CHAPTER VIII

REPAIRABLE TWO PHASE SERVICE $M^X/G/1$ QUEUEING MODELS WITH INFINITE NUMBER OF IMMEDIATE FEEDBACKS UNDER BERNOULLI SCHEDULE VACATION

INTRODUCTION

The model of the present section differs from that of section (7.1) only in the following aspects :

The customers who finish the first round of service (i.e., first phase service (FPS) followed by an optional second phase service) may either demand a re-service from the first phase followed by a second phase service with probability f_1 (or) repeat any i^{th} service only from second phase with probability $f_2\, r_i$ $(1 \leq i \leq C)$ or leave the system with probability $(1 – (f_1 + f_2))$. As in section (7.1) the service demand from the second phase, during feedback need not be the same.

The server is allowed to feedback any number of times to his satisfaction before leaving the system.

Kalidass and Kasturi (2013) have considered a reliable Poisson arrival M/G/1 queueing system with two phases of heterogeneous services and finite number (m) of immediate Bernoulli feedbacks. They have assumed that, all the arriving customers are provided with the same type of service in the first phase and allowed to choose one of the optional services from the second phase. After having completed services in both phases, the customer can make an immediate feedback. In the feedback service, the first phase of service is of the same type as in the previous service and in the second phase, the customer is permitted to choose an optional service (may be different from the one chosen earlier). In the present chapter a generalization of their model in the sense that the feedback customers can demand re-services either from the first phase or from the second phase alone is considered for a repairable $M^X/G/1$ queueing system with infinite number of feedbacks under Bernoulli Schedule Vacation. The results of Kalidass and

Kasturi are verified under the conditions, $f_2 = 0$, $m \to \infty$ and by setting the breakdown parameters and the vacation control parameter to zero for the single arrival Poisson queue. It is also justified that with $f_1 = 0$ ($f_2 = f$) the results of the present section coincide with that of Section 7.1 as $m \to \infty$.

SECTION : 8.1

$M^X/G/1$ QUEUE WITH INFINITE NUMBER OF FEEDBACKS AND RESUMPTION OF INTERRUPTED SERVICE

8.1.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

8.1.1.1 Model Description

The present chapter deals with $M^X/G/1$ queueing system with two-phases of service and Bernoulli vacation schedule for an unreliable server which consists of a breakdown period. The customers arrive at the system in batches of variable size in accordance with time-homogeneous compound Poisson process with group arrival rate λ.

Service is provided one by one according to FCFS basis. Every customer has to undergo two stages of services following different general (arbitrary) distributions. The arriving customers first receive the First Phase Service S (FPS), which is followed by any of the second phase services (SPS) S_i ($1 \le i \le C$). The customers after completing the first phase service, can choose any of the i^{th} optional services (available in second phase) with probability r_i, where $\sum_{i=1}^{C} r_i = 1$. The second phase service commences immediately after the completion of first phase service, and all the services are provided by the same server. A customer is said to complete the first round of service if he undergoes the FPS and anyone of the i^{th} second optional services with probability r_i ($1 \le i \le C$). The first round of service may be termed as primary (or) fresh service.

The customer, who finishes the first round of service either feeds back immediately from the first phase with probability f_1 (or) from any of the ℓ^{th} second optional services with probability $f_2\, r_\ell$ ($1 \le \ell \le C$) or leaves the system with probability $(1 - (f_1 + f_2))$. After the completion of the first feedback service, the customer may again go in for a third round of service and so on in a similar way. Thus the first round service consists of services of length $S + S_i$ $(1 \leq i \leq C)$. Second round of service (i.e., first feedback service) either is of length $S + S_\ell$ with probability f_1 (or) S_ℓ alone with probability $f_2\, r_\ell$ (or) else does not exist with probability $(1 - (f_1 + f_2))$. This process will continue any number of times until the customer is satisfied. The next customer in the queue can go into the system only after the successful completion of all feedback rounds of the preceding customer. The distribution functions, density functions, LST of FPS and SPS are respectively denoted by S(t), $S_i(t)$; s(t), $s_i(t)$, $S^*(\theta)$, $S_i^*(\theta)$ with finite moments.

During busy period, the server is subject to breakdowns. The lifetime of the server follows exponential distribution with parameters : a^0 in the primary FPS ; a in the feedback FPS ; a_i^0 in the primary i^{th} second phase service and a_i in the i^{th} second optional feedback service.

The server whenever breaks down is sent for repair immediately and the customer just being served, waits in the corresponding service facility to complete the remaining service.

The repair time distributions of the server follow arbitrary distributions $R_1^0(t)$, $R_1(t)$, $R_i^0(t)$ and $R_i(t)$ respectively according as the breakdowns occur in first phase due to primary service or feedback service (or) i^{th} primary or feedback services. It is also assumed that after completing a service to a customer (i.e., when the customer leaves the system) the server may take a Bernoulli Scheduled Single Vacation (BSV) with probability **p** or continue to serve the next customer in the queue if any (or) stay idle in the system for the next batch to arrive, with probability **(1 – p)**. The vacation time V follows general distribution with its distribution function V(t), density function v(t), LST $V^*(\theta)$ with finite first and second moments $E(V^k)$, k = 1, 2.

Thus a cycle consists of primary services, feedback services, breakdown period and vacation period. Various stochastic processes involved in the queueing system are assumed to be independent of each other. The customers continue to arrive and join the system independent of the system states, following the compound Poisson process. Using supplementary variable technique the steady state system equations under the steady-state condition are analysed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it.

The notations of Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RVs are listed below :

Table 8.1.1.

	RV	**CDF**	**PDF**	**LST**	**k^{th} moments k = 1, 2**
Service time in first phase	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$
i^{th} optional service time of the second phase	S_i	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S_i^k)$, i = 1 to C
Repair time in first primary service	R_1^0	$R_1^0(y)$	$r_1^0(y)$	$R_1^{0*1}(\theta_1)$	$E((R_1^0)^k)$
Repair time in first feedback service	R_1	$R_1(y)$	$r_1(y)$	$R_1^{*1}(\theta_1)$	$E(R_1^k)$
Repair time in second primary service	$R_2^{(i,0)}$	$R_2^{(i,0)}(y)$	$r_2^{(i,0)}(y)$	$R_2^{(i,0)*1}(\theta_1)$	$E((R_2^{(i,0)})^k)$
Repair time in second feedback service	R_2^i	$R_2^i(y)$	$r_2^i(y)$	$R_2^{i*1}(\theta_1)$	$E((R_2^i)^k)$
Vacation time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$

Let $N_S(t)$ denote the system size at time t and $S^o(t), S_i^o(t), (R_1^0)^o(t), (R_1)^o(t), (R_2^{(i,0)})^o(t), (R_2^i)^o(t), V^o(t)$ respectively denote the remaining times of the random variables ; service time in first stage, second stage, repair time in first fresh service, first feedback service, second fresh service, second feedback service and vacation time at time t. Further the server states are denoted by the random variable Y(t) at time t. Then the state space is $\{N_S(t), \delta(t)\}$ where $\delta(t) = (0, S^o(t), S_i^o(t), (R_1^0)^o(t), (R_1)^o(t), (R_2^{(i,0)})^o(t), (R_2^i)^o(t), V^o(t))$ according as Y(t) = 0, 1, 2, 3, 4, 5, 6 and 7 respectively. The following joint probability functions are defined, for further analysis of the model.

$P_I(t)$	=	Pr $\{N_S(t) = 0, Y(t) = 0\}$, when the server is idle.

