Fuzzy Topological Boolean Algebras

By

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INTRODUCTION

In 1954 G.Nobeling [21] in his clasified work "Grundlagen der Analytischen Topologie" has studied topological concepts on Boolean algebras. In 1949 Sikorski [27] has published a paper on closure algebras. Almost all topological theorems which can be expressed in terms of Boolean algebras hold for closure algebras. The concepts of open sets, closed sets, nowhere dense sets, F_σ and G_δ sets can be generalised directly to closure algebras.

In 1965 Zadeh introduced the concept of fuzzy sets which is a generalisation of set. In 1968 Chang [4] introduced the concept of fuzzy topological spaces. Lowen [15] modified this definition in 1976 and developed theory of fuzzy topological spaces. Since then various notions in topology like separation axioms, compactness, connectedness, uniformity, proximity, metric spaces, convergence, neighborhood spaces, etc have been extended to fuzzy topological spaces.

The problem of generalising fuzzy topological spaces to fuzzy topological Boolean algebras leads to some interesting results. A detailed study of fuzzy topological Boolean algebras is given in the Ph.D thesis of Parvathi (Bharathiar University, 1994).

In this dissertation we shall study some fundamental properties of fuzzy topological Boolean algebras discussed in the paper, "Fuzzy Topologcial Boolean Algebras" by Parvathi and Meenakshi [22].

The study of fuzzy topological Boolean algebras is of interest in that the underlying set has a Boolean structure. This gives a new dimension to the study of fuzzy topological spaces.

The passage from a topological space (X, τ) to the topological Boolean algebra $(S(X), \tau)$ is relatively smooth when compared to that from a fuzzy topological space (X, δ) to the fuzzy topological Boolean algebra $(S(X), \delta^{0})$.

Chapter 1 is devoted to the study of "Closure algebras" by Sikorski [27]. In section 1 of this chapter some preliminary definitions and results on Boolean algebras in general and Quotient Boolean algebras are collected.

In section 2 of this chapter the article "Closure algebras" by Sikorski is discussed.

A Boolean algebra with a closure operation is called a closure algebra.

Given a set X, S(X) denotes the Boolean algebra of all subsets of X with usual Boolean operations union, intersection and complementation.

If X is a topological space, then S(X) is a closure algebra.

Given a closure algebra $\mathcal{B}$ and a suitable ideal $\mathfrak{I}$ of $\mathcal{B}$, the Quotient algebra $\mathcal{B}/\mathfrak{I}$ is a closure algebra where the closure of $[A] \in \mathcal{B}/\mathfrak{I}$ is given by the class determined by $A^* = (\vee \{ G / G$ open in $\mathcal{B}$ and $G \wedge A \in \mathfrak{I} \})^{C}$.

Some of the interesting results in chapter one are as follows.

1) An isomorphism h of $\mathcal{A}$ on $\mathcal{B}$ is a homeomorphism of $\mathcal{A}$ on $\mathcal{B}$ if and only if $h/\mathfrak{B}(\mathcal{A})$ is a homeomorphism of $\mathfrak{B}(\mathcal{A})$ on $\mathfrak{B}(\mathcal{B})$, where $\mathfrak{B}(\mathcal{A})$ is the least subalgebra containing the closed elements of $\mathcal{A}$

2) The following three conditions are equivalent for arbitrary topological spaces $\mathcal{X}$ and $\mathcal{Y}$.

   i) the spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic.

ii) the closure algebras $S(\mathcal{X})$ and $S(\mathcal{Y})$ are homeomorphic.

iii) the closure algebras $S(\mathcal{X})$ and $S(\mathcal{Y})$ are weakly homeomorphic.

(i.e. $\mathfrak{B}(\mathcal{X})$ and $\mathfrak{B}(\mathcal{Y})$are homeomorphic).

3. Let $\mathfrak{I}_1$ be a $\mathfrak{m}$ - additive ideal of $\mathcal{B}/\mathfrak{I}$ and $\mathfrak{I}_2$ be an $\mathfrak{m}$ - additive ideal of all $A \in \mathcal{B}$ such that $[A] \in \mathfrak{I}_1$. Then the closure algebras $(\mathcal{B}/\mathfrak{I})/\mathfrak{I}_1$ and $\mathcal{B}/\mathfrak{I}_2$ are homeomorphic.

4. A topolocial space $\mathcal{X}$ is separable and meterizable if and only if $S(\mathcal{X})$ is a C-algebra.

5. A representation theorem on C-algebras : -

If a C-algebra $\mathcal{A}$ is of the form $\mathcal{A} = \mathcal{B}/\mathfrak{I}$, where $\mathcal{B}$ is a σ-complete Boolean algebra and $\mathfrak{I}$ is a σ-ideal of $\mathcal{B}$, then it is possible to define in $\mathcal{B}$ a closure operation in such a way that

i) $\mathcal{B}$ with this closure operation is a C-algebra.

ii) the C-algebra $\mathcal{A}$ is identical with the closure algebra which we obtain by the division of the C-algebras $\mathcal{B}$ by the ideal $\mathfrak{I}$.

Chapter 2 is devoted to the study of the article "Fuzzy Topological Boolean Algebras" by Parvathi and Meenakshi [22].

In section 1 of this chapter some fundamental definitions and results on fuzzy topological spaces that are needed for our study are collected.

In section 2 of this chapter the article "Fuzzy Topological Boolean Algebras" is analysed.

A fuzzy topological Boolean algebra is a pair $(\mathcal{B}, \sigma)$, where $\mathcal{B}$ is a Boolean algebra and σ is a subset of $I^{\mathcal{B}}$ such that

i) all constant functions belong to σ

ii) arbitrary union of elements of σ is in σ

iii) finite intersection of elements of σ is in σ.

A general method of constructing a fuzzy topology $\mathbf{W}(\mathcal{K})$ on a Boolean algebra $\mathcal{B}$ is given, where $\mathcal{K}$ is a collection of elements of $\mathcal{B}$ closed for finite union.

Let $\mathcal{K}$ be a collection of elements of $\mathcal{B}$ containing the zero element and closed with respect to finite union. Then there exists a fuzzy topology $\mathbf{W}(\mathcal{K})$ such that

$f \in \mathbf{W}(\mathcal{K})$ iff $f(B) = \text{Sup}\,\{f(A) / A \in \mathcal{K}, A < B\} \forall B \in \mathcal{B}$.

1. When $\mathcal{K} = \{0\}$, $\mathbf{W}(\mathcal{K})$ = All constant functions on $\mathcal{B}$.
2. For $A \in \mathcal{B}$, if $\mathcal{K} = \{0, A\}$,

   $f \in \mathbf{W}(\mathcal{K}) \Rightarrow f$ can take only two values $f(0)$ and $f(A)$
3. When $\mathcal{K} = \tau$, a topology on $\mathcal{B}$

   then $f \in \mathbf{W}(\mathcal{K}) \Leftrightarrow f(A) = \text{Sup}\,\{f(B) / B < A, B \text{ open}\}$

   $= f(\text{int } A) \ \forall A \in \mathcal{B}$

An example where $W(C) \not\subset W(\tau)$, where C is the collection of closed sets on $\mathcal{B}$, is discussed.

It is proved that for a given fuzzy topology $\mathbf{W}(\mathcal{K})$ on $\mathcal{B}$, the collection

$\{f_{A,\varepsilon} / A \in K, \varepsilon \in [0,1]\}$ forms a base for $\mathbf{W}(\mathcal{K})$ where

$f_{A,\varepsilon}(B) = \varepsilon$, if $B < A$ for $B \in \mathcal{B}$

$= 0$, otherwise.

With every fuzzy set f on x, a fuzzy set $f^{\Diamond}$ on S(X) is defined as follows.

$$f^{\Diamond}(A) = \text{Sup}\ \{f(x) / x \in A\}\ \text{for every}\ A \in S(X).$$

Given a fuzzy set f on S(X), f_x is the fuzzy set on X defined by $f_x(x) = f(x)$.

The fuzzy set $f^{\Diamond}$ on S(X) has the following properties.

I) If $h = g^{\Diamond}$ for some $g \in I^X$, then

$$h_x = (g^{\Diamond})_x = g$$

ii) If $h = g_1^{\Diamond} \wedge g_2^{\Diamond}$, for $g_1, g_2 \in I^X$, then

$$h_x = (g_1^{\Diamond} \wedge g_2^{\Diamond})_x = g_1 \wedge g_2\ .$$

iii) If $\{f_\alpha / \alpha \in \Delta\}$ is sub collection of I^X, then

$$\bigvee_{\alpha\in\Delta} f_\alpha^{\Diamond} = (\bigvee_{\alpha\in\Delta} f_\alpha)^{\Diamond}$$

$\delta^{\Diamond}$ is the fuzzy topology with the subbase $\{f^{\Diamond} / f \in \delta\}$. With this definition the following two results are immediate.

(1) Given $f \in I^X$, $f^{\Diamond} \in \delta^{\Diamond} \Leftrightarrow f \in \delta$

(2) $\delta^{\Diamond} \subseteq W(\mathcal{K}_0)$, where $\mathcal{K}_0$ is the collection of all finite subsets of X.

Generalising the concept of the characteristic function a new class of function $\lambda_A : \mathcal{B} \rightarrow I$, $A \in \mathcal{B}$ is defined as follows :

$$\text{For}\ B \in \mathcal{B},\ \lambda_A(B) = 1\ \text{if}\ A \wedge B \neq 0$$

$$= 0\ \text{otherwise.}$$

Some of the interesting properties of this class of functions are

(1) For $A_1, A_2 \in \mathcal{B}$, $\lambda_{A_1 \wedge A_2} \leq \lambda_{A_1} \wedge \lambda_{A_2}$

(2) If $\lambda_{A_1} \wedge \lambda_{A_2} = \lambda_B$, then $B = A_1 \wedge A_2$

(3) Let (X,δ) be a fuzzy topological space. Then the two topologies $t(\delta)$ and $T(\delta^\circ)$ on X coincide where $t(\delta) = \{A \,/\chi_A \in \delta\}$ and $T(\delta^\circ)$ is the topology generated by $\{A \,/\, \lambda_A \in \delta^\circ\}$.

To study separation axioms in fuzzy topological Boolean algebras, two important concepts new order relation and generalised quasi coincident are introduced.

A new order relation $(\leq)_n$ is defined on $I^{\mathcal{B}}$ as follows :

$f\ (\leq)_n\ g$, if given $A \in \mathcal{B}$ there exists a non zero $B < A$ such that for every $C < B$, $f(C) \leq g(C)$.

Two fuzzy sets f and g on $\mathcal{B}$ are said to be generalised quasi-coincident in symbols $f\ Q\ g$, if there exists A in $\mathcal{B}$ such that for every non zero $B < A$, there exists a non zero $C < B$ such that $f(C) + g(C) > 1$.

It is interesting to note that when f and g are fuzzy sets on X,

$f^\circ\ (\leq)_n\ g^\circ \Leftrightarrow f \leq g \Leftrightarrow f^\circ \leq g^\circ$.

For $A, B \in \mathcal{B}$, $\lambda_A\ Q\ \lambda_B \Leftrightarrow A \wedge B \neq 0$.

$f\,Q^C\, g \Leftrightarrow f\ (\leq)_n\ 1 - g$ for $f, g \in I^{\mathcal{B}}$.

Section 3 is devoted to the study of separation axioms and fuzzy continuity. Generalising the separation axioms T_1 and T_2 introduced by Nobeling, the separation axioms FT_1 and FT_2 on fuzzy topological Boolean algebras are defined as follows .

1. A fuzzy topological Boolean $(\mathcal{B}, \sigma)$ is said to be fuzzy T_1 or FT_1 iff for every pair of nonzero fuzzy sets f, g on $\mathcal{B}$ with $f(\not\leq)_n\, g$, there exists an open fuzzy set h on $\mathcal{B}$ such that $f(\not\leq)_n\, h$ and $g\ (\leq)_n\ h$.

2. A fuzzy topological Boolean algebra ($\mathcal{B}$, σ) is said to be fuzzy T_2 or FT_2 iff the following condition is satisfied.

Given two fuzzy sets f, g on $\mathcal{B}$ with $f\ Q^C\ g$ there exists two open fuzzy sets h_1, h_2 such that $f \zeta h_1$, $g \zeta h_2$ and $h_1\ Q^C\ h_2$, where the fuzzy sets f and g are said to be ζ related if there exists $A \in \mathcal{B}$ such that for every non zero $B<A$, there exists a non zero $C<B$ such that $\min(f(C), g(C)) > 0$.

The important results proved here are as follows :

(1) Let (X, δ) be a fuzzy topological space. Then (X,δ) is FT_i iff $(S(X), \delta^o)$ is FT_i for $i=1,2$.

(2) Let $(S(X), \sigma)$ be a fuzzy topological Boolean algebra. Then, $\sigma_x = \{(f)_x / f\in\sigma\}$ is a fuzzy topology on X having the following property:

$$(X, \sigma_x) \text{ is } FT_i,\ i = 1,2 \Leftrightarrow (S(X), \sigma) \text{ is } FT_i,\ i=1,2.$$

Let ($\mathcal{B}_1$, σ_1) and ($\mathcal{B}_2$, σ_2) be two fuzzy topological Boolean algebras. A function $\theta : (\mathcal{B}_1, \sigma_1) \rightarrow (\mathcal{B}_2, \sigma_2)$ is said to be fuzzy continous iff for every $f\in\sigma_2$, $(\theta)^{-1}(f)\in\sigma_1$, where $\theta^{-1}(f)(A) = f(\theta(A))\ \forall\ A \in \mathcal{B}_1$.

It is interesting to note that every fuzzy continous function $\theta:(X,\delta_1)\rightarrow(Y, \delta_2)$ associates a fuzzy continous function $S(\theta) : (S(X), \delta^o) \rightarrow (S(Y),\delta_2^o)$ and vice versa, where $S(\theta)(A) = \{\theta(x) / x\in A\}$

Section 4 is devoted to the study of fuzzy topologies on Quotient algebras.

Before discussing fuzzy topology on Quotient algebras we shall see the fundamental relations between fuzzy sets on $\mathcal{B}$ and fuzzy sets on $\mathcal{B}/\mathfrak{I}$, where $\mathfrak{I}$

is an ideal of $\mathscr{B}$. With every fuzzy set f on $\mathscr{B}/\Im$, one can associate a fuzzy set f_* on $\mathscr{B}$ as follows :

$$f_*(A) = f[A], \forall A \in \mathscr{B}$$

If f is a fuzzy set on $\mathscr{B}$ which preserves congruences (i.e.) $f(A) = f(B)$ if $A \equiv B \bmod \Im$, we can assoicate a function $*f$ on $\mathscr{B}/\Im$ as follows :

$$*f\,[A] = f(A), \forall\, [A] \in \mathscr{B}/\Im.$$

It is intersting to note that

(1) Let f, $g : \mathscr{B}/\Im \rightarrow I$ be two fuzzy sets such that $f(\leq)_n\, g$. Then $f^*\ (\leq)_n\, g^*$.

(2) Let f and g be fuzzy sets on $\mathscr{B}$ which preserve congruences modulo an ideal $\Im$ of $\mathscr{B}$ and $f(\leq)_n\, g$. Then $*f\ (\leq)_n\ *g$.

(3) Let f and g be fuzzy sets on $\mathscr{B}/\Im$ such that $f\ Q^C\ g$. Then $f_*\ Q^C\ g_*$.

(4) Let f and g be fuzzy sets on $\mathscr{B}$ such that they preserve congruences modulo an ideal $\Im$ of $\mathscr{B}$ and $f\zeta\, g$. Then $*f\ \zeta\ *g$.

Given a fuzzy topology σ on $\mathscr{B}$ and an ideal $\Im$ of elements of $\mathscr{B}$, two Quotient topologies $\mathcal{K}_1(\sigma)$ and $\mathcal{K}_2(\sigma)$ on $\mathscr{B}/\Im$ can be defined as follows :

(1) $\mathcal{K}_1(\sigma) = \{*f \,/\, f \in \sigma,\ f$ preserves congruences$\}$

(2) $\mathcal{K}_2(\sigma)$ is the fuzzy topology having $\{f_\Im \,/ f \in \sigma\}$ as a subbase, where

$$f_\Im\,[A] = \text{Sup}\,\{f(B) \,/\, B \equiv A \bmod \Im.\}$$

The important result proved here are

(1) Let $\mathscr{B}$ be a Boolean algebra and $\Im$ be an ideal of $\mathscr{B}$. Let $\mathcal{K}$ be a family of elements closed for finite union. Let $[\mathcal{K}]_\Im$ be the equivalence class of elements

of $\mathcal{K}$ modulo $\mathfrak{I}$. Let $W[\mathcal{K}]_{\mathfrak{I}}$ be the fuzzy topology on $\mathcal{B}/\mathfrak{I}$ induced by $[\mathcal{K}]_{\mathfrak{I}}$.

Then $\mathcal{K}_1(W(\mathcal{K})) \subset \mathcal{K}_2(W(\mathcal{K})) \subset W[\mathcal{K}]_{\mathfrak{I}}$.

When $\mathcal{K}$ is an ideal of $\mathcal{B}$, $\mathcal{K}_1(W(\mathcal{K})) = \mathcal{K}_2(W(\mathcal{K})) = W([\mathcal{K}]_{\mathfrak{I}})$

(2) Let $(\mathcal{B}, \sigma)$ be a fuzzy topological Boolean algebra and $\mathfrak{I}$ be an ideal of . elements of $\mathcal{B}$. Then $\theta: (\mathcal{B}, \sigma) \rightarrow (\mathcal{B}/\mathfrak{I}, \mathcal{K}_1(\sigma))$ defined by $\theta(A) = [A]\ \forall\ A \in \mathcal{B}$ is fuzzy continous.

