}}^{+}\right)_{\lambda}(x) / \lambda \in \Lambda\right\} = 1$,

$$\wedge\left\{\left(A_{bp}{}^{-}\right)_{\lambda}(x)\ /\lambda\in\Lambda\right\}=-1, \text{ for every } x\in X$$

implies $\vee\left\{\left(A_{bp}{}^{+}\right)_{\lambda}\ /\lambda\in\Lambda\,,\left(A_{bp}\right)_{\lambda}\in\tau_{\mathfrak{B}}\right\}=\mathbf{1},$

$$\wedge\left\{\left(A_{bp}{}^{-}\right)_{\lambda}\ /\lambda\in\Lambda,\left(A_{bp}\right)_{\lambda}\in\tau_{\mathfrak{B}}\right\}=\mathbf{-1}$$

implies $\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}(x)\ /\lambda\in\Lambda\,,\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{I}\in\hat{\tau}_{\mathfrak{B}_I}\right\}=\mathbf{1}$

$$\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}(x)\ /\lambda\in\Lambda,\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{I}\in\hat{\tau}_{\mathfrak{B}_I}\right\}=\mathbf{-1}, \text{ for every } x\in X.$$

Then there exists a finite subfamily $\Lambda_0(x)\subseteq\Lambda$ such that

$$\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}(x)/\lambda\in\Lambda_0(x)\right\}=\mathbf{1},\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}(x)/\ \lambda\in\Lambda_0(x)\right\}=\mathbf{-1}.$$

The subfamily $\Lambda_0(x)$ will be independent of x, because for any $x, y\in X$

$$\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}(x)=\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}(y)=A_{bp}{}^{+}$$

$$\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}(x)=\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}(y)=A_{bp}{}^{-}.$$

Therefore denote $\Lambda_0(x)$ as Λ_0

$$\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{I}(x)\ /\lambda\in\Lambda_0\right\}=\mathbf{1},\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{I}(x)/\lambda\in\Lambda_0\right\}=\mathbf{-1}, \text{ for every } x\in X$$

$$\vee\left\{\left(A_{bp}{}^{+}\right)_{\lambda}\ /\lambda\in\Lambda_0\right\}=\mathbf{1},\wedge\left\{\left(A_{bp}{}^{-}\right)_{\lambda}/\lambda\in\Lambda_0\right\}=\mathbf{-1}$$

$$\vee\left\{\left(A_{bp}{}^{+}\right)_{\lambda}(x)/\lambda\in\Lambda_0\right\}=1,\wedge\left\{\left(A_{bp}{}^{-}\right)_{\lambda}(x)/\lambda\in\Lambda_0\right\}=-1, \text{ for every } x\in X.$$

Therefore $(I,\tau_{\mathfrak{B}})$ is bipolar fuzzy compact.

Theorem:5.3.13

$(X,\hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2- compact with respect to $\alpha\in I$ iff $(X,(\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is bipolar fuzzy compact where $(X,(\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is from $(X,\hat{\tau}_{\mathfrak{B}})$ through the relation R_3.

Proof:

Given $(X, \hat{\tau}_{\mathfrak{B}})$, $(\hat{\tau}_{\mathfrak{B}})_{\alpha} = \left\{\left(\widehat{A}_{bp}\right)_{\alpha} / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$ where

$\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}(x) = \widehat{A}_{bp}{}^{+}(x)(\alpha)$, $\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}(x) = \widehat{A}_{bp}{}^{-}(x)(\alpha)$, for every $x \in X$ and for some $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}$.