For $n \geq 1$ and $1 \leq i \leq C$

$P^0_{1,n}(x,t)\,dt$	=	Pr $\{N_S(t) = n,\ x < S^o(t) \leq x + dt,\ Y(t) = 1\}$, a customer is being served in the primary first phase service
$P_{1,n}(x, t)\,dt$	=	Pr $\{N_S(t) = n,\ x < S^o(t) \leq x + dt,\ Y(t) = 1\}$, a customer is being served in the first phase – feedback service
$P^{(i,0)}_{2,n}(x,t)\,dt$	=	Pr $\{N_S(t) = n,\ x < S^o_i(t) \leq x + dt,\ Y(t) = 2\}$, a customer is being served in the i^{th} second phase primary service.
$P^i_{2,n}(x,t)\,dt$	=	Pr $\{N_S(t) = n,\ x < S^o_i(t) \leq x + dt,\ Y(t) = 2\}$, a customer is being served in the i^{th} optional service of the second phase, during feedback.
$BR^0_{1,n}(x,y,t)\,dt$	=	Pr $\{N_S(t) = n,\ S^o(t) = x,\ y < (R^0_1)^o(t) \leq y + dt,\ Y(t) = 3\}$, a customer is waiting for the first primary service due to breakdown.
$BR_{1,n}(x, y, t)\,dt$	=	Pr $\{N_S(t) = n,\ S^o(t) = x,\ y < (R_1)^o(t) \leq y + dt,\ Y(t) = 4\}$, a customer is waiting for the first phase – feedback services due to breakdown.

For $1 \leq i \leq C$; $n \geq 1$

$BR^{(i,0)}_{2,n}(x,y,t)\,dt$	=	Pr $\{N_S(t) = n,\ S^o_i(t) = x,\ y < (R^{(i,0)}_2)^o(t) \leq y + dt,\ Y(t) = 5\}$, a customer is waiting for the second i^{th} phase fresh service due to breakdown.
$BR^i_{2,n}(x,y,t)\,dt$	=	Pr $\{N_S(t) = n,\ S^o_i(t) = x,\ y < (R^i_2)^o(t) \leq y + dt,\ Y(t) = 6\}$, a customer is waiting for the second i^{th} phase service, during feedback due to breakdown.
$Q_n(x, t)\,dt$	=	Pr $\{N_S(t) = n,\ x < V^o(t) \leq x + dt,\ Y(t) = 7\}$, when the server is in vacation state. $(n \geq 0)$

8.1.1.2 System Size Distribution at Random Epoch

Observing the changes of states during the interval (t, t + Δt) for any time t, the steady state equations are given by :

System in Empty State

$$\lambda \, PI = Q_0(0) + \sum_{i=1}^{C} (P_{2,1}^{(i,0)}(0) + P_{2,1}^{i}(0))(1-f)(1-p)$$

Vacation State

$$-\frac{d}{dx} Q_n(x) = -\lambda \, Q_n(x) + \lambda\,(1-\delta_{0,n}) \sum_{k=1}^{n} Q_{n-k}(x) g_k$$

$$+ \sum_{i=1}^{C} (P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^{i}(0))(1-f)\, p\, v(x) \qquad n \geq 0$$

Busy with First Phase – Primary Service

$$-\frac{d}{dx} P_{1,n}^{0}(x) = -(\lambda + a_1^0)\, P_{1,n}^{0}(x) + PI\, \lambda\, g_n\, s(x) + BR_{1,n}^{0}(x,0)$$

$$+ \sum_{i=1}^{C} (P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^{i}(0))(1-f)(1-p)\, s(x) + Q_n(0)\, s(x)$$

$$+ \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{1,n-k}^{0}(x) g_k$$

Busy with First Phase Feedback Service

$$-\frac{d}{dx} P_{1,n}(x) = -(\lambda + a_1)\, P_{1,n}(x) + BR_{1,n}(x, 0) + \sum_{i=1}^{C} (P_{2,n}^{(i,0)}(0) + P_{2,n}^{i}(0)) f_1\, s(x)$$

$$+ (1-\delta_{1,n})\, \lambda \sum_{k=1}^{n-1} P_{1,n-k}(x) g_k$$

Busy with Second Phase Primary Service

$$-\frac{d}{dx} P_{2,n}^{(i,0)}(x) = -(\lambda + a_2^{(i,0)})\, P_{2.n}^{(i,0)}(x) + (1-\delta_{1,n})\, \lambda \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)}(x) g_k$$

$$+ P_{1,n}^{o}(0)\, r_i\, S_i(x) + BR_{2,n}^{(i,0)}(x,0), \qquad 1 \leq i \leq C, \;\; n \geq 1$$

Busy with Second Phase Feedback Service (1 ≤ i ≤ C)

$$-\frac{d}{dx} P_{2,n}^{i}(x) = -(\lambda + a_2^{i})\, P_{2,n}^{i}(x) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{i}(x) g_k + BR_{2,n}^{i}(x,0)$$

$$+ \sum_{=1}^{C} (P_{2,n}(0) + P_{2,n}^{(,0)}(0))\, f_2\, r_i\, S_i(x) + P_{1,n}(0)\, r_i\, S_i(x),$$

Breakdown in First Phase Fresh Service

$$-\frac{\partial}{\partial y} BR^{0}_{1,n}(x,y) = -\lambda\, BR^{0}_{1,n}(x,y) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR^{0}_{1,n-k}(x,y)\, g_k + a^{0}_{1}\, r^{0}_{1}(y)\, P^{0}_{1,n}(x), \quad (n \geq 1)$$

Breakdown in First Phase Feedback Services

$$-\frac{\partial}{\partial y} BR_{1,n}(x,y) = -\lambda\, BR_{1,n}(x,y) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}(x,y)\, g_k + a_1\, r_1(y)\, P_{1,n}(x), \quad n \geq 1$$

Breakdown in Second Phase Primary Service

$$-\frac{\partial}{\partial y} BR^{(i,0)}_{2,n}(x,y) = -\lambda\, BR^{(i,0)}_{2,n}(x,y) + (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR^{(i,0)}_{2,n-k}(x,y)\, g_k + a^{(i,0)}_{2}\, r^{(i,0)}_{2}(y)\, P^{(i,0)}_{2,n}(x), \quad 1 \leq i \leq C$$

Breakdown in Second Phase Feedback Services

$$-\frac{\partial}{\partial y} BR^{i}_{2,n}(x,y) = -\lambda\, BR^{i}_{2,n}(x,y) + (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR^{i}_{2,n-k}(x,y)\, g_k + a^{i}_{2}\, r^{i}_{2}(y)\, P^{i}_{2,n}(x), \quad 1 \leq i \leq C$$