Review of Literature

REVIEW OF LITERAUTE

In 1965, Zadeh[29] introduced the notion of fuzzy sets. Chang[4] introduced the notion of fuzzy topological space in 1968. Lowen[15] modified the definition of chang by including all constant functions in the definition of fuzzy topology and studied fuzzy compactness. Since then, various notions in topology like separation axioms, compactness, connectedness, uniformity, proximity, metric spaces, convergences, neighbourhood spaces, etc have been extended to fuzzy topological spaces.

Nobeling[21] in his classic "Grundlagen der Analytischen Topologie" has deveoped a detailed study of topological Boolean algebras generalising many topological concepts to Boolean algebras. The study of separation axioms on Boolean algebras necessiate new methods of defining these concepts which when applied to topological spaces coincide with the classical notions.

Another important article on topological Boolean algebra is "On topological Boolean Algebras" by Meenakshi[19] in 1968. Meenakshi has extended some non-elementary parts of point-set topology to classical Boolean algebras, by considering M-algebra the class of regular topological Boolean algebras which have a σ-locally finite bases.

The important results proved here are,

(1) A T_1 topological space (X, τ) is metrizable iff $(S(x),\tau)$ is an M-algebra.

(2) Every Quotient M-algebra of topological weight $\mathcal{M}$ is weakly homeomorphic to a closure algebra of the form $S(X)/\mathfrak{I}$ where X is a metric space of topological weight $\mathcal{M}$ and $\mathfrak{I}$ is an $\mathcal{M}$ -additive ideal of S(X).

In 1968 Chang[4] introduced the concept of fuzzy topology in the article "Fuzzy topological spaces". Chang has defined fuzzy topology δ on a set X as a subset of I^X such that

i) the constant functions **0,1** belong to δ

ii) arbitrary union of fuzzy sets in δ is in δ

iii) finite intersection of fuzzy sets in δ is in δ

Lowen[15] in his article "Fuzzy topological spaces and Fuzzy compactness", has modified Chang's defintions by including all constant functions and analysed fuzzy compactness. Lowen has associated with every topology τ on X, a fuzzy topology δ on X and with every fuzzy topology δ on X, a topology $i(\delta)$ on X and proved that $i(\omega(\tau)) = \tau$ and $\delta \subset \omega(i(\delta))$.

Lowen has generalised continuity and compactness to fuzzy situation and defined fuzzy compact set and fuzzy compact space as follows:

1) A fuzzy set $\gamma \in I^X$ is fuzzy compact iff for all family $\beta \subset \delta$ such that $\sup\limits_{\mu \in \beta} \mu \geq \gamma$.

and for all $\varepsilon > 0$, there exists a finite sub family $\beta_0 \subset \beta$ such that $\sup\limits_{\mu \in \beta_0} \mu \geq \gamma - \varepsilon$

2) A fuzzy topological space (X,δ) is fuzzy compact iff each constant fuzzy set in (X,δ) is fuzzy compact.

The following are the important theorems established in this article.

(1)The fuzzy topological space $(X, \omega(\tau))$ is fuzzy compact iff the space (X, τ) is compact.

(2) If $f : (X,\delta) \rightarrow (Y, \gamma)$ is fuzzy continous and γ is a fuzzy compact set in (X,δ) then $f(\gamma)$ is fuzzy compact in (Y,γ)

(3) If (X, δ) is fuzzy compact and f is fuzzy continous mapping from (X,δ) on to (Y,γ) then (Y,γ) is fuzzy compact.

Some of the important topics studied under fuzzy topological spaces are as follows:

Fuzzy separation axioms, Fuzzy proximity, Fuzzy uniformity and Fuzzy convergence.

A number of articles have been published on these topics. We shall mention a few.

Important contribution on fuzzy separation axioms are due to HUTTON. B[3], RODABAUGH.S.E[26], MIRA SARKAR[20], REKHA SRIVASTAVA, S.N. LAL and ARUN.K SRIVASTAVA[24], GANGULY.S and SAHA.S[5], SINHA.S.P[28].

On fuzzy proximities KATSARAS.A.K.[11,12], GHANIM.M.H[6], KANDIL. A and EL-SHAFEE.M.E[10] have published interesting articles.

Fuzzy uniformity has been studied by HUTTON.B[8], LOWEN.R and WUYTS.P[17], RODABOUGH.S.E[25], ARTICO.G and MORESCO.R[1] and KATSARAS. A.K[13].

Fuzzy convergence has been studied by LOWEN.R.[16,18], PU PAO-MING and LIU YING-MING[23].

Chapter 1

CHAPTER -1

Nobeling [21] has studied topological concepts on Boolean algebras in 1954. Sikorski[27] has published a paper on closure algebras in 1949. This chapter is devoted to the study of the article "Closure Algebras" by Sikorski[27].

Preliminary definitions and results on Boolean algebras are collected in the first section.

The results established in the article "Closure Algebras" by Sikorski[27] are studied in section 2. Here the concepts of open sets, closed sets, nowhere dense sets, F_σ and G_δ sets are generalised to closure algebras. A representation theorem for closure algebras is established.

SECTION:1

PRELIMINARY DEFINITIONS ON BOOLEAN ALGEBRAS

Definition : 1.1.1.

A Boolean algebra is a non-empty set $\mathscr{B}$ in which two binary operations V, $\wedge$ and one unary operation $'$ are defined such that for $A,B,C \in \mathscr{B}$

1. $A \vee B = B \vee A$, $A \wedge B = B \wedge A$
2. $A \vee (B \vee C) = (A \vee B) \vee C$, $A \wedge (B \wedge C) = (A \wedge B) \wedge C$
3. $(A \wedge B) \vee B = B$, $(A \vee B) \wedge B = B$
4. $A \wedge (B \vee C) = (A \wedge B) \vee (A \vee C)$, $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
5. $(A \wedge A') \vee B = B$, $(A \vee A') \wedge B = B$

i.e., A Boolean algebra $\mathscr{B}$ is a complemented distributive lattice.

Example: 1.1.2.

Let X be a topological space. Let $\mathcal{B}$ be the class of all regular closed subsets of X. i.e., subsets that are closures of open subsets of X. The join $A \cup B$ of sets $A, B \in \mathcal{B}$ is set theoritical union of A and B. The meet $A \cap B$ of sets $A, B \in \mathcal{B}$ is the closure of the interior of the set theoritical intersection of A and B. The Boolean complement A' of a set $A \in \mathcal{B}$ is the closure of the set theoritical complement of A. Then $\mathcal{B}$ is a Boolean algebra.

Let S be a set of elements of a Boolean algebra $\mathcal{B}$. An element $A_s \in \mathcal{B}$ ($A_p \in \mathcal{B}$) is called the sum (product) of all elements $A \in S$ if it is the least (greatest) element containing (contained in) all elements $A \in S$.

Definition : 1.1.3.

A Boolean algebra $\mathcal{B}$ is said to be complete if arbitrary sum of elments of $\mathcal{B}$ is in $\mathcal{B}$

Definition : 1.1.4

A Boolean algebra $\mathcal{B}$ is said to be $\mathfrak{m}$ -complete if the sum of any collection of elements of $\mathcal{B}$ with cardinal $\leq \mathfrak{m}$ is in $\mathcal{B}$

Definition 1.1.5

A Boolean algebra $\mathcal{B}$ is said to be σ -complete if it is $\aleph_0$ complete.

Definition : 1.1.6

A class $\mathfrak{I}$ of elements of a Boolean algebra $\mathcal{B}$ is said to be an ideal of $\mathcal{B}$ if,

i) $A \in \mathfrak{I}$, $A_1 \in \mathcal{B}$ and $A_1 \subset A$, then $A_1 \in \mathfrak{I}$

ii) $A1 \in \mathfrak{I}$ and $A_2 \in \mathfrak{I}$ then the sum $A_1 \vee A_2 \in \mathfrak{I}$

Definition : 1.1.7

An ideal $\mathfrak{I}$ of an $\mathscr{M}$-complete Boolean algebra $\mathscr{B}$ is $\mathscr{M}$-additive if the sum of elements belonging to an aribitrary set $S \subset \mathfrak{I}$ of cardinal $\leq \mathscr{M}$ belongs to $\mathfrak{I}$.

Definition : 1.1.8

Let $\mathfrak{I}$ be an ideal of a Boolean algebra $\mathscr{B}$. The symbol $\mathscr{B}/\mathfrak{I}$ denotes the Boolean algebra defined in the following way;

Elements of $\mathscr{B}/\mathfrak{I}$ are disjoint classes of elements of $\mathscr{B}$ such that two elements $A_1, A_2 \in \mathscr{B}$ belong to the same class iff $(A_1' \wedge A_2) \vee (A_1 \wedge A_2') \in \mathfrak{I}$.

The class $\mathbf{C} \in \mathscr{B}/\mathfrak{I}$ containing an element $A \in \mathscr{B}$ is denoted by $[A]$. The Boolean operations of $\mathscr{B}/\mathfrak{I}$ are defined as follows :

For $[A_1], [A_2] \in \mathscr{B}/\mathfrak{I}$

(i) $[A_1] \vee [A_2] = [A_1 \wedge A_2]$

(ii) $[A_1] \vee [A_2] = [A_1 \wedge A_2]$

(iii) $[A]' = [A']$

(iv) $[A_1] \wedge [A_2]' = [A_1 \wedge A_2']$

Example : 1.1.9

Let $\mathscr{B}$ be a Boolean algebra and E be an element of an Boolean algebra. Then the set of all subelements of E deonoted by $\mathscr{B}/E$ is a Boolean algebra under the binary operations meet and join as defined in $\mathscr{B}$ and the complement of A in $\mathscr{B}/E$ is given by the meet of E and the complement of A in the Boolean algebra $\mathscr{B}$.

Definition : 1.1.10

A set S of elements of a σ-complete Boolean algebra $\mathcal{B}$ is said to be a σ-sub-algebra of $\mathcal{B}$ provided

(i) if $A \in S$, then $A' \in S$

(ii) if $A_n \in S$ $(n=1,2..)$ then $\sum_{n=1}^{\infty} A_n \in S$

SUB ALGEBRAS AND ALGEBRAS E$\mathcal{B}$

Let S be a set of elements of a Boolean algebra $\mathcal{B}$ and $E \in \mathcal{B}$. Then the symbol ES denotes the set of all elements $E \wedge A$. where $A \in S$.

Remark: 1.1.11

$E\mathcal{B} = \{ A / A \in \mathcal{B} \text{ and } A \subset E\}$

$E\mathcal{B}$ is a Boolean algebra with respect to the same binary operations defined in $\mathcal{B}$ and the complement of an element $A \in E\mathcal{B}$ in the Boolean algebra $E\mathcal{B}$ is EA'

If $\mathcal{B}$ is $\mathfrak{m}$ -complete, then $E\mathcal{B}$ is $\mathfrak{m}$ -complete.

Remark : 1.1.12

A σ-subalgebra of a σ-complete Boolean algebra is also a σ-complete Boolean algebra.

Field of sets:

By a field of sets of a fixed space $\mathcal{X}$ we mean a class $S(\mathcal{X})$ of subsets of $\mathcal{X}$ such that $S(\mathcal{X})$ is closed for finite $\vee$, $\wedge$ and complementation.

SECTION : 2

CLOSURE ALGEBRAS

Definition 1.2.1

A closure algebra is a σ - complete Boolean algebra $\mathcal{B}$ in which with every $A \in \mathcal{B}$ there is associated an element $\overline{A} \in \mathcal{B}$ such that for $A_1, A_2 \in \mathcal{B}$

i) $\overline{A_1 \vee A_2} = \overline{A}1 \vee \overline{A}_2$

ii) $\overline{0} = 0$

iii) $A \subset \overline{A}$

iv) $\overline{(\overline{A})} = \overline{A}$

The element $\overline{A}$ is called the closure of A.

The interior and Frontier of A are denoted respectively as int A and F_r (A) and are defined as int $A = (\overline{A'})'$ and $F_r(A) = \overline{A} \wedge \overline{A'}$

An element $A \in \mathcal{B}$ is called respectively.

Closed , if $A = \overline{A}$

Open , if $A = \text{int}(A)$

nowherr dense, if $A \subset F_r\overline{(A)}$

dense, if $\overline{A} = /\mathcal{B}/$, where $/\mathcal{B}/$ is the greatest element of $\mathcal{B}$.

a G_δ , if $A = \prod_n A_n$ where A_n is open, n = 1,2,.....................

an F_σ, if $A = \sum_n A_n$ where A_n is closed, n = 1,2,....................

Remark 1.2.2

If X is a topological space, then the field S(X) is a closure algebra.

Notation

If $\mathcal{B}$ is a closure algebra, then the class of all open elements and closed elements of $\mathcal{B}$ are respectively denoted by $\mathcal{O}(\mathcal{B})$ and $\mathcal{F}(\mathcal{B})$.

The least σ - subalgebra of $\mathscr{B}$ which contains $\mathscr{F}(\mathscr{B})$ is denoted by $\mathfrak{B}(\mathscr{B})$ and the elements of $\mathfrak{B}(\mathscr{B})$ are called Borel elements.

Proposition 1.2.3

Let $\mathscr{B}$ be a Boolean algebra and E an element of $\mathscr{B}$. Then the subalgebra $E\mathscr{B}$ is a closure algebra, where the closure of any $A \in E\mathscr{B}$ (denoted as $\overline{A}_E$) with respect to $E\mathscr{B}$ is defined as

$$\overline{A}_E = E \wedge \overline{A}.$$

The closed elements, open elements and the Borel elements of $E\mathscr{B}$ are respectively given by

$$\mathscr{F}(E\mathscr{B}) = E\,\mathscr{F}(\mathscr{B}),\ \mathcal{O}(E\mathscr{B}) = E\,\mathcal{O}(\mathscr{B}).\text{ and }\mathfrak{B}(E\mathscr{B}) = E\,\mathfrak{B}(\mathscr{B}).$$

Definition 1.2.4

If $\mathscr{X}$ is a topological space then $\mathfrak{B}(\mathbf{S}(\mathscr{X}))$ is a closure algebra.

Basis of a closure algebra:-

Definition : 1.2.5

A class $\mathfrak{R}$ of open elements of a closure algebra $\mathscr{B}$ in called a basis of $\mathscr{B}$ if

i) $\mathscr{B}$ is $\overline{\overline{\mathfrak{R}}}$ complete , where $\overline{\overline{\mathfrak{R}}}$ is the cardinal of $\mathfrak{R}$.

ii) every open element $G \in \mathscr{B}$ is the sum of elements belonging to subclass of $\mathfrak{R}$.

Proposition : 1.2.6

Every complete closure algebra $\mathscr{B}$ possesses a basis.

Proposition : 1.2.7

If $\mathscr{X}$ is a topological space, then the field $\mathbf{S}(\mathscr{X})$ possesses a basis.

Proposition : 1.2.8

If $\mathscr{X}$ is a metric separable space, then field $S(\mathscr{X})$ possesses an enumerable basis.

Proposition : 1.2.9

If $\mathfrak{R}$ is a basis of a closure algebra $\mathscr{B}$ and $E \in \mathscr{B}$, then $E\mathfrak{R}$ is a basis of $E\mathscr{B}$ and $\overline{\overline{E\mathfrak{R}}} \leq \overline{\overline{\mathfrak{R}}}$.

Proposition : 1.2.10

If a closure algebra $\mathscr{B}$ possesses a basis, then the sum (product) of an arbitrary set of open (closed) elements exists and is open (closed).

Definition : 1.2.11

Let $\mathbf{A}_0$ be a σ- subalgebra of a closure algebra $\mathscr{B}$. Then $\mathbf{A}_0$ is a closure subalgebra $\mathscr{B}$ of if $A \in \mathbf{A}_0 \Rightarrow \overline{A} \in \mathbf{A}_0$

Proposition : 1.2.12

A closure subalgebra $\mathbf{A}_0$ of a closure algebra is also a closure algebra (with the same operation of closure) and the open elements, closed elements and the Borel elements of the closure subalgebra, $\mathbf{A}_0$ are respectively given by

$$\mathcal{O}(\mathbf{A}_0) = \mathbf{A}_0\mathcal{O}(\mathscr{B}),$$

$$\mathscr{F}(\mathbf{A}_0) = \mathbf{A}_0\mathscr{F}(\mathscr{B})$$

$$\mathcal{B}(\mathbf{A}_0) = \mathbf{A}_0\mathcal{B}(\mathscr{B})$$

Theorem : 1.2.17

Let $\mathcal{X}$ and $\mathcal{Y}$ be topologicl spaces and let **f** be a σ-homomorphism of $S(\mathcal{Y})$ in $S(\mathcal{X})$ induced by a mapping ϕ of $\mathcal{X}$ in $\mathcal{Y}$. Then **f** is continous if and only if ϕ is continous.

Theorem :1.2.18

A σ-homomorphism **f** of $\mathscr{A}$ in $\mathscr{B}$ is continous if and only if the homomorphism **f**/ $\mathscr{B}(\mathscr{A})$ (the restriction of **f** on $\mathscr{B}(\mathscr{A})$) is continous.

Definition : - 1.2.19

Two closure algebras $\mathscr{A}$ and $\mathscr{B}$ are homeomorphic if there exists an isomorphism **h** of $\mathscr{A}$ on $\mathscr{B}$ such that both **h** and $\mathbf{h}^{-1}$ are continous. The isomorphism **h** is then called a homeomorphism of $\mathscr{A}$ on $\mathscr{B}$.

Result : 1.2.20

An isomorphism **h** of a closure algebra $\mathscr{A}$ on a closure algebra $\mathscr{B}$ is a homeomorphism if and only if $\mathbf{h}(\overline{A}) = \overline{\mathbf{h}(A)}$ for every $A \in \mathscr{A}$.