Assume

$$\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}(x) / \lambda \in \Lambda, \left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}\right\} = 1$$

$$\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}(x) / \lambda \in \Lambda, \left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}\right\} = -1, \text{ for every } x \in X$$

implies

$$\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda, \left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = 1$$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda, \left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = -1, \text{ for every } x \in X, \text{ for every } \alpha \in I.$$

Therefore there exists a finite subfamily $\Lambda_0(\alpha) \subseteq \Lambda$ such that

$$\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0(\alpha)\right\} = 1$$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha) / \lambda \in \Lambda_0(\alpha)\right\} = -1, \text{ for every } x \in X, \text{ for every } \alpha \in I$$

implies

$$\vee\left\{\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}(x) / \lambda \in \Lambda_0(\alpha)\right\} = 1$$

$$\wedge\left\{\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}(x) / \lambda \in \Lambda_0(\alpha)\right\} = -1, \text{ for every } x \in X.$$

Therefore $(X, (\hat{\tau}_{\mathfrak{B}})_{\alpha})$ is bipolar fuzzy compact.

Conversely,

Given a family of second order bipolar fuzzy open sets $\left\{\left(\widehat{A}_{bp}\right)_{\lambda} / \lambda \in \Lambda\right\}$.

Assume $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)\ /\lambda \in \Lambda, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = 1$

$\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = -1$, for every $x \in X$,for every $\alpha \in I$

implies $\vee\left\{\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}(x)\ /\lambda \in \Lambda, \left(\left(\hat{A}_{bp}\right)_{\lambda}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}\right\} = 1$

$\wedge\left\{\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}(x)\ /\lambda \in \Lambda, \left(\left(\hat{A}_{bp}\right)_{\lambda}\right)_{\alpha} \in (\hat{\tau}_{\mathfrak{B}})_{\alpha}\right\} = -1$, for every $x \in X$.

Therefore there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that

$\vee\left\{\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{\alpha}(x)\ /\lambda \in \Lambda_0\right\} = 1, \wedge\left\{\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{\alpha}(x)\ /\lambda \in \Lambda_0\right\} = -1$, for every $x \in X$

implies $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)(\alpha)\ /\lambda \in \Lambda_0\right\} = 1, \wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)(\alpha)/\lambda \in \Lambda_0\right\} = -1$,
for every $x \in X$ and for every $\alpha \in I$.

Therefore $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 2- compact with respect to $\alpha \in I$.

Theorem:5.3.14

If $(X, \hat{\tau}_{\mathfrak{B}})$ is second order bipolar fuzzy 1- compact at $x \in X$, then $(I, (\hat{\tau}_{\mathfrak{B}})_x)$ is bipolar fuzzy compact where $(I, (\hat{\tau}_{\mathfrak{B}})_x)$ is from $(X, \hat{\tau}_{\mathfrak{B}})$ through the association R_2.

Proof:

Given $(X, \hat{\tau}_{\mathfrak{B}})$, then for $x \in X$, $(\hat{\tau}_{\mathfrak{B}})_x = \left\{\left(\hat{A}_{bp}\right)_x/\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}}\right\}$, where

$\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{x} = \left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)$, $\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{x} = \left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(x)$.

Assume $\vee\left\{\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)_{x}\ /\lambda \in \Lambda, \left(\left(\hat{A}_{bp}\right)_{\lambda}\right)_{x} \in (\hat{\tau}_{\mathfrak{B}})_x\right\} = \mathbf{1}$

$\wedge\left\{\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)_{x}\ /\lambda \in \Lambda, \left(\left(\hat{A}_{bp}\right)_{\lambda}\right)_{x} \in (\hat{\tau}_{\mathfrak{B}})_x\right\} = \mathbf{-1}$

implies $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(x)\ /\lambda \in \Lambda, \left(\hat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = \mathbf{1}$

$\wedge\left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\lambda}(\mathrm{x})/\lambda \in \Lambda, \left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\} = \mathbf{-1}$, for every $\mathrm{x} \in \mathrm{X}$.