The LST of the Steady-State equations are given by

$$\lambda\, PI = Q_0(0) + \sum_{i=1}^{C} \left(P^{(i,0)}_{2,1}(0) + P^{i}_{2,1}(0)\right)(1-f)\,(1-p) \qquad (8.1.0)$$

$$\theta\, Q^{*}_{n}(\theta) - Q_n(0) = \lambda\, Q^{*}_{n}(\theta) - \lambda\,(1-\delta_{0,n}) \sum_{k=1}^{n} Q^{*}_{n-k}(\theta)\, g_k - \sum_{i=1}^{C} \left(P^{(i,0)}_{2,n+1}(0) + P^{i}_{2,n+1}(0)\right)(1-f)\, p\, V^{*}(\theta) \quad n \geq 0 \qquad (8.1.1)$$

$$\theta\, P^{0*}_{1,n}(\theta) - P^{0}_{1,n}(0) = (\lambda + a^{0}_{1})\, P^{0*}_{1,n}(\theta) - PI\,\lambda\, g_n\, S^{*}(\theta) - BR^{0*}_{1,n}(\theta,0) - \sum_{i=1}^{C} \left(P^{(i,0)}_{2,n+1}(0) + P^{i}_{2,n+1}(0)\right)(1-f)\,(1-p)\, S^{*}(\theta) - Q_n(0)\, S^{*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P^{0*}_{1,n-k}(\theta)\, g_k \qquad (8.1.2)$$

$$\theta\, P^{*}_{1,n}(\theta) - P_{1,n}(0) = (\lambda + a_1)\, P^{*}_{1,n}(\theta) - BR^{*}_{1,n}(\theta,0) - \sum_{i=1}^{C} \left(P^{(i,0)}_{2,n}(0) + P^{i}_{2,n}(0)\right) f_1\, S^{*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P^{*}_{1,n-k}(\theta)\, g_k \qquad (8.1.3)$$

$$\theta\, P_{2,n}^{(i,0)^*}(\theta) - P_{2,n}^{(i,0)}(0) = (\lambda + a_2^{(i,0)})\, P_{2,n}^{(i,0)^*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)^*}(\theta) g_k$$

$$- P_{1,n}^{0}(0) r_i\, S_i^*(\theta) - BR_{2,n}^{(i,0)^*}(\theta, 0), \quad 1 \le i \le C \tag{8.1.4}$$

$$\theta\, P_{2,n}^{i*}(\theta) - P_{2,n}^{i}(0) = (\lambda + a_2^{i})\, P_{2,n}^{i*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{2,n-k}^{i*}(\theta) g_k$$

$$- BR_{2,n}^{i*}(\theta, 0) - \sum_{=1}^{C} (P_{2,n}(0) + P_{2,n}^{(\ ,0)}(0))\; f_2\, r_i S_i^*(\theta)$$

$$- \;\; P_{1,n}(0)\, r_i\, S_i^*(\theta) \qquad 1 \le i \le C \tag{8.1.5}$$

$$\theta_1\, BR_{1,n}^{0**_1}(\theta, \theta_1) - BR_{1,n}^{0*}(\theta, 0) = \lambda BR_{1,n}^{0**_1}(\theta, \theta_1) - \lambda(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{0**_1}(\theta, \theta_1) g_k$$

$$- a_1^0\; R_1^{0*_1}(\theta_1)\, P_{1,n}^{0*}(\theta) \tag{8.1.6}$$

$$\theta_1\, BR_{1,n}^{**_1}(\theta, \theta_1) - BR_{1,n}^{*}(\theta, 0) = \lambda BR_{1,n}^{**_1}(\theta, \theta_1) - \lambda(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{**_1}(\theta, \theta_1) g_k$$

$$- a_1\, R_1^{*_1}(\theta_1)\, P_{1,n}^{*}(\theta) \tag{8.1.7}$$

$$\theta_1\, BR_{2,n}^{(i,0)**_1}(\theta, \theta_1) - BR_{2,n}^{(i,0)^*}(\theta, 0) = \lambda\, BR_{2,n}^{(i,0)^{**_1}}(\theta, \theta_1)$$

$$- \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,0)**_1}(\theta, \theta_1) g_k$$

$$- a_2^{(i,0)}\; R_2^{(i,0)*_1}(\theta_1)\, P_{2,n}^{(i,0)^*}(\theta) \tag{8.1.8}$$

$$\theta_1\, BR_{2,n}^{i**_1}(\theta, \theta_1) - BR_{2,n}^{i*}(\theta, 0) = \lambda BR_{2,n}^{i**_1}(\theta, \theta_1) - \lambda(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{2,n-k}^{i**_1}(\theta, \theta_1) g_k$$

$$- a_2^{i}\, R_2^{i*_1}(\theta_1)\;\; P_{2,n}^{i*}(\theta) \tag{8.1.9}$$

8.1.1.3 Probability Generating Functions

The following partial PGFs are introduced to analyse the model :

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta)\, z^n, \qquad Q(z, 0) = \sum_{n=0}^{\infty} Q_n(0)\, z^n$$

$$P_1^{0*}(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^{0*}(\theta)\, z^n, \qquad P_1^{0}(z,0) = \sum_{n=1}^{\infty} P_{1,n}^{0}(0)\, z^n$$

$$P_1^{*}(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^{*}(\theta)\, z^n, \qquad P_1(z, 0) = \sum_{n=1}^{\infty} P_{1,n}(0)\, z^n$$

$$P_2^{(i,0)^*}(z,\theta) = \sum_{n=1}^{\infty} P_{2,n}^{(i,0)^*}(\theta)\, z^n, \qquad P_2^{(i,0)}(z,0) = \sum_{n=1}^{\infty} P_{2,n}^{(i,0)}(0)\, z^n$$

$$P_2^{i*}(z,\theta) = \sum_{n=1}^{\infty} P_{2,n}^{i*}(\theta)\, z^n, \qquad P_2^{i}(z,0) = \sum_{n=1}^{\infty} P_{2,n}^{i}(0)\, z^n \quad 1 \le i \le C$$

$$BR_1^{0**_1}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{1,n}^{0**_1}(\theta,\theta_1)\, z^n, \quad BR_1^{0*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{1,n}^{0}(\theta,0)\, z^n$$

$$BR_1^{**_1}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{1,n}^{**_1}(\theta,\theta_1) z^n, \quad BR_1^{*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{1,n}(\theta,0) z^n$$

$$BR_2^{(i,0)^{**_1}}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,0)^{**_1}}(\theta,\theta_1) z^n, \quad BR_2^{(i,0)^{*}}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,0)}(\theta,0) z^n$$

$$BR_2^{i**_1}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{2,n}^{i**_1}(\theta,\theta_1) z^n, \quad BR_2^{i*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{2,n}^{i}(\theta,0) z^n$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, partial generating functions are derived, through some algebraic manipulations. Equation (8.1.1) implies

$$(\theta - w_X(z))Q^*(z,\theta) = Q(z,0) - \frac{(1-f)p\,V^*(\theta)}{z} \sum_{i=1}^{C} (P_2^i(z,0) + P_2^{(i,0)}(z,0))$$

At $\theta = w_X(z)$,

$$Q(z,0) = \frac{(1-f)p\,V^*(w_X(z))}{z} \sum_{i=1}^{C} (P_2^i(z,0) + P_2^{(i,0)}(z,0)), \text{ hence} \tag{8.1.10}$$

$$Q^*(z,\theta) = \frac{(1-f)p}{z} \frac{(V^*(w_X(z)) - V^*(\theta))}{(\theta - w_X(z))} \sum_{i=1}^{C} (P_2^i(z,0) + P_2^{(i,0)}(z,0)) \tag{8.1.11}$$

The partial probability generating functions of the system size, when the server is in breakdown state during first stage (primary service and feedback services) services are obtained using the equations (8.1.6) and (8.1.7) given by,

$$BR_1^{0*}(z,\theta,0) = a_1^0\, P_1^{0*}(z,\theta)\, R_1^{0*_1}(w_X(z)) \tag{8.1.12}$$

$$BR_1^{o**_1}(z,\theta,\theta_1) = a_1^0\, P_1^{0*}(z,\theta) \frac{[R_1^{0*_1}(w_X(z)) - R_1^{0*_1}(\theta_1)]}{(\theta_1 - w_X(z))} \tag{8.1.12.1}$$

$$BR_1^{*}(z,\theta,0) = a_1\, P_1^{*}(z,\theta)\, R_1^{*_1}(w_X(z)) \tag{8.1.13}$$

$$BR_1^{**_1}(z,\theta,\theta_1) = a_1\, P_1^{*}(z,\theta) \frac{[R_1^{*_1}(w_X(z)) - R_1^{*_1}(\theta_1)]}{(\theta_1 - w_X(z))} \tag{8.1.13.1}$$