Result 1.2.21

An isomorphism **h** of a Boolean algebra $\mathscr{A}$ on a Boolean algebra $\mathscr{B}$ is a homeomorphism of $\mathscr{A}$ on $\mathscr{B}$ if and only if $\mathbf{h}/\mathfrak{B}(\mathscr{A})$ is a homeomorphism of $\mathfrak{B}(\mathscr{A})$ on $\mathfrak{B}(\mathscr{B})$.

Definition : 1.2.22

Two closure algebras $\mathscr{A}$ and $\mathscr{B}$ are weakly homeomorphic if the closure algebras $\mathfrak{B}(\mathscr{A})$ and $\mathfrak{B}(\mathscr{B})$ are homeomorphic.

Proposition : 1.2.23

The following three conditions are equivalent for arbitrary topological spaces $\mathcal{X}$ and $\mathcal{Y}$.

i) The spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic

ii) The closure algebras $\mathbf{S}(\mathcal{X})$ and $\mathbf{S}(\mathcal{Y})$ are homeomorphic

iii) The closure algebras $\mathbf{S}(\mathcal{X})$ and $\mathbf{S}(\mathcal{Y})$ are weakly homeomorphic.

i.e. $\mathfrak{B}(\mathcal{X})$ and $\mathfrak{B}(\mathcal{Y})$ are homeomorphic.

The localisation of topological properties :-

Let $\mathscr{B}$ denote a closure algebra with a basis $\mathfrak{R}$ and $\mathfrak{I}$ an ideal of $\mathscr{B}$.

Let A be an arbitrary element of $\mathscr{B}$. By proposition 1.2.10 there exists an open element G_0 which is the sum of all open elements G such that $G \wedge A \in \mathfrak{I}$. The closed element G_0' is denoted by A^*,

where $A^* = (\Sigma G)' = (\Sigma R)'$,

$R \in \mathfrak{R}$, and $R \wedge A \in \mathfrak{I}$.

Theorem : 1.2.24

If an $\mathfrak{m}$ -complete closure algebra $\mathscr{B}$ possesses a basis $\mathfrak{R}$ of cardinal $\leq \mathfrak{m}$ and if $\mathfrak{I}$ is an $\mathfrak{m}$ -additive ideal of $\mathscr{B}$, then

$$A \wedge (A^*)^C \in \mathfrak{I} \qquad \text{for every } A \in \mathscr{B}$$

The conditions : $A^* = 0$ and $A \in \mathfrak{I}$ are thus equivalent.

The division of closure algebra by ideals :-

Theorem :- 1.2.25

Let $\mathscr{B}$ be a $\mathfrak{m}$-complete Boolean algebra with a basis $\mathfrak{R}$ of cardinal $\leq \mathfrak{m}$ and $\Im$ an $\mathfrak{m}$-additive ideal of $\mathscr{B}$. Then

i) $[\mathfrak{R}]$ is a basis of $\mathscr{B}/\Im$ of cardinal $\leq \mathfrak{m}$

ii) $\mathcal{O}(\mathscr{B}/\Im) = [\mathcal{O}(\mathscr{B})]$

iii) $\mathscr{F}(\mathscr{B}/\Im) = [\mathscr{F}(\mathscr{B})]$

iv) $\mathfrak{B}(\mathscr{B}/\Im) = [\mathfrak{B}(\mathscr{B})]$

i.e an element of $\mathscr{B}/\Im$ is respectively an open, closed or Borel element of $\mathscr{B}/\Im$ if and only if it possesses a representative which is respectively an open, closed or Borel element of $\mathscr{B}$.

Theorem :- 1.2.26

Let $\mathscr{B}$ be a $\mathfrak{m}$-complete Boolean algebra with a basis $\mathfrak{R}$ of cardinal $\leq \mathfrak{m}$ and $\Im$ an $\mathfrak{m}$-additive ideal of $\mathscr{B}$. Then for $A \in \mathscr{B}$.

i) $\overline{[A]} = [A^*]$

ii) $\overline{[A]} \subset [\overline{A}]$

iii) $[\text{Int}(A)] \subset \text{Int}([A])$

iv) $\text{Fr}([A]) \subset [\text{Fr}(A)]$

The open element $(\overline{[A]})' \in \mathscr{B}/\Im$ is the sum of all elements $[R]$ where $R \in \mathfrak{R}$ and $[R \wedge A] = [R] \wedge [A] = 0$ i.e. $R \wedge A \in \Im$. Hence

$$(\overline{[A]})' = \Sigma\,[R] = [\Sigma\, R] = [(A^*)'] = [A^*]'$$

Where $R \in \mathfrak{R}$ and $R \wedge A \in \Im$.

Theorem 1.2.27

Let **h** be a homeomorphism of $\mathcal{A}$ on a closure algebra $\mathcal{B}$ and $\mathbf{J} = h(\mathfrak{I})$. Then $\mathcal{A}/\mathfrak{I}$ and $\mathcal{B}/\mathbf{J}$ are also homeomorphic.

Remark 1.2.28

The above theorem shows that the division of a closure algebra by an ideal is a topological operation.

Theorem : 1.2.29

i) The natural homomorphism of $\mathcal{B}$ on $\mathcal{B}/\mathfrak{I}$ is continous.

ii) The natural isomorphism of $E\mathcal{B}/E\mathfrak{I}$ on $[E](\mathcal{B}/\mathfrak{I})$ (where $E \in \mathcal{B}$) is a homeomorphism.

iii) The natural isomorphism of $\mathscr{B}(\mathcal{B})/\mathfrak{I}.\mathscr{B}(\mathcal{B}))$ on $\mathscr{B}(\mathcal{B}/\mathfrak{I}) = [\mathscr{B}(\mathcal{B})]$ is a homeomorphism.

Theorem : 1.2.30

Let $\mathfrak{I}_1$ be a $\mathcal{M}$-additive ideal of $\mathcal{B}/\mathfrak{I}$ and $\mathfrak{I}_2$ be the $\mathcal{M}$-additive ideal of all $A \in \mathcal{B}$ such that $[A] \in \mathfrak{I}_1$. Then the closure algebras $(\mathcal{B}/\mathfrak{I})/\mathfrak{I}_1$ and $\mathcal{B}/\mathfrak{I}_2$ are homeomorphic

C - algebras. Representation theorems :-

Definition : 1.2.31

A closure algebra $\mathcal{A}$ is called a C-algebra if it satisfies the following axiom:

There exists an enumerable sequence $\{R_n\}$ of open elements of $\mathscr{A}$ with the property that every open element $G \in \mathscr{A}$ is the sum of all elements $\overline{R_n}$ such that $\overline{R_n} \subset G$.

Definition :1.2.32.

Every Sequence $R_n \in \mathcal{O}(\mathscr{A})$ possessing the property in 1.2.31 is called a C - basis of $\mathscr{A}$.

It is interesting to note that C-algebras possess many interesting properties of separable metric spaces.

The following theorem is a natural generalisation of separable metric spaces :

Theorem :1.2.33

A topological space $\mathscr{X}$ is separable and metrizable if any only if $S(\mathscr{X})$ is a C-algebra

If $\mathscr{A}$ is a C-algebra and if $\mathfrak{I}$ is a σ-field of $\mathscr{A}$, then $\mathscr{A}/\mathfrak{I}$ is also a C-algebra.

If $\mathfrak{R}$ is a C-basis of $\mathscr{A}$, then $[R]$ is C-basis of $\mathscr{A}/\mathfrak{I}$.

Theorem : 1.2.34.

A closure algebra $\mathscr{A}$ is a C-algebra if and only if $\mathscr{B}(\mathscr{A})$ is a C-algebra. Every C-basis of $\mathscr{A}$ is a C-basis of $\mathscr{B}(\mathscr{A})$ and conversely.

Theorem :1.2.35.

Let $\mathfrak{I}$ be a σ-ideal of Borel subsets of separable metric space $\mathscr{X}$. Then $S(\mathscr{X})/\mathfrak{I}$ is C-algebra. $S(\mathscr{X})/\mathfrak{I}$ is homeomorphic to a C-algebra of all Borel subsets of a topological space if and only if $\mathfrak{I}$ is a principal. $S(\mathscr{X})/\mathfrak{I}$ is weakly homeomorphic to a topological iff $\mathfrak{I}$. $S(\mathscr{X})$ is a semi-principal ideal of $S(\mathscr{X})$.

Theorem : 1.2.36.

If $\mathscr{A}$ is a C -algebra and $E \in \mathscr{A}$, then $E\mathscr{A}$ is also a C-algebra. If $\mathfrak{R}$ is a C-basis of $\mathscr{A}$ then $E\mathfrak{R}$ is a C-basis of $E\mathscr{A}$

Definition :1.2.37.

A closure algebra $\mathscr{B}$ is said to be normal provided that for any two closed elements F_1, F_2, with $F_1 \wedge F_2 = 0$, there exists an open element G such that $F_1 \subset G$ and $\overline{G} \wedge F_2 = 0$

Properties of C-algebras :1.2.38

1) Every C-algebra is normal.
2) Let A and B two elements of a C-algebra $\mathscr{A}$ such that $(\overline{A} \wedge B) \vee (A \wedge \overline{B}) = 0$. Then there exist two open elements G and H such that $A \subset G$, $B \subset H$ and $G \wedge H = 0$.
3) Let $\mathfrak{I}$ be a σ-ideal of a C-algebra $\mathscr{A}$. In order that an element $B \in \mathscr{A}/\mathfrak{I}$ to be a Borel element of an additive (multiplicative) class α, it is necessary and sufficient that B possess a representative $A \in \mathscr{A}$ which is a Borel element of an additive (multiplicative) class α in $\mathscr{A}$.

Definition :1.2.39.

If a C-algebra is σ-field of sets, then it is called a C-field.

Proposition :1.2.40.

If $\mathscr{Y}$ is metric separable space, then $\mathbf{S}(\mathscr{Y})$ and $\mathfrak{B}(\mathscr{B})$ are C-fields.

Thorem :1.2.41

Every C-field is weakly homeomorphic to a separable metric space.

A representation theorem on C-algebras:

Theorem :1.2.42.

If a C-algebra $\mathscr{A}$ is of the form $\mathscr{A} = \mathscr{B}/\mathfrak{I}$, where $\mathscr{B}$ is a σ-complete Boolean algebra and $\mathfrak{I}$ is a σ-ideal of $\mathscr{B}$, then it is possible to define in $\mathscr{B}$ a closure operation in such a way that

i) $\mathscr{B}$ with this closure operation is a C-algebra

ii) the C-algebra $\mathscr{A}$ is identical with the closure algebra which we obtain by the division of the C-algebra by the ideal $\mathfrak{I}$.

Theorem :1.2.43.

Every C-algebra $\mathscr{A}$ is homeomorphic to a C-algebra $\mathbf{X}/\mathfrak{I}$ where **X** is a C-field and $\mathfrak{I}$ is a σ-ideal.

A generalisation of Urysohn's theorem may be stated as follows :

Thorem :1.2.44.

For every C-algebra $\mathscr{B}$ there exists a metric separable space $\mathscr{Y}$ and a σ-ideal **J** of $\mathrm{S}(\mathscr{Y})$ such that $\mathscr{B}$ is weakly homeomorphic to the C-algebra $\mathrm{S}(\mathscr{Y})/\mathbf{J}$ i.e.,$\mathfrak{B}(\mathscr{B})$ is homeomorphic to $\mathfrak{B}(\mathscr{Y})/\mathbf{J}_0$, where $\mathbf{J}_0 = \mathbf{J} \wedge \mathfrak{B}(\mathscr{Y})$ is a σ-ideal of $\mathfrak{B}(\mathscr{Y})$.

Chapter II

CHAPTER - 2

This chapter is devoted to the study of "Fuzzy Topological Boolean Algebras" by Parvathi and Meenakshi[22].

In section 1 the fundamental definitions and results on fuzzy topological spaces due to Chang [4] and Lowen [15] that are needed for our study are discussed.

Section 2 is devoted to the study of fuzzy topological Boolean algebras. With every fuzzy topological space (X, δ) a fuzzy topological Boolean algebra $(S(X), \delta^{\varphi})$ is assigned and some important properties of δ^{φ} are discussed. To develop the theory three concepts namely a new class of function $\lambda_A : \mathscr{B} \rightarrow I$, new order relation $(\leq)_n$ and generalised quasi coincident Q are introduced and various properties of these concepts are analysed.

Fuzzy separation axioms FT_1 and FT_2 and fuzzy continuity on fuzzy topological Boolean algebras are discussed in third section.

In section 4 fuzzy topologies on quotient algebra are studied. With every fuzzy topology σ on a Boolean algebra $\mathscr{B}$, two Quotient topologies $\mathcal{K}_1(\sigma)$ and $\mathcal{K}_2(\sigma)$ are associated and the relationship between them are studied. Further it is proved that given a fuzzy topology σ on $\mathscr{B}/\mathfrak{I}$ there exists a fuzzy topology $E(\sigma)$ on $\mathscr{B}$ such that $\mathcal{K}_1(E(\sigma)) = \mathcal{K}_2(E(\sigma)) = \sigma$ and proved that if $E(\sigma)$ is FT_i $i = 1,2$ then σ is FT_i, $i = 1,2$.

SECTION 1

PRELIMINARIES ON FUZZY TOPOLOGICAL SPACES

Definition : 2.1.1

Let X be any set and I=[0,1]. A **fuzzy set** on X is an element of the set I^X of all function and from X to I.

Equality, order relations, union and intersection between fuzzy sets are defined as below:

Let f and g be two fuzzy sets on X.

(i) $f = g \Leftrightarrow f(x) = g(x)$ for all $x \in X$.

(ii) $f \leq g \Leftrightarrow f(x) \leq g(x)$ for all $x \in X$.

Let $\{f_i\}_{i \in \Delta}$ be a family of fuzzy sets on X. Then

(iii) $\left[\underset{i \in \Delta}{\vee} f_i\right](x) = \text{Sup}\,\{f_i(x)\}\ \forall x \in X$.

(iv) $\left[\underset{i \in \Delta}{\wedge} f_i\right](x) = \underset{i \in \Delta}{\inf}\,\{f_i(x)\}\ \forall\ x \in X$.

The **complement** f^c of f is defined by $f^c(x) = 1 - f(x)$ for all $x \in X$.

For any $\alpha \in [0,1]$, the **constant fuzzy set** α is a map from X to I given by

$\alpha(x) = \alpha$ for all $x \in X$.

Definition : [chang 4] : 2.1.2

Let X be a set. A subset $\delta \subset I^X$ is called a **chang fuzzy topology** on X iff δ satifies the following requirements.

(i) The constant functions **0,1** $\in \delta$

(ii) $f, g \in \delta \Rightarrow (f \wedge g) \in \delta$

(iii) If $\{f_i\}_{i \in \Delta}$ is a sub family of δ, then $\underset{i \in \Delta}{\vee} f_i \in \delta$

The pair (X, δ) is called a **chang fuzzy topological space.**

Definition : [Lowen 15] :2.1.3

Let X be any set. A subset $\delta \subset I^X$ is called a **fuzzy topology(Lowen)** on X iff the following conditions are satisfied.

(1) All constant fuzzy sets belong to δ
(2) f, $g \in \delta \Rightarrow f \wedge g \in \delta$
(3) If $(f_i)_{i \in \Delta}$ is a subfamily of δ then $\vee_{i \in \Delta} f_i \in \delta$

The pair (X, δ) is called a fuzzy topological space.

The elements in δ are called **open fuzzy sets** of the fuzzy topological space (X, δ).

A fuzzy set $f \in I^X$ is called a **closed fuzzy set** of (X,δ) iff f^c is open i.e. $f^c \in \delta$.

Definition : [Lowen 15] :2.1.4

The **closure** and interior of a fuzzy set g denoted by cl g and int g in a fuzzy topological space (X, δ) are defined as follows.

$$\text{cl } g = \inf\{h / h \geq g,\ h^c \in \delta\}$$

$$\text{int } g = \sup\{h / h \leq g,\ h \in \delta\}$$

Note : 2.1.5

Cl g is the smallest closed fuzzy set larger than g and int g is the largest open fuzzy set smaller than g.

Definition : [Chang 4] : 2.1.6

Let X and Y be two sets and θ be a function from X into Y. The **inverse** of any fuzzy set h on Y under θ, denoted as $\theta^{-1}(h)$, is the fuzzy set on X defined as $\theta^{-1}(h)(x) = h(\theta(x))$ for all $x \in X$.

The image $\theta(g)$ of a fuzzy set g on X is the fuzzy set on Y defined by

$$\theta(g)(y) = \sup\{g(z) / \theta(z) = y\}$$
$$= 0 \text{ if } \theta^{-1}(y) \text{ is empty.}$$

Definition : [Lowen 15] : 2.1.7

Let (X, δ) be a fuzzy topological space. A subset $\mathcal{B} \subset \delta$ is a base for δ iff $\forall \mu \in \delta$ there exists a subfamily $(\mu_j)_{j \in \Delta} \subset \mathcal{B}$ such that $\mu = \sup\limits_{j \in \Delta} \mu_j$

Definition : [Lowen 15] : 2.1.8

Let (X, δ) be a fuzzy topological space. A subset $S \subset \delta$ is a **sub-base** for δ iff the family of finite intersection of members of S is a base for δ.

Example : 2.1.9

Consider the interval $I = [0,1]$. Then the collection $S = \{\alpha \,/\, \alpha \text{ constant}\} \cup$ the idenity function on I forms a sub-base for the fuzzy topology.

With every topology τ on X, a fuzzy topology $\omega(\tau)$ can be associated as follows.

$\omega(\tau)$ = the set of all lower semi-continous functions from (X,τ) to the unit interval $[0,1]$. Here $[0,1]$ is considered with the usual topology as a subsapce of the real line.

Conversely given a fuzzy topology δ on X the collection of all sets $\{f^{1}(\in,1] \,/\, \in>0, f\in\delta\}$ generates a topology $i(\delta)$ on X.