Therefore there exists a finite subfamily $\Lambda_0(\mathrm{x}) \subseteq \Lambda$ such that

$\vee\left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\lambda}(\mathrm{x})/\lambda \in \Lambda_0(\mathrm{x})\right\} = \mathbf{1}, \wedge\left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\lambda}(\mathrm{x})/\lambda \in \Lambda_0(\mathrm{x})\right\} = \mathbf{-1}$, for every $\mathrm{x} \in \mathrm{X}$

implies $\vee\left\{\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_{\lambda}\right)_{\mathrm{x}}/\lambda \in \Lambda_0(\mathrm{x})\right\} = \mathbf{1}, \wedge\left\{\left(\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_{\lambda}\right)_{\mathrm{x}}/\lambda \in \Lambda_0(\mathrm{x})\right\} = \mathbf{-1.}$

Therefore $(\mathrm{I}, (\hat{\tau}_{\mathfrak{B}})_{\mathrm{x}})$ is bipolar fuzzy compact.

Theorem:5.3.15

If $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$ is $\mathbb{P}$ iff $(\mathrm{X}, (\hat{\tau}_{\mathfrak{B}})_c)$ is $\mathbb{P}$ where $(\mathrm{X}, (\hat{\tau}_{\mathfrak{B}})_c)$ is from $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$ through the relation R_5 and $\mathbb{P}$ denotes any one of the five concepts of second order bipolar fuzzy compactness, namely, second order bipolar fuzzy 1-compact, second order bipolar fuzzy 1*-compact, second order bipolar fuzzy 2-compact, second order bipolar fuzzy 2*-compact and second order bipolar fuzzy 3-compact.

Proof:

Given $(\mathrm{X}, \hat{\tau}_{\mathfrak{B}})$, $(\hat{\tau}_{\mathfrak{B}})_c = \left\{\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right)_c / \widehat{\mathrm{A}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}}\right\}$ where

$\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\right)_c(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(1-\alpha)$

$\left(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}\right)_c(\mathrm{x})(\alpha) = \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{-}(\mathrm{x})(1-\alpha)$, for every $\mathrm{x} \in \mathrm{X}$ and for every $\alpha \in \mathrm{I}$.

Proof of this theorem follows immediately from the definition of $(\hat{\tau}_{\mathfrak{B}})_c$.

Theorem:5.3.16

Let $\theta: \left(\mathrm{X}, \hat{\tau}_{\mathfrak{B}_1}\right) \rightarrow \left(\mathrm{Y}, \hat{\tau}_{\mathfrak{B}_2}\right)$ be a second order bipolar fuzzy continuous onto function. If $\left(\mathrm{X}, \hat{\tau}_{\mathfrak{B}_1}\right)$ is second order bipolar fuzzy 1-compact, then $\left(\mathrm{Y}, \hat{\tau}_{\mathfrak{B}_2}\right)$ is also second bipolar fuzzy 1-compact.

Proof:

For $\mathrm{y} \in \mathrm{Y}$, assume

$\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(y)\,/\lambda\in\Lambda\,,\left(\hat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}_2}\right\}=\mathbf{1}$

$\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(y)\,/\lambda\in\Lambda,\left(\hat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}_2}\right\}=\mathbf{-1.}$

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}\left(\left(\hat{A}_{bp}\right)_{\lambda}\right)\in\hat{\tau}_{\mathfrak{B}_1}$ and since θ is onto, there exists $x\in X$ such that $y=\theta(x)$.

Assume $\vee\left\{\theta^{-1}\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\,/\lambda\in\Lambda\right\}=\mathbf{1},\wedge\left\{\theta^{-1}\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\,/\lambda\in\Lambda\right\}=\mathbf{-1.}$

Since $\hat{\tau}_{\mathfrak{B}_1}$ is second order bipolar fuzzy 1-compact, there exists a finite subfamily $\Lambda_0(x)\subseteq\Lambda$ (depending on x and hence depending on y) such that

$\vee\left\{\theta^{-1}\left(\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\,/\lambda\in\Lambda_0(x)\right\}=\mathbf{1},\wedge\left\{\theta^{-1}\left(\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\,/\lambda\in\Lambda_0(x)\right\}=\mathbf{-1}$

implies $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(y)\,/\lambda\in\Lambda_0(x)\right\}=\mathbf{1},\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(y)\,/\lambda\in\Lambda_0(x)\right\}=\mathbf{-1.}$

Therefore $\left(Y,\hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 1-compact at $y\in Y$.