The partial probability generating functions of the system size, when the server is in breakdown state during second stage (primary and feedback) services are obtained using the equations (8.1.8) and (8.1.9) and given by,

$$BR_2^{(i,0)^*}(z,\theta,0) = a_2^{(i,0)}\, P_2^{(i,0)^*}(z,\theta)\, R_2^{(i,0)^{*_1}}(w_X(z)) \tag{8.1.14}$$

$$BR_2^{(i,0)^{**_1}}(z,\theta,\theta_1) = a_2^{(i,0)}\, P_2^{(i,0)^*}(z,\theta) \frac{[R_2^{(i,0)^{*_1}}(w_X(z)) - R_2^{(i,0)^{*_1}}(\theta_1)]}{(\theta_1 - w_X(z))} \tag{8.1.15}$$

$$BR_2^{i*}(z,\theta,0) \quad = \quad a_2^i \; P_2^{i*}(z,\theta)\; R_2^{i*_1}(w_X(z)) \qquad (8.1.16)$$

$$BR_2^{i**_1}(z,\theta,\theta_1) \quad = \quad a_2^i \; P_2^{i*}(z,\theta)\frac{[R_2^{i*_1}(w_X(z)) - R_2^{i*_1}(\theta_1)]}{(\theta_1 - w_X(z))} \qquad (8.1.17)$$

Equation (8.1.4) gives the generating functions of the system size when the server is busy with second stage fresh service, at the service completion epoch and at arbitrary epoch :

$$P_2^{(i,0)}(z,0) \quad = \quad r_i \; P_1^0(z,0)\; S_i^*(h_{a_{2\,(i,0)}}(w_X(z))) \qquad (8.1.18)$$

where $h_{a_{2\,(i,0)}}(w_X(z)) = w_X(z) + a_2^{(i,0)}(1 - R_2^{(i,0)*_1}(w_X(z)))$ (8.1.18.1)

$$P_2^{(i,0)*}(z,\theta) \quad = \quad r_i \; P_1^0(z,0)\frac{(S_i^*(h_{a_{2\,(i,0)}}(w_X(z))) - S_i^*(\theta))}{(\theta - h_{a_{2\,(i,0)}}(w_X(z)))} \qquad (8.1.19)$$

Similarly Equation (8.1.5) gives the generating functions of the system size when the server is busy with second stage i[th] feedback service, at the service completion epoch and at arbitrary epoch :

$$P_2^i(z,0) \quad = \quad r_i\, S_i^*(h_{a_2^i}(w_X(z)))\,[f_2(\sum_{=1}^{C}(P_2(z,0) + P_2^{(\;,0)}(z,0)) + P_1(z,0)] \quad (8.1.20)$$

$$P_2^{i\,*}(z,\theta) \quad = \quad \frac{r_i\,(S_i^*(h_{a_2^i}(w_X(z))) - S_i^*(\theta))}{(\theta - h_{a_2^i}(w_X(z)))}[f_2(\sum_{=1}^{C}(P_2(z,0) + P_2^{(\;,0)}(z,0)) + P_1(z,0)]$$

(8.1.21)

where $h_{a_2^i}(w_X(z)) = w_X(z) + a_2^i(1 - R_2^{i*_1}(w_X(z)))$ (8.1.21.1)

The equations (8.1.14) to (8.1.21) are obtained for $1 \le i \le C$.

Adding the equations (8.1.18) and (8.1.20) over i = 1 to C,

$$(\sum_{=1}^{C}(P_2^{(\;,0)}(z,0) + P_2(z,0)) \quad = \quad \frac{k_0(z)P_1^0(z,0) + k(z)P_1(z,0)}{1 - f_2\,k(z)} \qquad (8.1.22)$$

where $k_0(z) = \sum_{i=1}^{C} r_i\, S_i^*(h_{a_2^{(i,0)}}(w_X(z)))$ (8.1.23)

and $k(z) = \sum_{i=1}^{C} r_i\, S_i^*(h_{a_2^i}(w_X(z)))$ (8.1.23.1)

The PGF corresponding to the state, when the server is busy in first stage feedback service, is obtained by using the equation (8.1.3) as,

$$(\theta - h_{a_1}(w_X(z)))\,P_1^*(z,\theta) = P_1(z,0) - f_1\; S^*(\theta)\,(\sum_{i=1}^{C}(P_2^i(z,0) + P_2^{(i,0)}(z,0))\,(8.1.24)$$

Using the equation (8.1.22) in (8.1.24) and simplifying, we have at $\theta = h_{a_1}(w_X(z))$,

$$P_1(z, 0) = \frac{f_1\, S^*(h_{a_1}(w_X(z)))\, k_0(z)\, P_1^0(z,0)}{1 - k(z)\,(f_2 + f_1\, S^*(\, h_{a_1}(w_X(z))))} \qquad (8.1.25)$$

Substituting the value of $P_1(z, 0)$ in (8.1.24) and simplifying we get

$$P_1^*(z,\theta) = \frac{f_1\, k_0(z)\, P_1^0(z,0)}{1 - k(z)\,(f_2 + f_1\, S^*(h_{a_1}(w_X(z))))}\; \frac{(S^*(h_{a_1}(w_X(z))) - S^*(\theta))}{(\theta - h_{a_1}(w_X(z)))} \qquad (8.1.26)$$

Using the equations (8.1.22) and (8.1.25) in (8.1.11),

$$Q^*(z,\theta) = \frac{(1-f)\,p}{z}\; \frac{k_0(z)\, P_1^0(z,0)}{[1 - k(z)\,(f_2 + f_1\, S^*(h_{a_1}(w_X(z))))]}\; \frac{(V^*(w_X(z))) - V^*(\theta))}{(\theta - w_X(z))}$$

(8.1.26.1)

Next to calculate the PGF corresponding to the state when the server is busy in first stage fresh service, the equation (8.1.2) is used and it is found that,

$$\theta\, P_1^{0^*}(z,\theta) - P_1^0(z,0)$$

$$= (\lambda + a_1^0)\, P_1^{0*}(z,\theta) - PI\,\lambda\, X(z)\, S^*(\theta) - BR_1^{0^*}(z,\theta,0)$$

$$- (1 - f)\,(1 - p)\, \frac{S^*(\theta)}{z} \sum_{i=1}^{C} [P_2^{(i,0)}(z,0) - P_{2,1}^{(i,0)}(0)\, z + P_2^i(z,0) - P_{2,1}^i(z,0)\, z]$$

$$- \lambda\, X(z)\, P_1^{0^*}(z,\theta) - S^*(\theta)\,[Q(z, 0) - Q_0(0)] \qquad (8.1.27)$$

The equations (8.1.12), (8.1.22), (8.1.0), (8.1.10) together with (8.1.27) give

$$(\theta - h_{a_1^0}(w_X(z)))\, P_1^{0^*}(z,\theta)$$

$$= P_1^0(z,0) \left(\frac{\begin{matrix}[z[1 - k(z)\,(f_2 + f_1\, S^*(h_{a_1}(w_X(z))))] \\ - S^*(\theta)\,(1-f)\,(1 - p + p\, V^*(w_X(z)))\, k_0(z)]\end{matrix}}{z\,(1 - k(z)\,[f_2 + f_1\, S^*(h_{a_1}(w_X(z)))])} \right) + S^*(\theta)\, PI w_X(z) \qquad (8.1.28)$$