Definition : [Lowen 15] : 2.1.10

If $\delta = \omega(\tau)$ for some topology τ, then δ is said to be **topologically generated.**

The operators i and ω are related as follows : [Lowen 15]

(i) $i(\omega(\tau)) = \tau$ for every topology τ.

(ii) $\omega(i(\delta))$ is the smallest topologically generated fuzzy topology which contains δ.

Definition : 2.1.11

Two fuzzy sets f, g on X are said to be quasi-coincident, denoted by f q g if there exists an $x \in X$, such that $f(x)+g(x)>1$.

Definition : 2.1.12

Let X be any set. A fuzzy point with support x in X and value t in (0,1] (denoted by x_t) is a fuzzy set defined by

$$x_t(y) = t, \quad \text{if } y = x \qquad = o, \ \text{otherwise}$$

Definition : 2.1.13

A **fuzzy point** x_t is said to belong to a fuzzy set f on X, denoted by $x_t \in f$, iff $t<f(x)$.

Definition : [chang 4] : 2.1.14

Let (X, δ_1) and (Y, δ_2) be two fuzzy topological spaces. A map $\sigma : X \rightarrow Y$ is said to be **fuzzy continous** if $\theta^{-1}(f) \in \delta$, for every $f \in \delta_2$.

SECTION : 2

FUZZY TOPOLOGICAL BOOLEAN ALGEBRAS

Definition : 2.2.1

A fuzzy topological Boolean algebra is a pair $(\mathscr{B}, \sigma)$ where $\mathscr{B}$ is a Boolean algebra and σ is a subset of $I^{\mathscr{B}}$, satisfying the following axioms.

1. All constant functions belong to σ.
2. Arbitrary union of elements of σ is in σ.
3. Finite intersection of elements of σ is in σ.

Terminology :

If $(\mathscr{B}, \sigma)$ is a fuzzy topological Boolean algebra, then σ is called a fuzzy topological structure on $\mathscr{B}$

Definition : 2.2.2

Let $\mathscr{K}$ be a collection of elements of $\mathscr{B}$ A fuzzy set f on $\mathscr{B}$ is said to be compatible with the family $\mathscr{K}$ if the following condition is satisfied.

Given A and B and $f(A) > \varepsilon$, there exists a $B \in \mathscr{K}$ such that $f(B) > \varepsilon$ and $B < A$.

Remark : 2.2.3

When $\mathscr{K}$ contains the element 0 then the constant functions are compatible with $\mathscr{K}$.

Proof :

Let $f : \mathscr{B} \to I$ be such that $f(A) = \alpha \in [0,1]$ $\forall$ $A \in \mathscr{B}$

Let $f(A) > \varepsilon$ for some $\varepsilon > 0$.

Then $0 \in \mathscr{K}$ such that $f(0) > \varepsilon$ and $0 < A$.

$\therefore$ f is compatible with $\mathscr{K}$.

Example : 2.2.4

An example of a fuzzy topological structure on $\mathscr{B}$

Let $(\mathscr{B}, \tau)$ be a topological Boolean algebra. Consider the collection $W(\tau)$ of all increasing fuzzy sets on $\mathscr{B}$ which are compatible with τ. Then $W(\tau)$ is a fuzzy topological structure on $\mathscr{B}$.

Proof :

(i) It is obvious that all the constant functions belong to $W(\tau)$ as all the constant functions are compatible with τ.

(ii) Let $\{ f_\alpha / \alpha \in \Delta\}$ be an arbitrary collection of elements of $W(\tau)$.

Claim : $\vee f_\alpha \in W(\tau)$

Given $A \in \mathscr{B}$

let $\left[\underset{\alpha \in \Delta}{\vee f_\alpha}\right](A) > \varepsilon$

Then $\underset{\alpha \in \Delta}{\vee} f_\alpha(A) > \varepsilon$

$\Rightarrow$ there exists some $\alpha \in \Delta$ such that $f_\alpha(A) > \varepsilon$. As $f_\alpha \in W(\tau)$, there exists a $B \in \tau$ such that $f_\alpha(B) > \varepsilon$ and $B < A$.

$$\therefore \left[\bigvee_{\alpha\in\Delta} f_\alpha\right](B) > \varepsilon \text{ and hence}$$

$$\Rightarrow \left[\bigvee_{\alpha\in\Delta} f_\alpha\right](B) > \varepsilon$$

$$\Rightarrow \left[\bigvee_{\alpha\in\Delta} f_\alpha\right] \in W(\tau)$$

(iii)Let $\{f_i \,/\, i=1 \text{ to } n\}$ be any finite collection of elements of $W(\tau)$.

Claim : $\left[\bigwedge_{i=1}^{n} f_i\right] \in W(\tau)$.

Given $A \in \mathscr{B}$ let $\left[\bigwedge_{i=1}^{n} f_i\right](A) > \varepsilon$

Then for every i, $f_i(A) > \varepsilon$.

As each $f_i \in W(\tau)$, there exists a $B_i \in \tau$ such that $f_i(B_i) > \varepsilon$ and $B_i < A$.

Since $B_i \in \tau$, we have $\bigvee_{i=1}^{n} B_i \in \tau$.

As f_i is increasing and $B_i < A$

We have $f_i\left[\bigvee_{i=1}^{n} B_i\right] > \varepsilon$ and $\bigvee_{i=1}^{n} B_i > \varepsilon$ and

$$\Rightarrow \bigwedge_{i=i}^{n} f_i\left[\bigvee_{i=i}^{n} B_i\right] > \varepsilon$$

$$\Rightarrow \left[\bigwedge_{i=i}^{n} f_i\right]\left[\bigvee_{i=i}^{n} B\right] > \varepsilon$$

$\Rightarrow \bigwedge_{i=i}^{n} f_i \; \varepsilon \; W(\tau)$. Hence the result.

General method of constructing fuzzy topologies on a Boolean algebra $\mathcal{B}$

Theorem : 2.2.5

Let $\mathcal{K}$ be a collection of elements of $\mathcal{B}$ containing the zero elelment and closed with respect to finite union. Then there exists a fuzzy topology W($\mathcal{K}$) having the following property :

For every $f \in W(\mathcal{K})$, $f(B) = \sup\{f(A) / A \in \mathcal{K},\ A<B,\ B\in \mathcal{B}\}$.

Proof :

Let W($\mathcal{K}$) consists of all increasing fuzzy sets on $\mathcal{B}$ which are compatible with $\mathcal{K}$.

Claim : W ($\mathcal{K}$) is a fuzzy topology on $\mathcal{B}$

Let $f \in W(\mathcal{K})$ and for $B\in \mathcal{B}$, $f(B) = \alpha > \varepsilon, \rightarrow (1)$ where $\varepsilon > 0$

Then there exists a $A \in \mathcal{K}$ such that $f(A) > \varepsilon$ and $A < B$.

Let $\sup\{f(A) / A \in \mathcal{K},\ A < B\} = \beta$ (say) $\rightarrow$ (2)

It is clear that $\beta \leq \alpha$.

Suppose $\beta < \alpha$,

Let $\alpha-\beta = \varepsilon$ (say), i.e. $\beta = \alpha-\varepsilon$.

From (1) $f(B) = \alpha > \alpha-\varepsilon = \beta$

$\Rightarrow f(B) > \beta$

Since $f \in W(\mathcal{K})$, there exists a $A\in \mathcal{K}$ such that

$f(A) > \beta$ such that $A < B$., a contradiction to (2)

$\therefore\ \alpha = B$.

i.e. $f(B) = \text{Sup}\{f(A) / A\in \mathcal{K}$ and $A < B\}$.

Study of W(K) for some important families :-

1) $\mathcal{K} = \{0\}$

W ($\mathcal{K}$) = All constant functions on $\mathcal{B}$.

2) $\mathcal{K} = \{0, A\}$, $A \in \mathcal{B}$

$f \in W(\mathcal{K}) \Rightarrow f$ can take only two values $f(0)$ and $f(A)$

3) $\mathcal{K} = \{B_1, B_2, \ldots\ldots B_n\}$, $B_1, B_2, \ldots\ldots B_n \in \mathcal{B}$

$f \in W(\mathcal{K}) \Rightarrow f$ takes atmost n values $f(B_1), f(B_2) \ldots\ldots f(B_n)$.

4) Let $\mathcal{B} = S(X)$ and $\mathcal{K}$ be the collection of all finite sets of S(X).

Then $f \in W(\mathcal{K}) \Leftrightarrow f(A) = \text{Sup}\,\{f(B) / B \text{ finite}, B<A\}\ \forall\ A \in \mathcal{B}$

5) Let $\mathcal{K}$ be the collection of all countable sets of S(X).

Then $f \in W(\mathcal{K}) \Leftrightarrow f(A) = \text{Sup}\,\{f(B) / B \text{ countable}, B<A\}\ \forall\ A \in \mathcal{B}$

6) Let $\mathcal{K} = \tau$, a topology on $\mathcal{B}$

Then $f \in W(\tau) \Leftrightarrow f(A) = \text{Sup}\,\{f(B) / B<A, B \text{ open}\}$

$$= f(\text{int } A)\ \forall\ A \in \mathcal{B}$$

7) Let $\mathcal{K} = C$, the collection of all closed sets on $\mathcal{B}$

Then $f \in W(C) \Leftrightarrow f(A) = \text{Sup}\,\{f(B), B<A, B \text{ closed}\}\ \forall\ A \in \mathcal{B}$

Example : 2.2.6

An example where $W(C) \not\subseteq W(\tau)$ can be constructed as follows:

$(\mathcal{B}, \tau)$ be a topological Boolean algebra, where $\mathcal{B}$ is the collection of all subsets of X=[0,1] and τ the usual topology on [0,1] derived from the standard topology on R.

Consider $f : [0,1] \to [0,1]$ such that $f(x) = x$ for every $x \in [0,1]$

Let $g : S(X) \to I$ be defined as

$g(A) = \text{Sup}\,\{f(x) / x \in A\}$ for every $A \in \delta(x)$.

As [0,1] is T1, every singleton is closed

Let $A \in S(X)$ and $g(A) > \in$

As $g(A) = \text{Sup}_{x \in A} f(x)$, there exists some $x \in A$,

Such that $f(x) > \varepsilon$.

Choose $\{x\} = B \in C$, the collection of all closed elements of $S(X)$.

Then $\{x\} < A$ and $g(x) > \varepsilon$

$\therefore$ g is compatiable with C,

$\therefore$ $g \in W(C)$.

To show : $g \notin W(\tau)$

Consider B = the set of rationals in $[0,1]$

Then $g(B) = 1$

But there is no open set of rational numbers $A < B$

Such that $g(A) > \varepsilon$

$\therefore$ $g \in W(\tau)$

$\therefore$ $W(c) \not\subseteq W(\tau)$

Definition 2.2.7

A subcollection $\mathcal{B} \subset I^{\mathcal{S}}$ is said to be a base for a fuzzy topology σ on $\mathcal{S}$ if every $f \in \sigma$ can be expressed as $f = \vee\{g/g \in \mathcal{B},\ g \leq f\}$

Example : 2.2.8

Construction of a base for a fuzzy topology $W(\mathcal{K})$ on $\mathcal{S}$.

Let $W(\mathcal{K})$ be a fuzzy topology on $\mathcal{S}$ where $\mathcal{K}$ is a collection of elements of $\mathcal{S}$ closed for finite union

Let $A \in \mathcal{S}$ and $\varepsilon = [0,1]$

$$\text{Define } f_{A,\varepsilon}(B) = \varepsilon \text{ if } B > A, \text{ for } B \in \mathcal{S}$$

$$= 0 \text{ otherwise} \rightarrow (1)$$

Let $f \in W(\mathcal{K})$, then for $B \in \mathcal{S}$

$$f(B) = \sup\{f(A)/A \in \mathcal{K},\ A < B\}$$

$$= \bigvee_{\substack{A \in \mathcal{K} \\ A < B}} f(A)$$

$$= \bigvee_{\substack{A \in \mathcal{K} \\ A < B}} f_{A,f(A)}(B) \quad [\therefore \text{ From } (1)\ f_{A,f(A)}(B) = f(A)]$$

$$\therefore f = \bigvee_{A \in \mathcal{K}} f_{A,f(A)}$$

Thus every $f \in W(\mathcal{K})$ can be expressed as

$$f = \vee\{f_{A,f(A)} / A \in \mathcal{K}\}$$

Hence $\{f_{A,\varepsilon} / A \in \mathcal{K}\ \varepsilon = [0,1]\}$ forms a base for $W(\mathcal{K})$.

Definition : 2.2.9

A subcollection $\mathcal{S} \subset \sigma$ is said to be a subbase for fuzzy topology σ on $\mathscr{B}$ iff the family of finite intersection of members of $\mathcal{S}$ is a base for σ.

Remark : 2.2.10

A family $\mathcal{B}$ of fuzzy sets on $\mathscr{B}$ is a base for some fuzzy topology σ on $\mathscr{B}$ if

(i) every constant function can be expressed as the union of members of $\mathcal{B}$

(ii) given $f\ g \in \mathcal{B}$, $f \wedge g$ can be expressed as the union of all members of $\mathcal{B}$ contained in $f \wedge g$.

In this case the fuzzy topology having $\mathcal{B}$ as a base is denoted by $\sigma(\mathcal{B})$

(i.e) $\sigma(\mathcal{B}) = \{ f / f$ can be expressed as union of members of $\mathcal{B} \}$

Remarks: 2.2.11.

Given any collection $\mathcal{S}$ of fuzzy sets on a Boolean algebra $\mathscr{B}$ containing all constant functions, the collection $\mathcal{B}$ of all finite intersections of members of $\mathcal{S}$ is a base for a fuzzy topology. This fuzzy topology is called the fuzzy topology determined or generated by the collection $\mathcal{S}$.

Definition: 2.2.12

With every fuzzy set f on X, a fuzzy set f° on S(X) is defined as follows:

$f^{\circ}(A) = \text{Sup}\{f(x) / x \in A\}$ for every $A \in S(X)$.

The correspondence $f \to f^{\circ}$ has the following properties.

(i) If $h = g^{\circ}$ for some $g \in I^X$ then

$$(\mathcal{h})_x = (g^{\diamond})_x = g$$

Proof:

Given $\mathcal{h} = g^{\diamond}$

Claim:

$$(\mathcal{h})_x = (g^{\diamond})_x = g$$

Let $x \in X$

Then $g^{\diamond}(x) = \sup\limits_{y \in \{x\}} g(y) = g(x)$

Hence $(\mathcal{h})_x = (g^{\diamond})_x = g$.

(ii) If $\mathcal{h} = (g_1^{\diamond} \wedge g_2^{\diamond})$ for $g_1, g_2 \in I^X$, then

$$(\mathcal{h})_x = (g_1^{\diamond} \wedge g_2^{\diamond})_x = g_1 \wedge g_2.$$

Proof :

Given $\mathcal{h} = g_1^{\diamond}$

Claim :

$$(\mathcal{h})_x = (g_1^{\diamond} \wedge g_2^{\diamond})_x = g_1 \wedge g_2.$$

Let $x \in X$.

$$\begin{aligned} \text{Then } (\mathcal{h})_x(x) &= (g_1^{\diamond} \wedge g_2^{\diamond})_x(x) \\ &= \min(g_1^{\diamond}(x), g_2^{\diamond}(x)) \\ &= \min(g_1(x), g_2(x)) \\ &= (g_1 \wedge g_2)(x). \end{aligned}$$

$$\therefore \quad (\mathcal{h})_x = (g_1^{\diamond} \wedge g_2^{\diamond})_x = g_1 \wedge g_2.$$

(iii) If $\{f_\alpha / \alpha \in \Delta\}$ is a subcollection of I^X, then

$$\bigvee_{\alpha\in\Lambda} f_\alpha^{\Diamond} = \left[\bigvee_{\alpha\in\Delta} f_\alpha\right]^{\Diamond}$$

Proof :

Let $A \subset I$, then $f_\alpha^{\Diamond}(A) = \sup f_\alpha (x)$

Claim :

$$\bigvee_{\alpha\in\Lambda} f_\alpha^{\Diamond} = \left[\bigvee_{\alpha\in\Delta} f_\alpha\right]^{\Diamond}$$

$$\begin{aligned}
\left[\bigvee_{\alpha\in\Delta} f_\alpha\right]^{\Diamond}(A) &= \sup_{\alpha\in\Delta} (\vee f_\alpha)(x) \\
&= \sup_{\alpha\in\Lambda} \{\vee f_\alpha (x)\} \\
&= \sup_{x\in A} \{\sup_{\alpha\in\Delta} f_\alpha (x)\} \\
&= \sup_{\alpha\in\Delta} \{\sup_{x\in A} f_\alpha(x)\} \qquad \rightarrow (1)
\end{aligned}$$

$$\begin{aligned}
\left[\bigvee_{\alpha\in\Lambda} f_\alpha^{\Diamond}\right](A) &= \bigvee_{\alpha\in\Lambda} f_\alpha^{\Diamond}(A) \\
&= \bigvee_{\alpha\in\Delta} \{\sup_{x\in A} f_\alpha(x)\} \\
&= \sup_{\alpha\in\Lambda} \{\sup_{x\in A} f_\alpha(x)\} \qquad \rightarrow (2)
\end{aligned}$$

From (1) & and (2) $\left[\bigvee_{\alpha\in\Delta} f_\alpha^{\Diamond}\right] = \left[\bigvee_{\alpha\in\Delta} f_\alpha\right]^{\Diamond}$

Remark : 2.2.13

$(g_1^{\Diamond} \wedge g_2^{\Diamond})$ need not be equal to $(g_1 \wedge g_2)^{\Diamond}$.