Hence $\left(Y,\hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 1-compact.

Theorem : 5.3.17

Let $\theta:\left(X,\hat{\tau}_{\mathfrak{B}_1}\right)\to\left(Y,\hat{\tau}_{\mathfrak{B}_2}\right)$ be a second order bipolar fuzzy continuous onto function. If $\left(X,\hat{\tau}_{\mathfrak{B}_1}\right)$ is second order bipolar fuzzy 1*-compact, then $\left(Y,\hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 1*-compact.

Proof:

Given $y\in Y$, assume $\vee\left\{\left(\hat{A}_{bp}{}^{+}\right)_{\lambda}(y)\,/\lambda\in\Lambda\,,\left(\hat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}_2}\right\}=\mathbf{1}$

$\wedge\left\{\left(\hat{A}_{bp}{}^{-}\right)_{\lambda}(y)\,/\lambda\in\Lambda,\left(\hat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}_2}\right\}=\mathbf{-1.}$

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}\left(\left(\hat{A}_{bp}\right)_{\lambda}\right)\in\hat{\tau}_{\mathfrak{B}_1}$ and since θ is onto, there exists $x\in X$ such that $y=\theta(x)$.

Assume $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\ /\lambda \in \Lambda\right\} = \mathbf{1}$, $\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\ /\lambda \in \Lambda\right\} = \mathbf{-1}$**.**

Since $\hat{\tau}_{\mathfrak{B}_1}$ is second order bipolar fuzzy 1* -compact, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)\ /\lambda \in \Lambda_0\right\} = \mathbf{1}$, $\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)\ /\lambda \in \Lambda_0\right\} = -\mathbf{1}$

implies $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(y)\ /\lambda \in \Lambda_0\right\} = \mathbf{1}$, $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(y)\ /\lambda \in \Lambda_0\right\} = \mathbf{-1}$**.**

Therefore $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 1*-compact.

Theorem : 5.3.18

Let $\theta: \left(X, \hat{\tau}_{\mathfrak{B}_1}\right) \to \left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be a second order bipolar fuzzy continuous onto function. If $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ is second order bipolar fuzzy 2-compact, then $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 2-compact.

Proof:

For $\alpha \in I$, assume

$$\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(y)(\alpha)\ /\lambda \in \Lambda, \left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}_2}\right\} = 1$$

$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(y)(\alpha)\ /\lambda \in \Lambda, \left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}_2}\right\} = -1$, for every $y \in Y$.

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}_1}$ and since θ is onto, there exists $x \in X$ such that $y = \theta(x)$.

Assume $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha)\ /\lambda \in \Lambda\right\} = 1$,

$\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha)\ /\lambda \in \Lambda\right\} = -1$, for every $x \in X$.

Since $\hat{\tau}_{\mathfrak{B}_1}$ is second order bipolar fuzzy 2 -compact, there exists a finite subfamily $\Lambda_0(\alpha) \subseteq \Lambda$ such that

$$\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha)\ /\lambda \in \Lambda_0(\alpha)\right\} = 1$$

$\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha)\ /\lambda \in \Lambda_0(\alpha)\right\} = -1$, for every $x \in X$

implies $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(y)\,(\alpha)/\lambda\in\Lambda_0(\alpha)\right\}=1$,

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(y)(\alpha)\,/\lambda\in\Lambda_0(\alpha)\right\}=-1, \text{ for every } y\in Y.$$

Therefore $(Y,\hat{\tau}_{\mathfrak{B}_2})$ is second bipolar fuzzy 2-compact with respect to $\alpha\in I$. Hence $(Y,\hat{\tau}_{\mathfrak{B}_2})$ is second bipolar fuzzy 2-compact.