At $\theta = h_{a_1}^0(w_X(z))$

$$P_1^0(z,0) = \frac{-w_X(z)\, PI\, S^*(h_{a_1^0}(w_X(z)))\, z\, S_F(w_X(z))}{D_{1,IF}^{BV}(z)} \qquad (8.1.29)$$

where $S_F^*(w_X(z)) = [1 - k(z)\,(f_2 + f_1\, S^*(h_{a_1}(w_X(z))))]$

and $D^{BV}_{1,IF}(z) = zS^*_F(w_X(z)) - S^*(h_{a_1^0}(w_X(z)))\,(1-f)\,(1-p+pV^*(w_X(z)))\,k_0(z)$

(8.1.30)

Substituting (8.1.29) in (8.1.28) and simplifying we get

$$P_1^{0^*}(z,\theta) = \frac{w_X(z)\,PI\,z\,S^*_F(w_X(z))[S^*(\theta) - S^*(h_{a_1^0}(w_X(z)))]}{D^{BV}_{1,IF}(z)(\theta - h_{a_1^0}(w_X(z)))} \quad (8.1.31)$$

Thus the partial generating functions corresponding to different states at arbitrary epochs are calculated using the respective equations and are given by :

$$P_1^{0^*}(z,0) = \frac{z\,PI\,w_X(z)\,S^*_F(w_X(z))[S^*(h_{a_1^0}(w_X(z))) - 1]}{D^{BV}_{1,IF}(z)\;h_{a_1^0}(w_X(z))} \quad (8.1.32.1)$$

$$P_1^{*}(z,0) = \frac{z\,PI\,w_X(z)\,f_1\,k_0(z)\,S^*(h_{a_1^0}(w_X(z)))(S^*(h_{a_1}(w_X(z))) - 1)}{D^{BV}_{1,IF}(z)\;h_{a_1}(w_X(z))} \quad (8.1.32.2)$$

For $1 \le i \le C$,

$$P_2^{(i,0)^*}(z,0) = \frac{z\,PI\,(w_X(z))\,r_i\,S^*(h_{a_1^0}(w_X(z)))\,S^*_F(w_X(z))\,[S^*_i(h_{a_2^{(i,0)}}(w_X(z))) - 1]}{D^{BV}_{1,IF}(z)\;h_{a_2^{(i,0)}}(w_X(z))}$$

(8.1.32.3)

$$P_2^{i^*}(z,0) = \frac{z\,PI\,w_X(z)\,r_i k_0(z) S^*(h_{a_1^0}(w_X(z)))\,[f_2 + f_1 S^*(h_{a_1}(w_X(z)))]\,[S^*_i(h_{a_2^i}(w_X(z))) - 1]}{D^{BV}_{1,IF}(z)\;h_{a_2^i}(w_X(z))}$$

(8.1.32.4)

$$Q^*(z,0) = \frac{PI\,(1-f)\,p\,k_0(z)\,S^*(h_{a_1^0}(w_X(z)))(V^*(w_X(z)) - 1)}{D^{BV}_{1,IF}(z)} \quad (8.1.32.5)$$

$$BR_1^{0^{**_1}}(z,0,0) = \frac{a_1^0\,P_1^{0^*}(z,0)(1 - R_1^{0^{*_1}}(w_X(z)))}{w_X(z)} \quad (8.1.32.6)$$

$$BR_1^{**_1}(z,0,0) = \frac{a_1\,P_1^{*}(z,0)(1 - R_1^{*_1}(w_X(z)))}{w_X(z)} \quad (8.1.32.7)$$

$$BR_2^{(i,0)^{**_1}}(z,0,0) = \frac{a_2^{(i,0)}\,P_2^{(i,0)^*}(z,0)(1 - R_2^{(i,0)^{*_1}}(w_X(z)))}{w_X(z)} \quad (8.1.32.8)$$

$$BR_2^{i^{**_1}}(z,0,0) = \frac{a_2^i\,P_2^{i^*}(z,0)(1 - R_2^{i^{*_1}}(w_X(z)))}{w_X(z)} \quad (8.1.32.9)$$

where $D^{BV}_{1,IF}(z)$, k(z), $k_0(z)$ are given by the equations (8.1.30), (8.1.23.1) and (8.1.23) respectively.

To derive the total PGF of the system size distribution, the following generating functions are considered.

$P_{Idle}(z)$ = Probability generating function of the system size when the server is idle in idle state

$$= PI + Q^*(z,0)$$

$$= \frac{PI}{D^{BV}_{1,IF}(z)}\left[z\, S^*_F(w_X(z)) - S^*(h_{a_1^0}(w_X(z)))(1-f)\,k_0(z)\right] \qquad (8.1.33)$$

$P_{Comp}(z)$ = The PGF of the system size when server is busy or in breakdown state

$$= P_1^{0*}(z,0) + BR_1^{0**_1}(z,0,0) + P_1^*(z,0) + BR_1^{**_1}(z,0,0)$$

$$+\sum_{i=1}^{C}\left[P_2^{(i,0)*}(z,0) + BR_2^{(i,0)**_1}(z,0,0) + P_2^{i*}(z,0) + BR_2^{i**_1}(z,0,0)\right]$$

$$= \frac{PI\, z}{D^{BV}_{1,IF}(z)}\left[S^*(h_{a_1^0}(w_X(z)))\,k_0(z)\,(1-f) - S^*_F(w_X(z))\right] \qquad (8.1.34)$$

Thus the total PGF of the system size distribution is given by

$$P^{BV}_{IF}(z) = P_{Idle}(z) + P_{Comp}(z)$$

$$= \frac{PI\,(z-1)(1-f)\,k_0(z)\,S^*(h_{a_1^0}(w_X(z)))}{D^{BV}_{1,IF}(z)} \qquad (8.1.35)$$

where $D^{BV}_{1,IF}(z)$ is given by the equation (8.1.30). PI can be calculated by using the normalizing condition $P^{BV}_{IF}(1)$ = 1 and found to be PI = 1 – $\rho^{BV}_{1,IF}$ where (8.1.36)

$$\rho^{BV}_{1,IF} = \lambda E(X)\left[pE(V) + E(H_1^0) + \frac{f}{1-f}\sum_{i=1}^{C} r_i\, E(H_2^i) + \sum_{i=1}^{C} r_i\, E(H_2^{(i,0)}) + \frac{f_1}{1-f}E(H_1)\right] \qquad (8.1.37)$$