Let $g_1 : I \rightarrow I$ such that $g_1(x) = 1$ if x is rational

$= 0$ otherwise

Let $g_2 : I \rightarrow I$ such that $g_2(x) = 1$ if x is irrational

$= 0$ otherwise

Let $A = [1/3, 1/4] \subset I$

Then $g_1^{\Diamond}(A) = 1$, $g_2^{\Diamond}(A) = 1$

and $(g_1^{\Diamond} \wedge g_2^{\Diamond})(A) = \min(g_1^{\Diamond}(A), g_2^{\Diamond}(A)$

$= \min(1,1) = 1$

For $x \in I$,

$$(g_1 \wedge g_2)(x) = \min(g_1(x), g_2(x))$$

$$= \left[\begin{array}{l} \min(1,0) = 0 \text{ of } x \text{ is rational} \\ \\ \min(0,1) = 0 \text{ if } x \text{ is irrational} \end{array}\right.$$

$$\therefore (g_1 \wedge g_2)(x) = 0.$$

$$\therefore (g_1 \wedge g_2)^{\Diamond}(A) = \operatorname*{Sup}_{\alpha \in \Delta} (g_1 \wedge g_2)(x)$$

$$= \operatorname*{Sup}_{\alpha \in \Delta} 0 = 0$$

Hence the result.

Proposition : 2.2.14

Let (X, δ) be a fuzzy topological space. Then $\{f^{\Diamond} / f \in \delta\}$ is a subbase for a fuzzy topology $\delta^{\Diamond}$ on $S(X)$.

Remark : 2.2.15

Given $f \in I^X$, $f^{\Diamond} \in \delta^{\Diamond} \Leftrightarrow f \in \delta$.

Proof :

Let $f \in \delta$,

Then by the definition of $\delta^{\Diamond}$, $f^{\Diamond} \in \delta^{\Diamond}$.

Conversely assume $f^{\Diamond} \in \delta^{\Diamond}$.

Claim :

$f \in \delta$.

$$f = (f^{\Diamond})_x = \left[\bigcup_{\alpha \in \Delta} \bigcap_{i=i}^{n} f^{\Diamond}_{\alpha_i}\right]_X$$

$$= \bigcup_{\alpha\in\Delta} \bigcap_{i=i}^{n} \left[f^{\Diamond}_{\alpha_i} \right]_{x}$$

$$= \bigcup_{\alpha\in\Delta} \bigcap_{i=i}^{n} \left[f_{\alpha_i} \right] \in \delta$$

Hence $f^{\Diamond} \in \delta^{\Diamond} \Leftrightarrow f \in \delta$.

Remark : 2.2.16

$\delta^{\Diamond} \subseteq W(\mathcal{K}_0)$, Where $\mathcal{K}_0$ is the collection of all finite subsets of X.

Proof :

To prove the result it is enough to consider any subbasis element $f^{\Diamond} \in \delta^{\Diamond}$.

Claim : $f^{\Diamond} \in W(\mathcal{K}_0)$

Let $A \subset X$ and $f^{\Diamond}(A) > \varepsilon$ where $\varepsilon > 0$

i.e. $\sup_{x\in A} f(x) > \varepsilon$

i.e for some $x \in X$, $f(x) > \varepsilon$.

$\therefore$ there exists a $\{x\} = B \in \mathcal{K}_0$ such that $f(x) > \varepsilon$ and $\{x\} \subset A$.

$\therefore f^{\Diamond} \in W(\mathcal{K}_0)$.

Hence $\delta^{\Diamond} \subseteq W(\mathcal{K}_0)$.

Example : 2.2.17

An example where $\delta^{\Diamond}$ is strictly contained in $W(\mathcal{K}_0)$.

Let δ be the fuzzy topology generated by all constant functions together with the identify function on [0,1].

Let $f : [0,1] \rightarrow [0,1]$ be such thhat $f(x) = x/3$.

Then $f^{\Diamond} \in W(\mathcal{K}_0)$

But $f^{\circ} \notin \delta$ as f cannot be expressed as the arbitrary union of finite intersection of elements of δ.

An important class of fuzzy sets :

Definition : 2.2.18

For $A \in \mathscr{B}$, $\lambda_A : \mathscr{B} \to I$ is defined as
$\lambda_A(B) = 1$, if $A \wedge B \neq 0$.
$= 0$,otherwise.

Remark : 2.2.19

When $\mathscr{B} = S(X)$, the function λ_A restricted to the points of X is the same as χ_A, in symbols $(\lambda_A)_x = \chi_A$

Properties of λ_A : 2.2.20

(1) λ_A is incresing for every $A \in \mathscr{B}$
i.e. If $B_1 < B_2$, then $\lambda_A(B_1) \leq \lambda_A(B_2)$

Proof :

Case (i) : Let $A \cap B_1 \neq 0$ and $A \cap B_2 \neq 0$
$\therefore \lambda_A(B_1) = 1$ and $\lambda_A(B_2) = 1$.

Case (ii) : $A \cap B_1 = 0$ and $A \cap B_2 \neq 0$
Then $\lambda_A(B_1) = 0$ and $\lambda_A(B_2) = 1$
Hence $\lambda_A(B_1) \leq \lambda_A(B_2)$

(2) $A_1 < A_2$ implies $\lambda_{A_1} \leq \lambda_{A_2}$ for $A_1, A_2, \in B$.

Proof :

Let $B \in \mathscr{B}$

Case (i) Let $A_1 \cap B \neq 0$ then $A_2 \cap B \neq 0$.
$\therefore \lambda_{A_1}(B) = 1 \Rightarrow \lambda_{A_2}(B) = 1$

Case (ii) suppose $A_1 \cap B = 0$ and $A_2 \cap B \neq 0$.
Then $\lambda_{A_1}(B) = 0$ and $\lambda_{A_2}(B) = 1$
Hence $\lambda_{A_1} \leq \lambda_{A_2}$.

(3) Given a finite collection of elements A_i in $\mathcal{B}$,

$\lambda_{\vee Ai} = \vee\lambda_{A_i}$, whenever $\vee A_i$ exists.

Proof:

Let $B \in \mathcal{B}$

Case (i) Assume $(\vee_{Ai}) \cap B \neq 0$

$\Rightarrow A_i \cap B \neq 0$ for some i.

$\therefore \lambda_{\vee A_i}(B) = 1$ and $\lambda_{A_i}(B) = 1$

$\therefore \vee\lambda_{A_i}(B) = 1$

$\therefore \lambda_{\vee A_i}(B) = \vee\lambda_{A_i}(B)$

$= (\vee\lambda_{A_i})(B)$

$\therefore \lambda_{\vee A_i} = \vee\,\lambda_{A_i}$

Case (ii) Let $(\vee A_i) \cap B = 0$

Then $A_i \cap B = 0$ for every i

$\therefore \underset{\alpha\in\Delta}{\lambda_{\vee A_i}}(B) = 0$ and $\lambda_{Ai}(B) = 0$ for every i

$\therefore \underset{\alpha\in\Delta}{\vee}\,\lambda_{A_i}(B) = 0$

$\therefore \lambda_{\vee A_i} = \vee\lambda_{A_i}$

Hence the result.

(4) For $A_1, A_2 \in \mathcal{B}$, $\lambda_{A_1 \wedge A_2} \leq \lambda_{A_1} \wedge \lambda_{A_2}$

Proof :

Let $B \in \mathcal{B}$

If $A_1 \cap A_2 \cap B \neq 0$ then $\lambda_{A_1 \wedge A_2}(B) = 1$

Then $A_1 \cap B \neq 0$ and $A_2 \cap B \neq 0$

$\therefore \lambda_{A_1}(B) = 1$ and $\lambda_{A_2}(B) = 1$

$\therefore (\lambda_{A_1} \wedge \lambda_{A_2})(B) = 1$

$\therefore (\lambda_{A_1} \wedge \lambda_{A_2})(B) = \lambda_{A_1 \wedge A_2}(B)$

$\therefore \quad \lambda_{A_1} \wedge \lambda_{A_2} = \lambda_{A_1 \wedge A_2}$

If $A_1 \cap A_2 \cap B = 0$ then $\lambda_{A_1 \cap A_2}(B) = 0$.

Case (i) :

$A_1 \cap A_2 \cap B = 0 \Rightarrow A_1 \cap B = 0$ or $A_2 \cap B = 0$

$\therefore$ Either $\lambda_{A_1}(B) = 0$ or $\lambda_{A_2}(B) = 0$

$\Rightarrow (\lambda_{A_1} \wedge \lambda_{A_2})(B) = 0$

In this case $\lambda_{A_1 \wedge A_2} = \lambda_{A_1} \wedge \lambda_{A_2}$.

Case (ii) :

If both $A_1 \cap B$ and $A_2 \cap B$ are non zero then $\lambda_{A_1}(B) = 1$ and $\lambda_{A_2}(B) = 1$

Then $(\lambda_{A_1} \wedge \lambda_{A_2})(B) = 1$

In this case $\lambda_{A_1 \wedge A_2} < \lambda_{A_1} \wedge \lambda_{A_2}$.

Hence $\lambda_{A_1 \wedge A_2} \leq \lambda_{A_1} \wedge \lambda_{A_2}$.

(5) If $\lambda_{A_1} \wedge \lambda_{A_2} = \lambda_B$, then $B = A_1 \wedge A_2$

Proof :

Suppose $\lambda_B(C) = 1$

$\Rightarrow (\lambda_{A_1} \wedge \lambda_{A_2})(C) = 1$

$\Rightarrow A_1 \cap C \neq 0$ and $A_2 \cap C \neq 0$

$\Rightarrow (A_1 \cap A_2) \cap C \neq 0$

$\Rightarrow \lambda_{A_1 \cap A_2}(C) = 1 \rightarrow (1)$

Suppose $\lambda_B(C) = 0$

Then $(\lambda_{A_1} \wedge \lambda_{A_2})(C) = 0$.

$\Rightarrow \min(\lambda_{A_1}(C), \lambda_{A_2}(C)) = 0$

$\Rightarrow$ Either $\lambda_{A_1}(C) = 0$ or $\lambda_{A_2}(C) = 0$

Then $A_1 \cap C = 0$ or $A_2 \cap C = 0$

$\Rightarrow (A_1 \cap A_2) \cap C = 0$

$\Rightarrow \lambda_{A_1 \wedge A_2}(C) = 0 \rightarrow (2)$

From (1) and (2)

$\lambda_B = \lambda_{A_1 \wedge A_2}$

$\Rightarrow B = A_1 \wedge A_2$.

Example : - 2.2.21

The following example shows that $\lambda_{A_1 \wedge A_2}$ need not be equal to $\lambda_{A_1} \wedge \lambda_{A_2}$.

Let $A_1 = \{1,2,3,.......10\}$

Let $A_2 = \{5,6,..........15\}$

$A_1 \wedge A_2 = \{5,6,7,8,9,10\}$

Take $B = \{1, 15\}$

Then $A_1 \cap B \neq \phi$, $A_2 \cap B \neq \phi$ and $A_1 \cap A_2 \cap B = 0$

$\therefore \lambda_{A_1}(B) = 1$, $\lambda_{A_2}(B) = 1$ and $\lambda_{A_1 \wedge A_2}(B) = 0$

$\therefore \lambda_{A_1 \wedge A_2}(B) < \lambda_{A_1}(B) \cap \lambda_{A_2}(B) = (\lambda_{A_1} \wedge \lambda_{A_2})(B)$

$\therefore \lambda_{A_1} \wedge \lambda_{A_2} > \lambda_{A_1 \wedge A_2}$

Definition : - 2.2.22

A new order relation $(\leq)_n$ is defined on $I^{\mathcal{B}}$ as follows:

$f\ (\leq)_n\ g$, if given $A \in \mathcal{B}$ there exists a non zero $B<A$ such that for every $C < B$, $f(C) \leq g(C)$

Remark : 2.2.23.

Let f, g, h be fuzzy sets defined on a Boolean algebra $\mathcal{B}$. . Then the following results hold:

(i) $f \leq g \Rightarrow f\ (\leq)_n g$

Proof :

Let $f \leq g$

Claim :

$f\ (\leq)_n\ g$

$f \leq g \Rightarrow f(A) \leq g(A)\ \forall\ A \in \mathcal{B}$

Then by the definition of $(\leq)_n$ we have $f(\leq)_n\ g$

(ii) $f\ (\leq)_n\ g\ ,\ g\ (\leq)_n\ h \Rightarrow f\ (\leq)_n\ h$

Proof:

Let $f\ (\leq)_n\ g\ ,\ g\ (\leq)_n\ h$

Claim :

$f\ (\leq)_n\ h$

$f\ (\leq)_n\ g \Rightarrow$

given $A \in \mathscr{B}$ there exists non zero $B \in \mathscr{B}$ such that $B < A$ and for every $C<B$ $f(C) \leq g\ (C)$

$g\ (\leq)_n\ h \Rightarrow$

for $B \in \mathscr{B}$ there exists a non zero $B_1 \in \mathscr{B}$ such that $B_1 < A$ and for every $C < B_1$

$f(C)\ \leq g\ (C) \leq h(C)$

$\Rightarrow \quad f(C) \leq h(C)$

$\Rightarrow \quad f\ (\leq)_n\ h$

Hence the result

(iii) $(\lambda_{A_1} \wedge \lambda_{A_2})\ (\leq)_n\ \lambda_{A_1 \wedge A_2}$

Let $B \in \mathscr{B}$

If $B \cap (A_1 \cap A_2) \neq 0$, then $A_1 \cap B \neq 0$ and $A_2 \cap B \neq 0$ and the result is obvious.

If $B \cap (A_1 \cap A_2) = 0$, then we have

(i) $A_1 \cap B = 0$ and $A_2 \cap B = 0$

(ii) $A_1 \cap B \neq 0$ and $A_2 \cap B \neq 0$

(iii) any one of $A_1 \cap B$ and $A_2 \cap B$ is not zero

In (i) & (ii) the result is obvious.

In (iii) choose $C = B \cap A_1^C \cap A_2$ or $B \cap A_1 \cap A_2^C$ the result holds.

(iv) When $\mathscr{B} = S(X)$,

$f\ (\leq)_n\ g \Leftrightarrow (f)_x \leq (g)_x$

Proof :

Let $f\ (\leq)_n\ g$

Claim :

$(f)_x \leq (g)_x$

$f\ (\leq)_n\ g \Rightarrow$

given $A \in \mathscr{B}$, there exists a non-zero $B = \{x\} < A$ such that for every $\{x\} < B$, $f(x) \leq g(x)$.

This $\Rightarrow$ $(f)_x\,(x) \leq (g)_x(x)$

i.e., $(f)_x \leq (g)_x$

Conversely assume $(f)_x \leq (g)_x$

Claim:

$f\ (\leq)_n\ g$

i.e, for a given $A\in \mathscr{B}$ there exist a non zero $B<A$ such that for every $C<B$ $f(C) \leq g(C)$.

Take $B = \{x\} < A$

Then $f\ (\leq)_n\ g$

$\therefore\ f(\leq)_n\ g \Leftrightarrow (f)_x \leq (g)_x$

When f and g are fuzzy sets on X

$f^{\diamond}\ (\leq)_n g^{\diamond} \quad \Leftrightarrow \quad f \leq g \quad \Leftrightarrow \quad f^{\diamond} \leq g^{\diamond}$.

Proof :

Given $f^{\diamond}\ (\leq)_n\ g^{\diamond}$.

Claim :

$f \leq g$.

i.e. $\forall\ x \in X,\ f(x) \leq g(x)$.

In the definition of $(\leq)_n$, take $A = \{x\} \in X$

Then for every $C < \{x\}$, $f(C) \leq g(C)$

i.e. $f(x) \leq g(x)$

i.e. $f \leq g$.

Conversely assume $f \leq g$

i.e. $f(x) \leq g(x)\ \forall\ x \in X$.

Claim :

$f^{\diamond}\ (\leq)_n g^{\diamond}$

Let $A \in S(x)$

Then for every $\{x\} < A$, Choose $B = \{x\}$.

Then for every $C < B$.

$$f^{\diamond}(x) = f(x) \leq g(x) = g^{\diamond}(x)$$

i.e. $f^{\diamond}(x) \leq g^{\diamond}(x)$

i.e. $f^{\diamond} \leq g^{\diamond}$.

$\therefore f^{\diamond} \leq g^{\diamond} \Leftrightarrow f \leq g$.

Proof :

Given $f \leq g$

i.e. $f(x) \leq g(x) \ \forall x \in X$

Claim :

$f^{\diamond} \leq g^{\diamond}$.

i.e. $f^{\diamond}(A) \leq g^{\diamond}(A) \ \forall A \in S(X)$

For $A \in S(X)$,

$$f^{\diamond}(A) = \sup_{x \in A} f(x) \leq \sup_{x \in A} g(x) = g^{\diamond}(A)$$

$\therefore f^{\diamond}(A) \leq g^{\diamond}(A)$

i.e. $f^{\diamond} \leq g^{\diamond}$.

Conversely assume $f^{\diamond} \leq g^{\diamond}$.

i.e. $f^{\diamond}(A) \leq g^{\diamond}(A) \ \forall A \in S(X)$.

Claim :

$f \leq g$

i.e. $f(x) \leq g(x) \ \forall x \in X$.

For $x \in X$, $f(x) = f^{\diamond}(x) \leq g^{\diamond}(x) = g(x)$

i.e. $f(x) \leq g(x)$

i.e. $f \leq g$.

$\therefore f \leq g \Leftrightarrow f^{\diamond} \leq g \Leftrightarrow f^{\diamond} \leq g^{\diamond}$.

Hence $f^{\diamond} (\leq)_n g^{\diamond} \Leftrightarrow f \leq g \Leftrightarrow f^{\diamond} \leq g^{\diamond}$.

Definition : 2.2.24

Two fuzzy sets f and g are said to be generalised quasi-coincident, in symbols $f \, Q \, g$, if there exists A in $\mathscr{B}$ such that for every non zero B<A, there exits a non zero C<B such that $f(C) + g(C) > 1$.