Theorem:5.3.19

Let $\theta\colon(X,\hat{\tau}_{\mathfrak{B}_1})\to(Y,\hat{\tau}_{\mathfrak{B}_2})$ be a second order bipolar fuzzy continuous onto function. If $(X,\hat{\tau}_{\mathfrak{B}_1})$ is second order bipolar fuzzy 2*-compact, then $(Y,\hat{\tau}_{\mathfrak{B}_2})$ is second order bipolar fuzzy 2*-compact.

Proof:

Given $\alpha\in I$, assume $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(y)(\alpha)\,/\lambda\in\Lambda\,,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}_2}\right\}=1$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(y)(\alpha)\,/\lambda\in\Lambda,\left(\widehat{A}_{bp}\right)_{\lambda}\in\hat{\tau}_{\mathfrak{B}_2}\right\}=-1, \text{ for every } y\in Y.$$

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right)\in\hat{\tau}_{\mathfrak{B}_1}$ and since θ is onto, there exists $x\in X$ such that $y=\theta(x)$.

Assume $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha)\,/\lambda\in\Lambda\right\}=1$

$$\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha)\,/\lambda\in\Lambda\right\}=-1, \text{ for every } x\in X.$$

Since $\hat{\tau}_{\mathfrak{B}_1}$ is second order bipolar fuzzy 2*-compact, there exists a finite subfamily $\Lambda_0\subseteq\Lambda$ such that

$$\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha)\,/\lambda\in\Lambda_0\right\}=1,$$

$$\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)(x)(\alpha)\,/\lambda\in\Lambda_0\right\}=-1, \text{ for every } x\in X$$

implies $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(y)\ (\alpha)/\lambda \in \Lambda_0\right\} = 1,$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}(y)(\alpha)\ /\lambda \in \Lambda_0\right\} = -1, \text{ for every } y \in Y.$$

Therefore $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 2*-compact.

Theorem : 5.3.20

Let $\theta: \left(X, \hat{\tau}_{\mathfrak{B}_1}\right) \to \left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be a second order bipolar fuzzy continuous onto function. If $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ be second order bipolar fuzzy 3-compact, then $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 3-compact.

Proof:

Assume $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\ /\lambda \in \Lambda, \left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}_2}\right\} = \hat{1}_{bp}{}^{+}$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\ /\lambda \in \Lambda, \left(\widehat{A}_{bp}\right)_{\lambda} \in \hat{\tau}_{\mathfrak{B}_2}\right\} = \hat{1}_{bp}{}^{-}$$

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}\left(\left(\widehat{A}_{bp}\right)_{\lambda}\right) \in \hat{\tau}_{\mathfrak{B}_1}$ and since θ is onto, there exists $x \in X$ such that $y = \theta(x)$.

Assume $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)/\lambda \in \Lambda\right\} = \hat{1}_{bp}{}^{+}$, $\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)/\lambda \in \Lambda\right\} = \hat{1}_{bp}{}^{-}$. Since $\hat{\tau}_{\mathfrak{B}_1}$ is second order bipolar fuzzy 3 -compact, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)\ /\lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{+}$, $\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)\ /\lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{-}$

implies $\vee\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}\right)(x)(\alpha)\ /\lambda \in \Lambda_0\right\} = 1$

$$\wedge\left\{\theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\right)\ (x)(\alpha)/\lambda \in \Lambda_0\right\} = -1, \text{ for every } x \in X \text{ and for every } \alpha \in I$$

implies $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda}(y)(\alpha)\ /\lambda \in \Lambda_0\right\} = 1$

$$\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda}\ (y)(\alpha)/\lambda \in \Lambda_0\right\} = -1, \text{ for every } y \in Y \text{ and for every } \alpha \in I$$

implies $\vee\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{\lambda} / \lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{+}$, $\wedge\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{\lambda} / \lambda \in \Lambda_0\right\} = \hat{1}_{bp}{}^{-}$.

Therefore $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy 3-compact.