The measures E(H) s' are obtained from the LST of random variables,

$$H_1^*(z) = S^*(h_{a_1}(w_X(z))), \quad H_1^{0*}(z) = S^*(h_{a_1^0}(w_X(z))),$$

$$H_2^{i*}(z) = S_i^*(h_{a_2^i}(w_X(z))), \quad H_2^{i,0*}(z) = S_i^*(h_{a_2^{(i,0)}}(w_X(z))) \text{ for } 1 \le i \le C,$$

and are given by :

$$E(H_1) = E(S)\,(1 + a_1\, E(R_1)) \qquad (8.1.37.1)$$

$$E(H_1^0) = E(S)\,(1 + a_1^0\ E(R_1^0)) \tag{8.1.37.2}$$

$$E(H_2^{(i,0)}) = E(S_i)\,(1 + a_2^{(i,0)}\,E(R_2^{(i,0)})) \tag{8.1.37.3}$$

$$E(H_2^i) = E(S_i)\,(1 + a_2^i\ E(R_2^i)) \tag{8.1.37.4}$$

$$E(H_1^2) = E(S)\,a_1\ E(R_1^2) + E(S^2)\,(1 + a_1\,E(R_1))^2 \tag{8.1.37.5}$$

$$E((H_1^0)^2) = E(S)\ a_1^0\ E((R_1^0)^2) + E(S^2)\,(1 + a_1^0\,E(R_1^0))^2 \tag{8.1.37.6}$$

$$E((H_2^{(i,0)})^2) = E(S_i)\qquad a_2^{(i,0)}\,E((R_2^{(i,0)})^2) + \qquad E(S_i^2)\,(1 + a_2^{(i,0)}\,E(R_2^{(i,0)}))^2$$

(8.1.37.7)

$$E((H_2^i)^2) = E(S_i)\ a_2^i\ E((R_2^i)^2) + E(S_i^2)\ (1 + a_2^i\,E(R_2^i))^2 \tag{8.1.37.8}$$

Hence

$$P_{IF}^{BV}(z) = \frac{(1-\rho_{1,IF}^{BV})\,(z-1)\,(1-f)\,k_0(z)\,S^*(h_{a_1^0}(w_X(z)))}{D_{1,IF}^{BV}(z)} \tag{8.1.38}$$

8.1.1.4 Decomposition Property

Using equation (8.1.33), equation (8.1.38) can be re-written as

$$P_{IF}^{BV}(z) = \frac{(z-1)\,(1-\rho_{1,IF})\ (1-f)\,k_0(z)\,S^*(h_{a_1^0}(w_X(z)))}{[z\,S_F^*(w_X(z)) - S^*(h_{a_1^0}(w_X(z)))\,(1-f)\,k_0]}\left(\frac{P_{Idle}(z)}{P_{Idle}(1)}\right) \tag{8.1.39}$$

where

$$\rho_{1,IF} = \lambda\,E(X)\,[E(H_1^0) + \sum_{i=1}^{C} r_i\,E(H_2^{(i,0)}) + \frac{f}{1-f}\ \sum_{i=1}^{C} r_i\,E(H_2^i) + \frac{f_1}{1-f}\ E(H_1)] \tag{8.1.40}$$

$$= \rho_{1,IF}^{BV} - p\,\lambda\,E(X)\,E(V)$$

and $E(H_1^0)$, $E(H_1)$, $E(H_2^i)$, $E(H_2^{(i,0)})$ are given by the equations (8.1.38.1) to (8.1.38.4).

Under the steady state condition $\rho_{1,IF}^{BV} < 1$, the PGF of the stationary system size of the queueing model under consideration is the product of the PGF of the system size of $M^X/G,G_{i(1 \le i \le C)}/1$ queueing system with infinite feedback and service interruption (without vacation) and the distribution of the conditional system size during the idle period given that the server is idle.

8.1.1.5 Queue Size Distribution at Departure Epoch

If π_n^+ denotes the probability that there are n customers in the system at departure epoch, then $\pi_n^+ = D_1 [(1-f) \sum_{i=1}^{C} P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^{i}(0)]$, with the normalizing constant D_1.

The PGF $\pi^+(z)$ of the queue size distribution $\{\pi_n^+ ; n \geq 0\}$ at departure epoch is given by

$$\pi^+(z) = \sum_{n=0}^{\infty} \pi_n^+ z^n = \frac{D_1}{z}(1-f) \sum_{i=1}^{C} (P_2^{(i,0)}(z,0) + P_2^{i}(z,0)) = \frac{D_1}{z-1} \lambda (X(z)-1)\, P_{IF}^{BV}(z)$$

(from (8.1.20) and (8.1.18)

Evaluating D_1 using normalizing condition, $\pi^+(z) = \frac{(X(z)-1)}{E(X)(z-1)} P_{IF}^{BV}(z)$

8.1.1.6 Performance Measures

(i) The probability that the server is on vacation state (P_V) is

$$P_V = \lim_{z \to 1} Q^*(z,0) = p\, \lambda\, E(X)\, E(V)$$

(ii) The probability that the server is busy is

$$P_{busy} = P_1^0 + P_1 + \sum_{i=1}^{C} (P_2^{(i,0)} + P_2^{i})$$

$$= \lim_{z \to 1} \left[P_1^{0^*}(z,0) + P_1^*(z,0) + \sum_{i=1}^{C} P_2^{(i,0)^*}(z,0) + \sum_{i=1}^{C} P_2^{i^*}(z,0) \right]$$

$$= \frac{\lambda E(X)}{1-f} \left[E(S)(1-f_2) + \sum_{i=1}^{C} r_i E(S_i) \right]$$

(iii) The probability that the server is in breakdown state (P_{br}) is obtained by,

$$P_{br} = \lim_{z \to 1} \left[P_{br_1^0} + P_{br_1} + \sum_{i=1}^{C} (P_{br_2^{(i,0)}} + P_{br_2^{i}}) \right]$$

$$= \lambda\, E(X) \left\{ E(S)\,[a_1^0 E(R_1^0) + \frac{f_1}{1-f} a_1 E(R_1)] \right.$$

$$\left. + \sum_{i=1}^{C} r_i E(S_i) \left[a_2^{(i,0)} E(R_2^{(i,0)}) + \frac{f}{1-f} a_2^{i} E(R_2^{i}) \right] \right\}$$

(iv) The expected system size for the model is given by

$$L_{IF}^{BV} = \left[\frac{d}{dz}(P_{IF}^{BV}(z)\right]_{z=1} = \lambda E(X)[E(H_1^0) + \sum_{i=1}^{C} r_i\, E(H_2^{(i,0)})] + \frac{(-D_{1,IF}^{BV\,''}(1))}{2\,(1-f)\,(1-\rho_{1,IF}^{BV})} \quad (8.1.41)$$

where $\rho_{1,IF}^{BV}$ is given by the equation (8.1.37) and

$$\begin{aligned}(-D_{1,IF}^{BV\,''}(1)) &= \lambda\, E(X(X-1)\,[f \sum_{i=1}^{C} r_i\, E(H_2^i) + f_1\, E(H_1) + (1-f)\,[E(H_1^0) + pE(V) + \sum_{i=1}^{C} r_i\, E(H_2^{(i,0)})]] \\ &+ (\lambda E(X))^2\{f\sum_{i=1}^{C} r_i\, E((H_2^i)^2) + f_1 E(H_1^2) + (1-f)[E((H_1^0)^2) + pE(V^2) + \sum_{i=1}^{C} r_i\, E((H_2^{(i,0)})^2)] \\ &+ 2[\sum_{i=1}^{C} r_i\, E(H_2^i)\, f_1\, E(H_1) + (1-f)[pE(V)E(H_1^0) + (pE(V)+E(H_1^0))\sum_{i=1}^{C} r_i\, E((H_2^{(i,0)}))]]\} \\ &+ 2\,\lambda\, E(X)\,[f \sum_{i=1}^{C} r_i\, E(H_2^i) + f_1\, E(H_1)]\end{aligned}$$

where $E(H_1)$, $E(H_1^0)$, $E(H_2^i)$, $E(H_2^{(i,0)})$, $E(H_1^2)$, $E((H_1^0)^2)$, $E((H_2^i)^2)$, $E((H_2^{(i,0)})^2)$ are given by the equations (8.1.37.1) to (8.1.37.8).