Remark : 2.2.25

Fuzzy sets f and g on $\mathscr{B}$ are not generalised quasi-coincident, in symbols $fQ^C g$, if the following condition holds good:

Given $A \in \mathscr{B}$ there exists a non zero B<A such that for every C<B, $f(C) + g(C) \leq 1$.

Proposition : 2.2.26

For $A, B \in \mathscr{B}$, $\lambda_A Q \lambda_B \Leftrightarrow A \wedge B \neq 0$.

Proof :

Let $C = A \wedge B \neq O$.
Then for every non zero D<C $\lambda_A(D) + \lambda_B(D) = 1+1 = 2>1$
$\therefore \lambda_A Q \, \lambda_B$.

Conversely assume $\lambda_A Q \, \lambda_B$.
Claim : $A \wedge B \neq O$.
Assume $A \wedge B = O$. $\Rightarrow A^C \vee B^C = 1$.

Given a non zero K,
Choose $D = A^C \wedge K$ if $A^C \wedge K \neq O$
$= B^C \wedge K$, otherwise.

If $D = A^C \wedge K$,

Then for every $L < D$,

$\lambda_A(L) = 0$ [Since $L \subset A^C \wedge K$ and $A \wedge L = 0$]
and therefore $\lambda_A(L) + \lambda_B(L) \leq 1$.
If $D = B^C \wedge K$
$\lambda_B(L) = 0$ [Since $L \subset B^C \wedge K$ and $B \wedge L = 0$]
and therefore $\lambda_A(L) + \lambda_B(L) \leq 1$.

$\therefore\ \lambda_A(L) + \lambda_B(L) \le 1.$

This implies $\lambda_A Q^C \lambda_B$, a contradiction.

$\therefore\ A \wedge B \neq 0.$

Hence then result.

Remark 2.2.27

$f\, Q^C\, g \Leftrightarrow f (\le)_n\, 1\text{-}g$ for $f, g \in I^{\mathcal{B}}$.

Proof :

Given $f\, Q^C\, g$

Claim :

$f\ (\le)_n\ 1\text{-}g.$

$f\, Q^C\, g \Rightarrow$

for every $A \in \mathcal{B}$ there exist a non zero $B<A$ such that for every $C < B$

$$f(C) + g(C) \le 1$$

$$\Rightarrow\quad f(C) \le 1\text{-}g(C)$$

$$\Rightarrow\quad f(C) \le (1\text{-}g)\,(C) \text{ for every C.}$$

$$\Rightarrow\quad f \le 1\text{-}g$$

$$\Rightarrow\quad f\ (\le)_n\ 1\text{-}g$$

Conversely assume $f\ (\le)_n\ 1\text{-}g$

Claim : $f\, Q^C\, g$

$f\ (\le)_n\ (1\text{-}g) \Rightarrow$

for every $A \in \mathcal{B}$ there exists a non zero $B<A$ such that for every $C<B$

$$f(C) \le 1\text{-}g\,(C)$$

$$\Rightarrow\quad f\,(C) + g\,(C) \le 1$$

$$\Rightarrow\quad f\, Q^C\, g$$

Remark : 2.2.28

Given three fuzzy sets f, g and h on $\mathcal{B}$, the following implication is easily verified:

$$f\ (\le)_n\ g,\ \ g\, Q^C\, h \Rightarrow f\, Q^C\, h$$

Proof :

Given $f \, (\leq)_n \, g$ and $g \, Q^C \, h$

Claim :

$f \, Q^C \, h$

$f \, (\leq)_n \, g \Rightarrow$

given $A \in \mathscr{B}$ there exists a non zero $B<A$ such that for every $C < B$,

$$f(C) \leq g(C) \rightarrow (1)$$

$g \, Q^C \, h \Rightarrow$

Given $B \in \mathscr{B}$, there exists a non zero $B_1 < B$ such that for every $C<B_1$

$$g(C) + h(C) \leq 1$$

$$\Rightarrow g(C) \leq 1 - h(C) \rightarrow (2)$$

From (1) and (2), Given $A \in \mathscr{B}$, there exists a non zero $B_1 < B$ such that for every $C < B_1$.

$$f(C) \leq g(C) \leq 1 - h(C)$$

$$\Rightarrow \quad f(C) \leq 1 - h(C)$$

$$\Rightarrow \quad f(C) + h(C) \leq 1$$

$$\Rightarrow \quad f \, Q^C \, h$$

$$\therefore \quad f \, (\leq)_n \, 1-g, \quad g \, Q^C \, h \Rightarrow f \, Q^C \, h.$$

Remark : 2.2.29

Let f_1, f_2, g_1, g_2 be fuzzy sets on $\mathscr{B}$ If $f_1 (\leq)_n \, g_1$, $f_2 (\leq)_n \, g_2$, and $f_1 \, Q \, f_2$ then $g_1 Q \, g_2$.

Proof :

Given $f_1 \, (\leq)_n \, g_1$, $f_2 \, (\leq)_n \, g_2$ and $f_1 \, Q \, f_2$.

Claim : $g_1 \, Q \, g_2$.

$f_1 \, Q \, f_2 \Rightarrow$

there exists a A in $\mathscr{B}$ such that for every non zero B<A, there exists a non zero C<B such that

$$f_1(C) + f_2(C) > 1 \rightarrow (1)$$

$f_1 \, (\leq)_n \, g_1 \Rightarrow$

For A $\in \mathscr{B}$, there exists a non zero B_1<A such that for every C < B_1.

$$f_1(C) \leq g_1(C) \rightarrow (2)$$

$f_2 \, (\leq)_n \, g_2 \Rightarrow$

Given $B_1 \in \mathscr{B}$, there exists a non zero $B_2 < B_1$ such that for every $C < B_2$,

$$f_2(C) \leq g_2(C) \rightarrow (3)$$

∴ From (1) and (2) and (3),

For A $\in \mathscr{B}$ there exists a non zero $B_2 < B_1$ such that for every $C < B_2$.

$$g_1(C) \leq g_2(C) \geq f_1(C) + f_2(C) \geq 1$$

$$\Rightarrow g_1(C) + g_2(C) \geq 1.$$

$$\Rightarrow g_1 Q \, g_2$$

Hence the result.

SECTION : 3

FUZZY SEPARATION AXIOMS AND FUZZY CONTINUITY

Definition : 2.3.1

A fuzzy topological space (X, δ) is said to be fuzzy T_1 or FT_1 iff for every pair of non zero fuzzy sets f, g on $\mathscr{B}$ with $f \nleq g$, there exists an open fuzzy set h such that $f \nleq h$ and $g \leq h$.

Definition : 2.3.2

A fuzzy topological Boolean algebra ($\mathscr{B}$, σ) is said to be fuzzy T_1 or FT_1 iff for every pair of non zero fuzzy sets f, g on $\mathscr{B}$ with $f(\leq)_n\, g$, there exists an open fuzzy set $\hbar$ on $\mathscr{B}$ such that $f(\leq)_n\, \hbar$, and $g\, (\leq)_n\, \hbar$.

Definition : 2.3.3

A fuzzy topological space (X, δ) is said to be fuzzy T_2 or FT_2 iff for every pair of non zero fuzzy sets f, g on X with $f\, q^C\, g$ there exist open fuzzy sets h_1 and h_2 such that $f \wedge h_1 > 0$, $g \wedge h_2 > 0$ and $h_1\, q^C\, h_2$.

Example : 2.3.4

(1) A fuzzy topological space with satisfies FT_1 but not FT_2.

Let X be any set, $x \in X$ and $t \in (0,1)$

Define the mapping

$$f_{(x,\, t)}\,(x) = t$$

and $f_{(x,t)}\,(y) = 1$ if $y \neq x$.

Let δ be the fuzzy topology on X generated by
$\{f_{(x,t)},\ x \in X,\ t \in (0,1)\} \cup \{\alpha \in [0,1],\ \alpha$ constant fuzzy set in X $\}$

Let α, β be non zero fuzzy sets in X with $\alpha \leq B$.
Then there exists an $x \in X$ such that $\alpha(x) > \beta\,(x)$.
Choose λ such that $\alpha(x) > \lambda\beta\,(x)$
Let $g = f_{(x,\, \lambda)}$. Then $\beta \leq g$ and $\alpha \leq g$.
Thus (X, δ) is an FT_1, fuzzy topological space.
Since $f \wedge g \neq 0$ for any two non zero open fuzzy sets the space cannot be FT_2.

Example : 2.3.5

(2) An example of an FT_2 space which is not discrete.

Let $\{I_\lambda, \lambda \in \wedge\}$ be the usual interval base of the subbase topology on $I = [0,1]$ induced by R, the set of reals.

Let $t \in [0,1]$

Define $f_{\lambda t} : X \rightarrow I$ by

$$f_{\lambda t}(x) = t \text{ if } x \in I_\lambda$$

$$= 0 \text{ if } x \notin I_x.$$

Let δ be the fuzzy topology on $[0,1]$ having as base the collection of all constant fuzzy set in $[0,1]$ together with $\{ f_{\lambda t}, \lambda \in \wedge, t \in [0,1] \}$

To prove : (I, δ) is FT_2.

Let α, β be non zero fuzzy sets in I with $\alpha \wedge \beta = 0$.
There exists $x, y \in X$ such that $\alpha(x) > 0$, $\beta(y) > 0$
As $\alpha \wedge \beta = 0$, $x \neq y$
Since the topology on $[0,1]$ is T_2, there exist $\lambda_1, \lambda_2 \in \wedge$
Such that $x \in I_{\lambda_1}$, $y \in I_{\lambda_2}$ and $I_{\lambda_1}, \cap I_{\lambda_2} = \phi$
Let $\alpha(x) = t_1$, $\beta(y) = t_2$
Let $f = f_{\lambda_1 t_1}$, $g = f_{\lambda_2 t_2}$
Then $(\alpha \wedge f)(x) = t_1 > 0$
Similarly $(\beta \wedge g)(y) = t_2 > 0$
As $I_{\lambda_1} \cap I_{\lambda_2} = \phi$, $f \wedge g = 0$
Hence (I, δ) is FT_2.
As all fuzzy sets are not included in δ, (I, δ) is not discrete.

The conditions $f \wedge h_1 > 0$ and $g \wedge h_2 > 0$ in the definition of FT_2 space have to be modified in the case of fuzzy topological Boolean algebras. This leads to the definition of a new relation ζ between fuzzy sets on $\mathcal{B}$

Definition : 2.3.6

Let f, g be two fuzzy sets on $\mathcal{B}$ f and g are said to be ζ related (i.e) $f \zeta g$ if there exists $A \in \mathcal{B}$ such that for every non zero $B < A$ there exists a non zero $C<B$ such that $\min (f(C), g(C)) >0$.

Remark : 2.3.7

$f \,\zeta\, g \Rightarrow \quad f \wedge g > 0$

Proof :

Given $f \,\zeta\, g$

Claim :

$f \wedge g > 0$

$f \,\zeta\, g \Rightarrow$

There exists A ∈ $\mathscr{B}$ such that for every non zero B<A there exists a non zero C<B such that min (f(C), g(C)) >0

each f (C) and g (C) >0

$\Rightarrow \quad f \wedge g > 0$

Hence the claim.

Remark : 2.3.8

Let f and g be two fuzzy sets on S(X)

Then $(f)_x \wedge (g)_x > 0 \Leftrightarrow f \,\zeta\, g$

When f and g are two fuzzy sets on X, the following implication holds:

$f \wedge g > 0 \Leftrightarrow f^{0} \,\zeta\, g^{0}$

Proof :

Given $(f)_x \wedge (g)_x > 0$

Claim :

$f \,\zeta\, g$

Let $(f)_x \wedge (g)_x > 0$, then there exists x ∈ X such that min $((f)_x (x), (g)_x (x)) > 0$

Choosing {x} = A, min (f(A), g (A)) >0

Then by the definition of ζ, $f \,\zeta\, g$.

Conversely assume $f \,\zeta\, g$

Claim : $(f)_x \wedge (g)_x > 0$

$f \,\zeta\, g \Rightarrow$

there exists A∈ $\mathscr{B}$ such that for every non zero B < A there exists a non zero C < B such that min ((f(C), g (C)) >0

This implies for every {x} ⊂ A, min (f(x), g(x)) >0

$\Rightarrow (f \wedge g)(x) > 0$

$\Rightarrow (f)_x \wedge (g)_x > 0.$

Hence the claim.

Assume $f \wedge g > 0$
Claim : $f^{\diamond} \zeta g^{\diamond}$.

$f \wedge g > 0 \Rightarrow$

there exists $x \in X$ such that $(f \wedge g)(x) > 0$

i.e. $\min(f(x), g(x)) > 0$

Take $A = \{x\}$, then $f^{\diamond} \zeta g^{\diamond}$.

Conversely assume $f^{\diamond} \zeta g^{\diamond}$.

Claim : $f \wedge g > 0$.

$f^{\diamond} \zeta g^{\diamond} \Rightarrow$

there exists a $A \in \mathcal{B}$ such that for every non zero $B < A$ there exists a non zero $C < B$ such that

$\min(f^{\diamond}(C), g^{\diamond}(C)) > 0$

i.e. for $\{x\} \subset A$,

$\min(f^{\diamond}(x), g^{\diamond}(x)) > 0$

i.e. $\min(f(x), g(x)) > 0$

i.e. $f \wedge g > 0$

Hence $f \wedge g > 0 \Leftrightarrow f^{\diamond} \zeta g^{\diamond}$.

Remark : 2.3.9

Let f and g be two fuzzy sets on X. Then $f^{\diamond} \wedge g^{\diamond} > 0$ need not imply $f \wedge g > 0$.

Let $f : [0,1] \rightarrow [0,1]$ such that

$f(x) = 1$ if x is rational

$= 0$ if x is irrational

Let $g : [0,1] \rightarrow [0,1]$ such that

$g(x) = 0$ if x is rational

$= 1/2$ if x is irrational

Let $A = [1/2, 1/4]$

$(f^{\diamond} \wedge g^{\diamond})(A) = \min(f^{\diamond}(A), f^{\diamond}(A))$

$= \min(1, 1/2) = 1/2 > 0$

But $\forall\, x \in [0,1] \qquad f(x) \wedge g(x) = 0$

$f \wedge g = 0$

$\therefore\ f^{\diamond} \wedge g^{\diamond} > 0$ need not imply $f \wedge g = 0$.

Definition : 2.3.10

A fuzzy topology Boolean algebra $(\mathcal{B}, \sigma)$ is said to be fuzzy T_2 or FT_2 iff the following condition is satisfied.

Given two fuzzy sets f, g on $\mathcal{B}$ with $f\ Q^C\ g$ there exists two open fuzzy sets $\hbar_1, \hbar_2$ such that $f\ \zeta\ \hbar_1$, $g\ \zeta\ \hbar_2$ and $\hbar_1\ Q^C\ \hbar_2$

Theorem : 2.3.11

Let (X, δ) be a fuzzy topological space. Then (X,δ) is FT_i iff $(S(X), \delta^{\diamond})$ is FT_i for $i = 1,2$

Proof :

Assume (X,δ) is FT_1

Claim :

$(S(X), \delta^{\diamond})$ is FT_1.

Let f, g be two fuzzy sets on $S(X)$ with $f\ (\nleq)_n\ g$

$\Rightarrow (f)_x \nleq (g)_x$.

As (X,δ) is FT_1, there exists a $h \in \delta$ such that

$(f)_x \nleq h$ and $(g)_x \leq h$

i.e. $f\ (\nleq)_n\ h^{\diamond}$ and $g\ (\leq)_n\ h^{\diamond}$.

$\therefore$ there exists a $h^{\diamond} \in \delta^{\diamond}$ such that $f\ (\nleq)_n\ h^{\diamond}$ and $g\ (\leq)_n\ h^{\diamond}$.

$\therefore\ (S(X), \delta^{\diamond})$ is FT_1.

Conversely assume $(S(X), \delta^{\diamond})$ is FT_1.

Claim :

(X, δ) is FT_1

Let $f, g \in I^x$ such that $f \leq g$. $\Rightarrow f^{\diamond}\ (\nleq)_n\ g^{\diamond}$ since $(S(X), \delta^{\diamond})$ is FT_1, there exists a $\hbar \in \delta^{\diamond}$ such that $f^{\diamond}\ (\nleq)_n\ \hbar$ and $g^{\diamond}\ (\leq)_n.\ \hbar$

$f^{\diamond}(\nleq)_n\ \hbar \qquad \Rightarrow (f^{\diamond})_x\ (\nleq)\ (\hbar)_x$

i.e. $f \nleq (\hbar)_x$

and $g^{\diamond}(\leq)_n\ \hbar \Rightarrow (g^{\diamond})_x \leq (\hbar)_x$

i.e. $g \leq (\hbar)_x$

∴ there exists a $(\hbar)_x \in \delta$ such that $f \nleq (\hbar)_x$ and $g \leq (\hbar)_x$

∴ (X,δ) is FT_1.

Assume (X, δ) is FT_2

Claim :

$(S(X), \delta^\diamond)$ is FT_2

Let f, g be two fuzzy sets on $S(X)$ with $f\ Q^C\ g$.

$\Rightarrow (f)_x\ q^C\ (g)_x$

Since (X, δ) is FT_2, there exists $h_1, h_2 \in \delta$ such that

$(f)_x \wedge h_1 > 0$ and $(g)_x \wedge h_2 > 0$ and $h_1\ q^C\ h_2$

i.e. $(f)_x \wedge (h_1{}^\diamond)_x > 0$ and $(g)_x \wedge (h_2{}^\diamond)_x$ and $h_1{}^\diamond\ \phi^C\ h_2{}^\diamond$.

∴ $f\ \zeta\ h_1{}^\diamond$, $g\ \zeta\ h_2{}^\diamond$ and $h_1{}^\diamond\ Q^C\ h_2{}^\diamond$.

∴ $(S(X), \delta^\diamond)$ is FT_2.