8.1.2 PARTICULAR CASES

The model of the present section (8.1) considers the case in which, the customers when discontented with their services may either demand re-service from phase 1 followed by phase 2 with probability (f_1) (or) demand re-service of phase 2 type alone with probability f_2 (or) leave the system with probability $1 - (f_1 + f_2) = 1 - f$, without demanding re-services.

Case 1:

The total PGFs of section 8.1 is compared with that of section 7.1, in case 1.

If $f_1 = 0$ (i.e., $f_2 = f$), then the PGF of the system size of the model, in which the feedback customers demand re-services only of phase 2 type is obtained from equation (8.1.38) and given by

$$P_{1,IF}^{BV}(z) = \frac{(1-\rho_{1,IF}^{BV})\,(z-1(\,(1-f)\,S^*(h_{a_1^0}(w_X(z)))\dfrac{k_0(z)}{1-f\,k(z)}}{z - S^*(h_{a_1^0}(w_X(z)))\,(1-f)\,(1-p+p\,V^*(w_X(z)))\dfrac{k_0(z)}{1-f\,k(z)}} \quad (8.1.42.1)$$

where,

$$\rho_{1,IF}^{BV} = \lambda\, E(X) \left(\frac{f}{1-f} \sum_{i=1}^{C} r_i\, E(H_2^i) + \sum_{i=1}^{C} r_i\, E(H_2^{i,0}) + E(H_1^0) + p\, E(V) \right) \quad (8.1.42.2)$$

The model discussed in section 7.1 also considers a similar case, allowing finitely many (m) such feedbacks. It is noted that the expressions for $k_S(z)$ and $k(z)$ of equations (7.1.22) and (8.1.23.1) coincide if $f_S = f$ for $1 \le s \le m-1$. Thus as $m \to \infty$, the PGF of system size of the model of section 7.1 given in (7.1.40) coincides with the PGF given by (8.1.42.1).

Case 2 :

If $f_2 = 0$ then the PGF of the system size of the model in which all the feedback customers repeat services from phase 1 followed by phase 2 is obtained from equation (8.1.38) and given by

$$P_{1,IF}^{BV}(z) = \frac{(1-\rho_{1,IF}^{BV})(z-1)(1-f)\,k_0(z)\,S^*(h_{a_1^0}(w_X(z)))}{z\,(1-k(z)+fS^*(h_{a_1}(w_X(z)))) - (1-f)\,S^*(h_{a_1^0}(w_X(z)))\,k_0(z)\,(1-p+pV^*(w_X(z)))} \quad (8.1.43.1)$$

$$\rho_{1,IF}^{BV} = \lambda\, E(X) \left(\frac{f}{1-f} \sum_{i=1}^{C} r_i\, E(H_2^i) + \sum_{i=1}^{C} r_i\, E(H_2^{i,0}) + \frac{f}{1-f} E(H_1) + E(H_1^0) + p\, E(V) \right)$$

If we assume that, the probability with which the server takes vacation after the completion of each service (p) is zero, the arrivals follow simple Poisson process (E(X) = 0) and the server never breaks down ($a_i = 0$), then equations (8.1.23) and (8.1.23.1) give :

$$k_0(z) = k(z)\,; \quad S^*(h_{a_1^0}(w_X(z))) = S^*(h_a(w_X(z))) = S^*(\lambda(1-z))$$

and
$$P(z) = \frac{(1-\rho)(z-1)(1-f)\,A(z)}{z\,(1-f\,A(z)) - (1-f)\,A(z)} \quad (8.1.44)$$

where
$$\rho = \frac{\lambda}{1-f}\left(E(S) + \sum_{i=1}^{C} r_i\, E(S_i)\right) \text{ and } A(z) = k(z)\; S^*(\lambda(1-z))$$

it is verified that under the condition $m \to \infty$, (8.1.44) coincides with the PGF of the system size of the two phase service reliable M/G/1 queueing model (with finite number of immediate feedback) analysed by Kalidass and Kasturi (2013).

SECTION : 8.2

M^X/G/1 QUEUE WITH INFINITE NUMBER OF FEEDBACKS AND REPETITION OF INTERRUPTED SERVICE

8.2.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

8.2.1.1 Model Description

The model of the present Section 8.2 differs from that of 8.1, in the behaviour of the customers during breakdown period. Whenever the service is interrupted due to breakdown, it was assumed in Section 8.1, that the service interrupted customers, resume their service from where they got interrupted. In the present section it is considered that the service gets started from the very beginning of the service independently of the earlier amount of service. Also in Section 8.1, the option of feedback of each customer is considered either from first phase service or from second phase services. But in the present section (Section 8.2) it is assumed that the customer can repeat only the optional services of second stage during feedback services.

The other assumptions, regarding batch arrivals, infinite number of feedbacks, occurrence of the breakdowns and the Bernoulli schedule vacation that the server takes between services are as similar as in Section 8.1. Thus the states of the system is Y(t) = 0, 1, 2, 3, 4, 5, 6 and 7 represents that the server is idle in the system, busy in first phase, busy in second service, in breakdown state during the first phase service, breakdown state during the second phase fresh service, breakdown state during the second feedback service and the server is in vacation respectively.

The notations of the Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the R.Vs are similar to Section 8.1, Table 8.1.1.

The joint probabilities corresponding to the breakdown states are explained below. The other system size probabilities are as given in section 8.1.

$BR_{1,n}(y, t)\,dt$ = Pr $\{N_S(t) = n,\ y < R_1^o(t) \le y + dt,\ Y(t) = 3\}$, $n \ge 1$, the service interrupted customer repeats a new first service and the repair time lies in the interval (y + y+dt).

$BR_{2,n}^{(i,0)}(y, t)\,dt$ = Pr $\{N_S(t) = n,\ y < (R_2^{(i,0)})^o(t) \le y + dt,\ Y(t) = 5\}$, $n \ge 1$, $1 \le i \le C$, the customer whose service is interrupted repeats a new second i^{th} primary service.

$BR_{2,n}^{i}(y, t)\,dt$ = Pr $\{N_S(t) = n,\ y < (R_2^{i})^o(t) \le y + dt,\ Y(t) = 6\}$, $n \ge 1$, $1 \le i \le C$, the customer whose service is interrupted repeats a new second i^{th} feedback service.

Using supplementary variable technique the steady state system equations under the steady-state condition are analysed and the PGF of the system size is obtained. Since the arguments are similar as in Section (8.1), only the LST of the steady state equations are listed below :

$$\theta\, Q_n^*(\theta) - Q_n(0) = \lambda\, Q_n^*(\theta) - \lambda\,(1-\delta_{0,n}) \sum_{k=1}^{n} Q_{n-k}^*(\theta)\, g_k - \sum_{i=1}^{C} (P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^{i}(0))(1-f)\, p\, V^*(\theta), \qquad n \ge 0 \qquad (8.2.1)$$

$$\theta\, P_{1,n}^*(\theta) - P_{1,n}(0) = (\lambda + a_1)\, P_{1,n}^*(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{1,n-k}^*(\theta)\, g_k - \sum_{i=1}^{C} (P_{2,n+1}^{(i,0)}(0) + P_{2,n+1}^{i}(0))\,(1-f)\,(1-p)\, S^*(\theta) - PI\,\lambda\, g_n\, S^*(\theta) - BR_{1,n}(0)\, S^*(\theta) - Q_n(0)\, S^*(\theta), \qquad n \ge 1 \qquad (8.2.2)$$