Conversely assume $(S(X), \delta^\diamond)$ is FT_2.

Claim :

(X, δ) is FT_2.

Let f, g be two fuzzy sets on X with $f\ q^C\ g$.

Then $f^\diamond$ and $g^\diamond$ are fuzzy sets on $S(X)$ with $f^\diamond\ Q^C\ g^\diamond$.

As $(S(X), \delta^\diamond)$ is FT_2, there exists $\hbar_1$, $\hbar_2$ in $\delta^\diamond$ such that

$f^\diamond\ \zeta\ \hbar_1$, $g^\diamond\ \zeta\ \hbar_2$ and $\hbar_1 Q^C \hbar_2$.

i.e. $(f^\diamond)_x \wedge (\hbar_1)_x > 0$, $(g^\diamond)_x \wedge (\hbar_2)_x > 0$ and $(\hbar_1)_x\ q^C\ (\hbar_2)_x$

i.e $f \wedge (\hbar_1)_x > 0$, $g_x \wedge (\hbar_2)_x > 0$ $(\hbar_1)_x q^C (\hbar_2)_x$

$\Rightarrow (\hbar_1)_x, (\hbar_2)_x \in \delta$

∴ (X, δ) is FT_2.

Theorem : 2.3.12

Let $(S(X)\ \sigma,)$ be a fuzzy topological Boolean algebra.

Then $\sigma_x = \{ (f)_x / f \in \sigma \}$ is a fuzzy topology on X having the following property :

(X, σ_x) is FT_i, $i = 1,2 \Leftrightarrow (S(X), \sigma)$ is FT_i, $i = 1,2$.

First claim : σ_X is a fuzzy topology

(1) Since σ consists of all constant functions on S(X), it is obvious that σ_X consists of all constant functions on X.

(2) Let $(f_\alpha)_X = \alpha \in \Delta$ be a collection of elements of σ_X.

Then $\vee f_\alpha \in \sigma$ and $(\vee f_\alpha)_X \in \sigma_X$

As $\vee (f_\alpha)_X = (\vee f_\alpha)_X$, $\vee (f_\alpha)_X \in \sigma_X$.

(3) Let $(f_i)_X$ for i = 1 to n be a finite collection of elements of σ_X.

Then $\wedge^n_{i=i} f_i \in \sigma$ and $\left[\wedge^n_{i=i} f_i\right]_X \in \sigma_X$.

As $\wedge^n_{i=i} (f_i)_X = \left[\wedge^n_{i=i} f_i\right]_X$, $\wedge^n_{i=i} (f_i)_X \in \sigma_X$.

Hence σ_X is a fuzzy topology on X.

Assume (X, σ_X) is FT_1

Claim :

$(S(X), \sigma)$ is FT_1

Let f, g be two fuzzy sets on S(X) with $f\ (\nleq)_n\ g$

$\Rightarrow (f)_X \nleq (g)_X$

Since (X, σ_X) is FT_1, there exists a $h \in \sigma$ such that

$(f)_X \nleq (h)_X$ and $(g)_X \leq (h)_X$

$\Rightarrow f\ (\nleq)_n\ h$ and $g\ (\leq)_n\ h$

$\therefore$ there exist a $h \in \sigma$ such that $f (\nleq)_n\ h$ and $g\ (\leq)_n\ h$

$\therefore$ $(S(X), \sigma)$ is FT_1.

Conversely assume $(S(X), \sigma)$ is FT_1.

Claim : (X, σ_X) is FT_1.

Let f and g fuzzy sets on X with $f \leq g$.

$f^\circ\ (\nleq)_n\ g^\circ$.

As $(S(X), \sigma)$ is FT_1, there exists a $h \in \sigma$ such that $f^{\diamond}\ (\nleq)_n\ h$ and $g^{\diamond}\ (\leq)_n\ h$.

$h \in \sigma \Rightarrow (h)_X \in \sigma_X$.

$f^{\diamond}\ (\nleq)_n\ h$ and $g^{\diamond}\ (\leq)_n\ h$

$\Rightarrow (f^{\diamond})_X\ (\nleq)_n\ (h)_X$ and $(g^{\diamond})_X\ (\leq)_n\ (h)_X$

$\Rightarrow f(\nleq)_n\ (h)_X$ and $g\ (\leq)_n\ (h)_X$

$\therefore$ there exists a $(h)_X \in \sigma_X$ such that

$f\ (\nleq)_n\ (h)_X$ and $g\ (\leq)_n\ (h)_X$

$\therefore (X, \sigma_X)$ is FT_1

Hence the claim.

Assume (X, σ_X) is FT_2

Claim :

$(S(X), \sigma)$ is FT_2

Let f and g be fuzzy sets on $S(X)$ such that $f\ Q^C\ g$. Then $(f)_X\ q^C\ (g)_X$

As (X, σ_X) is FT_2, there exists fuzzy sets $h_1, h_2 \in \sigma$ with $(f)_X \wedge (h_1)_X > 0$, $(g)_X \wedge (h_2)_X > 0$ and $(h_1)_X\ q^C\ (h_2)_X$

Hence

$f\ \zeta\ h_1$, $g\ \zeta\ h_2$ and $h_1\ Q^C\ h_2$

Hence $(S(X), \sigma)$ is FT_2

Conversely assume $(S(X), \sigma)$ is FT_2

Claim :

(X, σ_X) is FT_2

Let f and g be two fuzzy sets on X such that $f\ q^C\ g$ Then $f^{\diamond}\ Q^C\ g^{\diamond}$

As $(S(X), \sigma)$ is FT_2, there exist fuzzy sets $h_1, h_2 \in \sigma$

such that $f^{\diamond}\ \zeta\ h_1$, $g^{\diamond}\ \zeta\ h_2$ and $h_1\ Q^C\ h_2$

This implies $(h_1)_x$, $(h_2)_x \in \sigma_x$ and

$(f^{\lozenge})_x \wedge (h_1)_x > 0$, $(g^{\lozenge})_x \wedge (h_2)_x > 0$ and

$(h_1)_x\, q^C\, (h_2)_x$

i.e. $f \wedge (h_1)_x > 0$, $g \wedge (h_2)_x > 0$ and $(h_1)_x\, q^C\, (h_2)_x$

Hence (X, σ_x) is FT_2.

Note : 2.3.13

$\sigma_x^{\lozenge}$ need not be the same as σ

Let $\sigma_x = \{\, (f)_x \,/\, f \in \sigma \}$

Then $(\sigma_x)^{\lozenge}$ is denoted by $\{\, (f)_x^{\lozenge} \,/\, f \in \sigma \,\}$

Claim : $(\sigma_x)^{\lozenge} \neq \sigma$.

Let $f \in \sigma$

Then $(f)_x \in \sigma_x$

let $A \in S(X)$, then $$(f)_x^{\lozenge}(A) = \sup_{x \in A} (f)_x(x)$$

$$. \neq f\ (A)$$

Hence $(\sigma_x)^{\lozenge} \neq \sigma$.

Remark : 2.3.14

Let (X, δ) be a fuzzy topological space. Then $\{A \,/ \chi_A \in \delta\}$ is a topology $t(\delta)$ on X.

(1) As $0,1 \in \delta\ \phi, x \in t(\delta)$.

(2) Let A_α, $\alpha \in \Delta$ be a collection of elements in $t(\delta)$.

Then $\chi_{A_\alpha} \in \delta$ for $\alpha \in \Delta$

Then $$\bigcup_{\alpha \in \Delta} \chi_{A_\alpha} \in \delta$$

But $$\cup\, \chi_{A_\alpha} = \chi_{\underset{\alpha \in \Delta}{\cup} A_\alpha}$$

$$\therefore \chi_{\underset{\alpha \in \Delta}{\cup} A_\alpha} \in \delta$$

$$\Rightarrow \bigcup_{\alpha \in \Delta} A_\alpha \in t(\delta)$$

(3) Let Ai for i = 1 to n be a finite collection of elements of t (δ).

Then $\chi_{A_i} \in \delta$

Then $\bigcap_{i=i}^{n} \chi_{A_i} \in \delta$

But $\bigcap_{i=i}^{n} \chi_{A_i} = \chi_{\bigcap_{i=i}^{n} A_i} \in \delta$

This implies $\bigcap_{i=i}^{n} Ai \in t(\delta)$

$\therefore \{ A / \chi_A \in \delta\}$ is a topology t (δ) on X.

Remark : 2.3.15

Let σ be a fuzzy topology on S(X). The collection of elements $\{A / \lambda_A \in \sigma\}$ forms a subbase for a topology T(σ) on S(X).

Let $S = \{A / \lambda_A \in \sigma\}$

As $\lambda_{A_1 \cap A_2} \neq \lambda_{A_1} \wedge \lambda_{A_2}$, S forms a subbase for a topology T(σ) on S(X).

Proposition : 2.3.16

Let (X, δ) be a fuzzy topological space. Then the two topologies t(δ) and $T(\delta^\circ)$ on X coincide.

Claim : $t(\delta) = T(\delta^\circ)$

Let $A \in t(\delta)$

$\Rightarrow \chi_A \in \delta$

$\Rightarrow (\chi_A)^\circ \in \delta^\circ$

$\Rightarrow (\lambda_A) \in \delta^\circ$

$\Rightarrow A \in T(\delta^\circ)$

$\therefore t(\delta) \subseteq t(\delta^\circ) \rightarrow (1)$

Let $A \in T(\delta^\circ)$

$\Rightarrow \lambda_A \in \delta^\circ$

$\Rightarrow (\lambda_A)_x \in \delta$

$\Rightarrow \chi_A \in \delta$

$\Rightarrow A \in t(\delta)$

$\therefore T(\delta^\circ) \subseteq t(\delta) \rightarrow (2)$

∴ From (1) and (2) t(δ) and $T(\delta^\circ)$ coincide.

Proposition : 2.3.17

Let (S (X), σ) be a fuzzy topological Boolean algebra.
Then $\lambda_A \in \sigma$ implies $\lambda_A \in (W(T(\sigma)))^{\diamond}$

In this statement $T(\sigma)$ is considered as a topology on X. Hence $w(T(\sigma))$ is a fuzzy topology on X and therefore $(w(T(\sigma)))^{\diamond}$ is a fuzzy topology on S(X).

$$\begin{aligned} \lambda_A \in \delta \quad &\Leftrightarrow \quad A \in T(\sigma) \\ &\Leftrightarrow \quad \chi_A \in W(T(\sigma)) \\ &\Leftrightarrow \quad \lambda_A \in (W(T(\sigma)))^{\diamond} \end{aligned}$$

FUZZY CONTINUITY :

Definition : 2.3.18

Let $(\mathcal{B}_1, \sigma_1)$ and $(\mathcal{B}_2, \sigma_2)$ be two fuzzy topological Boolean algebras. A function $\theta : (\mathcal{B}_1, \sigma_1) \rightarrow (\mathcal{B}_2, \sigma_2)$ is said to be fuzzy continous iff for every $f \in \sigma_2$, $\theta^{-1}(f) \in \sigma_1$ where $\theta^{-1}(f)(A) = f(\theta(A))$, $\forall A \in \mathcal{B}_1$.

Remark : 2.3.19

$\theta : (\mathcal{B}_1, \sigma_1) \rightarrow (\mathcal{B}_2, \sigma_2)$ is fuzzy continous iff for every closed fuzzy set f on $\mathcal{B}_2$, $\theta^{-1}(f)$ is a closed fuzzy set on $\mathcal{B}_1$.

Theorem : 2.3.20

A function $\theta : (X, \delta_1) \rightarrow (Y, \delta_2)$ is fuzzy continous iff
$S(\theta) : (S(X), \delta_1^{\diamond}) \rightarrow (S(Y), \delta_2^{\diamond})$ is fuzzy continous,
where $S(\theta)(A) = \{ \theta(x) / x \in A \}$.

Proof :

Let $\theta : (X, \delta_1) \rightarrow (Y, \delta_2)$ be a fuzzy continous function.

Claim :

$S(\theta)$; $(S(X), \delta_1^{\diamond}) \rightarrow (S(y), \delta_2^{\diamond})$ is fuzzy continous.
Let $f \in \delta_2^{\diamond}$.

It is enough to consider the case when $f = g^{\diamond}$, for some $g \in \delta_2$
Then $(f)_x = (g^{\diamond})_x = g \in \delta_2$
and $\theta^{-1}((f)_x) = \theta^{-1}(g) \in \delta_1$

$$\begin{aligned} \text{Now consider } ((S(\theta))^{-1} f)_x(x) &= f(S(\theta)(x)) \\ &= f(\theta(x)) \\ &= \theta^{-1} f(x) \\ &= (\theta^{-1}(f)_x)(x) \end{aligned}$$

$$\therefore ((S(\theta))^{-1} f)_x = (\theta^{-1}(f)_x)$$

$$\therefore ((S(\theta))^{-1} f)_x \in \delta_1.$$

To prove $((S(\theta))^{-1} f) \in \delta_1^{\diamond}$, it is enough to observe that
$((S(\theta))^{-1}(g^{\diamond}) = (\theta^{-1}(g))^{\diamond}$

We know $S(\theta) : (S(X), \delta_1^{\diamond}) \rightarrow (S(Y), \delta_2^{\diamond})$

Let $g : Y \rightarrow I$, then $g^{\diamond} : S(y) \rightarrow I$.
and $(S(\theta))^{-1} g^{\diamond} : S(X) \rightarrow I$.

$$\begin{aligned} \text{For } A \in S(X),\ ((S(\theta))^{-1} g^{\diamond})(A) &= g^{\diamond}(S(\theta)A) \\ &= \sup_{x \in A} g(S(\theta)(x)) \\ &= \sup_{x \in A} g(\theta(x)) \rightarrow (1) \end{aligned}$$

$\therefore\ g : Y \rightarrow I$ then $\theta^{-1}(g) : X \rightarrow I$

and $(\theta^{-1}(g))^{\diamond} : S(X) \rightarrow I$

$$\begin{aligned} (\theta^{-1}(g))^{\diamond}(A) &= \sup_{x \in A} (\theta^{-1}(g)(x)) \\ &= \sup_{x \in A} g(\theta(x)) \rightarrow (2) \end{aligned}$$

From (1) and (2) we have $(S(\theta))^{-1}\ g^{\diamond}) = (\theta^{-1}(g))^{\diamond}$.

$\therefore\ (S(\theta))^{-1}(f) = (S(\theta))^{-1}(g^{\diamond}) = (\theta^{-1}(g))^{\diamond} \in \delta_1^{\diamond}$.

Conversely assume $S(\theta) : (S(X), \delta_1^{\diamond}) \rightarrow (S(Y), \delta_2^{\diamond})$ is fuzzy continous.

Claim :

$\theta : (X, \delta_1) \rightarrow (Y, \delta_2)$ is fuzzy continous.

Let $h \in \delta_2$ then $h^\diamond \in \delta_2^\diamond$.

As $S(\theta)$ is fuzzy continous, $(S(\theta))^{-1} (h^\diamond) \in \delta_1{}^\diamond$.

We know $S(\theta) : (S(X), \delta_1{}^\diamond) \rightarrow (S(Y), \delta_2{}^\diamond)$

Let $h : Y \rightarrow I$ then $h^\diamond : S(Y) \rightarrow I$

$\therefore (S(\theta))^{-1} (h^\diamond) : S(X) \rightarrow I$

$$\therefore ((S(\theta)^{-1} h^\diamond)(A) = h^\diamond (S(\theta)(A))$$

$$= \operatorname*{Sup}_{x \in A} h (S(\theta)(x))$$

$$= \operatorname*{Sup}_{x \in A} h (\theta(x)) \rightarrow (1)$$

Since $h : Y \rightarrow I, \theta^{-1}(h) : X \rightarrow I$

$\therefore (\theta^{-1}(h))^\diamond (A) : S(X) \rightarrow I$

$$\therefore (\theta^{-1}(h))^\diamond (A) = \operatorname*{Sup}_{x \in A} \theta^{-1} (h) (x)$$

$$= \operatorname*{Sup}_{x \in A} h(\theta(x)) \rightarrow (2)$$

From (1) and (2) $((S(\theta))^{-1} h^\diamond) = (\theta^{-1}) (h))^\diamond \in \delta_1{}^\diamond$

$\therefore \theta^{-1} (h) \in \delta_1$

Hence the claim.

Theorem : 2.3.21

Let $(S(X), \sigma_1)$ and $(S(Y), \sigma_2)$ be two fuzzy topological Boolean algebras, and $\theta : (S(X), \sigma_1) \rightarrow (S(Y), \sigma_2)$ be fuzzy continous.
Then $(\theta)_x : (X, (\sigma_1)_x) \rightarrow (Y, (\sigma_2)_y)$ is fuzzy continous.

Proof :
Let $\theta : (S(X), \sigma_1) \rightarrow (S(Y), \sigma_2)$ be a fuzzy continous function.

Claim :

$(\theta)_x : (X, (\sigma_1)_x) \rightarrow (Y, (\sigma_2)_y)$ is fuzzy continous.

Let $f \in (\sigma_2)_y$

Then there exists a $g \in \sigma_2$ such that $(g)_y = f$

As θ is fuzzy continous $\theta^{-1}(g) \in \sigma_1$

As $(\theta)_x^{-1}(f) = \theta^{-1}((g)_y) = (\theta^{-1}(g))_x$

$(\theta)_x^{-1}(f) \in (\sigma_1)_x$

$\therefore$ $(\theta)_x$ is fuzzy continous

Theorem : 2.3. 22

Composition of two fuzzy continous functions is fuzzy continous.

SECTION. 4

FUZZY TOPOLOGIES ON QUOTIENT ALGEBRAS.

Before discussing fuzzy topology on quotient algebras it is useful to study the fundamental relations between fuzzy sets on $\mathscr{B}$ and fuzzy sets on $\mathscr{B}/\mathfrak{I}$ where $\mathfrak{I}$ is an ideal of $\mathscr{B}$ With every fuzzy set f on $\mathscr{B}/\mathfrak{I}$, one can associate a fuzzy set f_* on $\mathscr{B}$ as follows :

$$f_*(A) = f[A], \forall A \in B.$$

If f is a fuzzy set on $\mathscr{B}$ which preserves congruences (i.e.) $f(A) = f(B)$ if $A \equiv B \bmod \mathfrak{I}$, we can associate a function ${}^*f$ on $\mathscr{B}/\mathfrak{I}$ as follows:

$${}^*f\,[A] = f(A), \forall [A] \in \mathscr{B}/\mathfrak{I}.$$

Proposition : 2.4.1

Let $f, g: \mathscr{B}/\mathfrak{I} \to I$ be two fuzzy sets such that

$f\,(\leq)_n\, g$. Then $f_*\,(\leq)_n\, g_*$.

Proof :

Let $f\,(\leq)_n\, g$. Then for every $[A] \in \mathscr{B}/\mathfrak{I}$, there exists $[B] < [A]$ such that for every $[C] < [B]$, $f[C] \leq g[C]$.

$\therefore$ For every $A \in \mathscr{B}$, there exists $B_1 \in [B]$ such that $B_1 < A$ and for every $C < B_1$,

$$f_*(C) = f[C] \leq g[C] = g_*(C)$$

$$\Rightarrow f_*\,(\leq)_n\, g_*$$

Proposition : 2.4.2

Let f and g be fuzzy sets on $\mathscr{B}$ which preserve congruences modulo an ideal $\Im$ of $\mathscr{B}$ and $f\ (\leq)_n\ g$. Then ${}^*f\ (\leq)_n\ {}^*g$.

Proof :

Let $f\ (\leq)_n\ g$.

Then $\forall$ A $\in$ $\mathscr{B}$, there exists B < A such that for every C < B, $f(C) \leq g(C)$.

$\therefore$ For every [A], there exists [B] < [A] such that for every [C] < [B],

$${}^*f\,[C] = f\,[C] = f(C) \leq g(c) = {}^*g\,[C]$$

$$\therefore {}^*f\ (\leq)_n\ {}^*g$$

Proposition : 2.4.3

Let f and g be fuzzy sets on $\mathscr{B}/\Im$ such that $f Q^c g$. Then $f_*\ Q^C\ g_*$

Proof :

$$f\ Q^C\ g \Leftrightarrow f\ (\leq)_n\ 1-g$$

$$\Leftrightarrow f_*\ (\leq)_n\ (1-g)_* = 1-g_* \Rightarrow f_*\, Q^C\ g_*.$$

Proposition : 2.4.4

Let f and g be fuzzy sets on $\mathscr{B}$ such that they preserve congruences modulo an ideal $\Im$ of $\mathscr{B}$ and $f\ \zeta\ g$. Then ${}^*f\ \zeta\ {}^*g$.

Proof :

Assume $f\ \zeta\ g$.

Then there exists A $\in$ $\mathscr{B}$ such that for every B <A, there exists C < B such that min (f(C), g(C)) >0.

$\therefore$ There exists [A] $\in$ $\mathscr{B}/\Im$ such that for every non zero [B] < [A], there exists [C] < [B] such that min (${}^*f$ [C], ${}^*g$ [C]) >0.

$$\therefore {}^*f\ \zeta\ {}^*g.$$

Definition : 2.4.5

Given a fuzzy topology σ on $\mathscr{B}$ and an ideal $\Im$ of elements of $\mathscr{B}$, two Quotient topologies $\mathcal{K}_1(\sigma)$ and $\mathcal{K}_2(\sigma)$ on $\mathscr{B}/\Im$ can be defined as follows :

(1) $\mathcal{K}_1(\sigma) = \{{}^*f \,/\, f \in \sigma,\ f$ preserves congruences $\}$

(2) $\mathcal{K}_2(\sigma)$ is the fuzzy topology having $\{ f_\Im \,/\, f \in \sigma\}$ as a sub base, where

$$f_\Im\,[A] = \text{Sup}\,\{f(B) \,/\, B \equiv A \bmod \Im\}$$

To prove $\mathscr{X}_1(\sigma)$ is a fuzzy topology :-

(1) It is obvious that all constant functions on $\mathscr{B}/\mathfrak{I}$ belongs to $\mathcal{K}_1(\sigma)$ as all constant functions on $\mathscr{B} \in \sigma$.

(2) Let $\{ {}^*f_\alpha / \alpha \in \Delta \}$ be a collection of elements of $\mathcal{K}_1(\sigma)$

$$ {}^*f_\alpha \in \mathcal{K}_1(\sigma) \quad \Rightarrow f_\alpha \in \sigma $$

$$ \Rightarrow \bigcup_{\alpha\in\Delta} f_\alpha \in \sigma $$

$$ \Rightarrow {}^*(\cup f_\alpha) \in \mathcal{K}_1(\sigma) $$

Consider $\left[\bigcup_{\alpha\in\Delta} {}^*f_\alpha\right][A]$

$$ = \bigcup_{\alpha\in\Delta} {}^*f_\alpha[A] $$

$$ = \bigcup_{\alpha\in\Delta} f_\alpha(A) \quad \rightarrow (1) $$

$$ {}^*\left[\bigcup_{\alpha\in\Delta} f_\alpha\right][A] $$

$$ = \left[\bigcup_{\alpha\in\Delta} f_\alpha\right](A) $$

$$ = \bigcup_{\alpha\in\Delta} f_\alpha(A) \quad \rightarrow (2) $$

From (1) and (2) ${}^*\left[\bigcup_{\alpha\in\Delta} f_\alpha\right] = \cup\, {}^*f_\alpha$

Since $\quad {}^*(\cup f_\alpha) \in \mathcal{K}_1(\sigma)$,

$$ \cup\, {}^*f_\alpha \in \mathcal{K}_1(\sigma) $$

$\therefore \mathcal{K}_1(\sigma)$ is a fuzzy topology.

Note :

(i) ${}^*f \in \mathcal{K}_1(\sigma) \Rightarrow f_{\mathfrak{I}} = {}^*f$

(ii) $\mathcal{K}_1(\sigma) \subset \mathcal{K}_2(\sigma)$

(iii) $f_{\mathfrak{I}} \in \mathcal{K}_2(\sigma) \Rightarrow {}^*((f_{\mathfrak{I}})_*) = f_{\mathfrak{I}}$

Theorem : 2.4.6

Let $\mathscr{B}$ be a Boolean algebra and $\mathfrak{I}$ be an ideal of $\mathscr{B}$ Let $\mathscr{K}$ be a family of elements of $\mathscr{B}$ closed for finite union. Let $[\mathscr{K}]_{\mathfrak{I}}$ be the equivalence class of elements of $\mathscr{K}$ modulo $\mathfrak{I}$. Let $W([\mathscr{K}]_{\mathfrak{I}})$ be the fuzzy topology on $\mathscr{B}/\mathfrak{I}$ induced by $[\mathscr{K}]_{\mathfrak{I}}$. Then $\mathcal{K}_1(W(\mathscr{K})) \subset \mathcal{K}_2(W(\mathscr{K})) \subset W([\mathscr{K}]_{\mathfrak{I}})$.

Proof :

It is obvious that $\mathcal{K}_1(W(\mathscr{K})) \subset \mathcal{K}_2(W(\mathscr{K}))$

Claim :

$\mathcal{K}_2(W(\mathscr{K})) \subset W([\mathscr{K}]_{\mathfrak{I}})$

Let $f_{\mathfrak{I}} \in \mathcal{K}_2(W(\mathscr{K}))$ for some $f \in W(\mathscr{K})$ and $f_{\mathfrak{I}}[B] > \in$.

$\therefore$ there exists $A \equiv B \mod \mathfrak{I}$ such that $f(A) > \in$.

$\therefore$ there exists $D \in \mathscr{K}$ such that $f(D) > \in$ and $D < A$.

Hence $f_{\mathfrak{I}}[D] > \varepsilon$ and $[D] < [A] = [B]$. Therefore $f_{\mathfrak{I}}$ is compatiable with $[\mathscr{K}]_{\mathfrak{I}}$

Hence the claim.

Example 2.4.7

An example where $\mathcal{K}_1(W(\mathscr{K}))$ is strictly contained in $\mathcal{K}_2(W(\mathscr{K}))$.

Consider $X = [0,1]$, with the usual topology on $[0,1]$ derived as a subspace of R with the usual topology. Let $\mathscr{K}$ be the collection of all open sets in $[0,1]$.

Let f be the identify function on $[0,1]$. For $A \in S(X)$,

$$\text{define } g(A) = \operatorname{Sup}\{f(x) / x \in \operatorname{int} A\}$$
$$= 0, \text{ if } \operatorname{int} A = \phi.$$

Then $g \in W(\mathscr{K})$

Let $\mathfrak{I}$ be the ideal of all countable subsets of X.

Define $g_{\mathfrak{I}}[A] = \operatorname{Sup}\{ g(B) / A \equiv B \mod \mathfrak{I}\}$

Then $g_{\mathfrak{I}} \in \mathcal{K}_2(W(\mathscr{K}))$

Claim :

$g_{\mathfrak{I}} \notin W(\mathscr{K})$

Let B be the set of irrationals.

Then $g\,(B) = \text{Sup}\,\{f(x) \,/\, x \in \text{int } B\,\} = 0$

If A is a countable set in X, int $A = \phi$ and $g\,(A) = 0$.

As $B \equiv B \cup \{1\} \bmod \mathfrak{I}$, $g_{\mathfrak{I}}\,[B] = \text{Sup}\,\{g(C) \,/\, C \equiv B \bmod \mathfrak{I}\,\} = 1$

$\therefore\ (g_{\mathfrak{I}})_* \notin W\,(\mathscr{K})$

$\therefore\ {}^*((g_{\mathfrak{I}})_*) = g_{\mathfrak{I}} \notin \mathcal{K}_1(W(\mathscr{K}))$

$\therefore\ \mathcal{K}_1(W(\mathscr{K}))$ is strictly contained in $\mathcal{K}_2(W(\mathscr{K}))$.

Note : 2.4.8

(i) When $\mathscr{K}$ is an ideal of $\mathcal{B}$, $\mathcal{K}_1(W(\mathscr{K})) = \mathcal{K}_2(W(\mathscr{K})) = W([\mathscr{K}\,]_{\mathfrak{I}})$

(ii) When $[\mathscr{K}]_{\mathfrak{I}} = [0]$, then $W([\mathscr{K}\,]_{\mathfrak{I}})$ consists of only the constant functions.

Proposition : 2.4.9

Let $(\mathcal{B}, \sigma)$ be a fuzzy topological Boolean algebra and $\mathfrak{I}$ be an ideal of elements of $\mathcal{B}$ Then $\theta{:}(\mathcal{B}, \sigma) \rightarrow (\mathcal{B}/\mathfrak{I}, \mathcal{K}_1(\sigma))$ defined by

$\theta\,(A) = [A]\ \ \forall\, A \in \mathcal{B}$, is fuzzy continous.

Proof :

Let ${}^*f \in \mathcal{K}_1\,(\sigma)$. Then $f \in \sigma$ and preserves congruences.

$\therefore\ \theta^{-1}\,({}^*f)\,(A) = {}^*f\,(\theta(A)) = {}^*f[A] = f\,(A)$

$\therefore\ \theta^{-1}\,({}^*f) = f$

Hence θ is continous.

Proposition : 2.4.10

Let $\mathcal{B}$ be a Boolean algebra and $\mathscr{K}$ and $\mathfrak{I}$ be two ideals of $\mathcal{B}$ Then $\theta : (\mathcal{B}, W(\mathscr{K})) \rightarrow \mathcal{B}/\,\mathfrak{I},\ W([\mathscr{K}])_{\mathfrak{I}})$ defined by

$\theta\,(A) = [A]$, $A \in \mathcal{B}$ is fuzzy continous.

Proof :

Obvious.

Proposition : 2.4.11

Let σ be a fuzzy topology on $\mathcal{B}/\mathfrak{I}$. Then there exits a fuzzy topology $E(\sigma)$ on $\mathcal{B}$ such that $\mathcal{K}_1(E(\sigma)) = \mathcal{K}_2(E(\sigma)) = \sigma$. Further if $E(\sigma)$ satisfies the FT_i axiom for $i = 1,2$ then σ satisfies the FT_i axiom for $i = 1,2$.

Proof :

(i) Let $(\mathcal{B}, E(\sigma))$ satisfy the FT_1 - axiom.

Claim :

$(\mathcal{B}/\mathfrak{I}, \sigma)$ is FT_1

Let f, g be fuzzy sets on $\mathcal{B}/\mathfrak{I}$ with $f_* (\leq)_n g_*$. As $(\mathcal{B}, E(\sigma))$ is FT_1, there exists $\hbar \in \sigma$ such that $f_*(\leq)_n \hbar_*$ and $g_* (\leq)\hbar_*$. This implies $f(\leq)_n \hbar$ and $f(\leq)_n\hbar$.

Hence the claim.

(ii) Let $(\mathcal{B}, E(\sigma))$ satisfy the FT_2 - axiom.

Claim :

$(\mathcal{B}/\mathfrak{I}, \sigma)$ is FT_2.

Let f and g be two fuzzy sets on $\mathcal{B}/\mathfrak{I}$ such that $f\, Q^C g$. Then $f_*\, Q^C\, g_*$. As $(\mathcal{B}, E(\sigma))$ is FT_2, there exists $\hbar_1, \hbar_2 \in \sigma$ such that $f_* \zeta (\hbar_1)_*$, $g_* \zeta (\hbar_2)_*$ and $(\hbar_1)_*\, Q^C\, (\hbar_2)_*$.

This implies $f \zeta \hbar_1$, $g \zeta \hbar_2$ and $\hbar_1\, Q^C\, \hbar_2$

$\therefore (\mathcal{B}/\mathfrak{I}, \sigma)$ is FT_2.

Summary and Conclusion

SUMMARY AND CONCLUSION

In this dissertation we have discussed the results contained in the following two articles.

1) "Closure algebras" by Sikorski [27]
2) "Fuzzy topological Boolean Algebras" by Parvathi and Meenakshi [22].

In chapter 1 some of the theorems on topological spaces are generalised to closure algebras and few results on Quotient algebras are discussed. Few important theorems are as follows:

1. An isomorphism h of $\mathcal{A}$ on $\mathcal{B}$ is a homeomorphism of $\mathcal{A}$ on $\mathcal{B}$ if and only if h $/\mathfrak{B}(\mathcal{A})$ is a homeomorphism of $\mathfrak{B}(\mathcal{A})$ on $\mathfrak{B}(\mathcal{B})$.

2. The following three conditions are equivalent for arbitrary topological spaces $\mathcal{X}$ and $\mathcal{Y}$
    i. the spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic.
    ii. the closure algebras $\mathbf{S}(\mathcal{X})$ and $\mathbf{S}(\mathcal{Y})$ are homeomorphic
    iii. the closure algebras $\mathbf{S}(\mathcal{X})$ and $\mathbf{S}(\mathcal{Y})$ are weakly homeomorphic (i.e. $\mathfrak{B}(\mathcal{X})$ and $\mathfrak{B}(\mathcal{Y})$ are homeomorphic)

3. A topological space $\mathcal{X}$ is separable and metrizable if and only if $\mathbf{S}(\mathcal{X})$ is a C- algebra.

In chapter two an introduction to the study of fuzzy topological Boolean algebra is given. Here, given a Boolean algebra $\mathcal{B}$ many ways of associating fuzzy topological structures on $\mathcal{B}$ are given.

With every fuzzy topological space (X,δ) a fuzzy topological Boolean algebra $(S(X), \delta^{\Diamond})$ is associated and interesting results on fuzzy separation axioms FT_1 and FT_2 and fuzzy continuity are discussed.

Study of quotient fuzzy topological Boolen algebras leads to some interesting results. With every fuzzy topology σ on a Boolean algebra $\mathcal{B}$ there exists two fuzzy Quotient topologies $\mathcal{K}_1(\sigma)$ and $\mathcal{K}_2(\sigma)$ on $\mathcal{B}/\mathfrak{I}$, where $\mathfrak{I}$ is an ideal of $\mathcal{B}$ such that $\mathcal{K}_1(\sigma) \subset \mathcal{K}_2(\sigma)$.

An important theorem proved in this connection is as follows :

Let $\mathcal{B}$ be a Boolean algebra and $\mathfrak{I}$ be an ideal of $\mathcal{B}$ Let $\mathcal{K}$ be a family of elements of $\mathcal{B}$ closed for finite union. Let $[\mathcal{K}]_{\mathfrak{I}}$ be the equivalence class of elements of $\mathcal{K}$ modullo $\mathfrak{I}$.Let $W([\mathcal{K}]_{\mathfrak{I}})$ be the fuzzy topology on $\mathcal{B}/\mathfrak{I}$ induced by $[\mathcal{K}]_{\mathfrak{I}}$ Then $\mathcal{K}_1(W(\mathcal{K})) \subset \mathcal{K}_2(W(\mathcal{K})) \subset (W[\mathcal{K}]_{\mathfrak{I}})$

When $\mathcal{K}$ is an ideal then

$\mathcal{K}_1(W(\mathcal{K})) = \mathcal{K}_2(W(\mathcal{K})) = W([\mathcal{K}]_{\mathfrak{I}})$

Fuzzy continuity and fuzzy separation axioms are studied for the Quotient fuzzy topological Boolean algebras.

The concepts of fuzzy proximity, fuzzy completely regular and fuzzy gradation of openness are discussed in the Ph.D thesis of Parvathi.

Study of fuzzy topological Boolean algebras provides lot of scope for further research. Fuzzy topological concepts like compactness, connectedness, etc can be devloped for fuzzy topological Boolen algebras.

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