$$\theta\, P_{2,n}^{(i,0)^*}(\theta) - P_{2,n}^{(i,0)}(0) = (\lambda + a_2^{(i,0)})\, P_{2,n}^{(i,0)^*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)*}(\theta)\, g_k - P_{1,n}(0)\, r_i\, S_i^*(\theta) - BR_{2,n}^{(i,0)}(0)\, S_i^*(\theta) \quad n \ge 1 \qquad (8.2.3)$$

$$\theta\, P_{2,n}^{i*}(\theta) - P_{2,n}^{i}(0) = (\lambda + a_2^{i})\, P_{2,n}^{i*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} P_{2,n-k}^{i*}(\theta)\, g_k - \sum_{=1}^{C} (P_{2,n}(0) + P_{2,n}^{(\ ,0)}(0))\, f\, r_i\, S_i^*(\theta) - BR_{2,n}^{i}(0) \qquad n \ge 1 \qquad (8.2.4)$$

$$\theta\, BR_{1,n}^*(\theta) - BR_{1,n}(0) = \lambda\, BR_{1,n}^*(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR_{1,n-k}^*(\theta)\, g_k - \left(\int^{\infty} P_{1,n}(w)\, dw\right) a_1\, R_1^*(\theta), \qquad n \ge 1 \qquad (8.2.5)$$

$$\text{i.e., } \theta\, BR^{*}_{1,n}(\theta) - BR_{1,n}(0) = \lambda\, BR^{*}_{1,n}(\theta) - (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR^{*}_{1,n-k}(\theta)\, g_k - P^{*}_{1,n}(0)\, a_1\, R^{*}_1(\theta), \quad n \ge 1 \qquad (8.2.5.1)$$

$$\theta\, BR^{(i,0)^*}_{2,n}(\theta) - BR^{(i,0)}_{2,n}(0) = \lambda\, BR^{(i,0)^*}_{2,n}(\theta) - (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR^{(i,0)}_{2,n-k}{}^{*}(\theta)\, g_k - \left(\int_0^\infty P^{(i,0)}_{2,n}(w)\, dw\right) a^{(i,0)}_2\, R^{(i,0)^*}_2(\theta), \; n \ge 1 \qquad (8.2.6)$$

$$\text{i.e., } \theta\, BR^{(i,0)^*}_{2,n}(\theta) - BR^{(i,0)}_{2,n}(0) = \lambda\, BR^{(i,0)^*}_{2,n}(\theta) - (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR^{(i,0)}_{2,n-k}{}^{*}(\theta)\, g_k - P^{(i,0)^*}_2(0)\; a^{(i,0)}_2\, R^{(i,0)^*}_2(\theta), \quad n \ge 1 \qquad (8.2.6.1)$$

$$\theta\, BR^{i*}_{2,n}(\theta) - BR^{i}_{2,n}(0) = \lambda\, BR^{i*}_{2,n}(\theta) - (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR^{i*}_{2,n-k}(\theta)\, g_k - \left(\int_0^\infty P^{i}_{2,n}(w)\, dw\right) a^{i}_2\, R^{i}_2{}^{*}(\theta), \quad n \ge 1 \qquad (8.2.7)$$

$$\theta\, BR^{i*}_{2,n}(\theta) - BR^{i}_{2,n}(0) = \lambda\, BR^{i*}_{2,n}(\theta) - (1-\delta_{1,n})\lambda \sum_{k=1}^{n-1} BR^{i*}_{2,n-k}(\theta)\, g_k - P^{i}_2{}^{*}(0)\, a^{i}_2\, R^{i}_2{}^{*}(\theta), \quad n \ge 1 \qquad (8.2.7.1)$$

$$\lambda\, PI = Q_0(0) + \sum_{i=1}^{C} \left(P^{(i,0)}_{2,1}(0) + P^{i}_{2,1}(0)\right)(1-f)\,(1-p) \qquad (8.2.8)$$

Proceeding as in Section 8.1, the partial generating functions corresponding to different states at arbitrary epochs are calculated using the respective equations and are given by

$$P^{*}_1(z,0) = \frac{PI\, z\,(S^{*}(g_{a_1}(w_X(z))) - 1)\, w_X(z)}{D^{BV}_{2,if}(z)} \qquad (8.2.9)$$

where $D^{BV}_{2,if}(z)$

$$= z\,[h_{a_1}(w_X(z)) + S^{*}(g_{a_1}(w_X(z)))\, a_1\, R^{*}_1(w_X(z))] - S^{*}(g_{a_1}(w_X(z)))\, g_{a_1}(w_X(z))\, \left[(1-f)\,(1-p+p\, V^{*}(w_X(z)))\, \frac{K_0(z)}{1-f\,K(z)}\right]$$

$$g_{a_1}(w_X(z)) = a_1 + w_X(z)\,;$$

$$h_{a_1}(w_X(z)) = g_{a_1}(w_X(z)) - a_1\, R^{*}_1(w_X(z))$$

For $1 \le i \le C$

$$g_{a^i_2}(w_X(z)) = a^{i}_2 + w_X(z)$$

$$h_{a_2^i}(w_X(z)) = g_{a_2^i}(w_X(z)) - a_2^i R_2^*(w_X(z)),$$

$$g_{a_2^{(i,0)}}(w_X(z)) = a_2^{(i,0)} + w_X(z)$$

$$h_{a_2^{(i,0)}}(w_X(z)) = g_{a_2^{(i,0)}}(w_X(z)) - a_2^{(i,0)} R_2^{(i,0)^*}(w_X(z))$$

$$K_0(z) = \sum_{i=1}^{C} \frac{r_i S_i^*(g_{a_2^{(i,0)}}(w_X(z)))\, g_{a_2^{(i,0)}}(w_X(z))}{h_{a_2^{(i,0)}}(w_X(z)) + S_i^*(g_{a_2^{(i,0)}}(w_X(z)))\, a_2^{(i,0)} R_2^{(i,0)^*}(w_X(z))}$$

$$K(z) = \sum_{i=1}^{C} \frac{r_i S_i^*(g_{a_2^i}(w_X(z)))\, g_{a_2^i}(w_X(z))}{h_{a_2^i}(w_X(z)) + S_i^*(g_{a_2^i}(w_X(z)))\, a_2^i R_2^{i^*}(w_X(z))}$$

$$P_2^{(i,0)^*}(z,0) = \frac{\mathrm{PI}\, z\, r_i\, (S_i^*(g_{a_2^{(i,0)}}(w_X(z))) - 1)\, S^*(g_{a_1}(w_X(z)))\, g_{a_1}(w_X(z))}{D_{2,if}^{BV}(z)\ [h_{a_2^{(i,0)}}(w_X(z)) + S_i^*(g_{a_2^{(i,0)}}(w_X(z)))\, a_2^{(i,0)} R_2^{(i,0)^*}(w_X(z))]} \quad (8.2.10)$$

$$P_2^{i^*}(z,0) = \frac{\mathrm{PI}\, z\, r_i\, (S_i^*(g_{a_2^i}(w_X(z))) - 1)\, S^*(g_{a_1}(w_X(z)))\, g_{a_1}(w_X(z))\, w_X(z)}{(h_{a_2^i}(w_X(z)) + S_i^*(g_{a_2^i}(w_X(z)))\, a_2^i R_2^{i^*}(w_X(z)))\, D_{2,if}^{BV}(z)} \left(\frac{f\, K_0(z)}{1 - f\, K(z)}\right) \quad (8.2.11)$$

$$Q^*(z,0) = \frac{\mathrm{PI}\, p\ (1-f)(V^*(w_X(z)) - 1)\, S^*(g_{a_1}(w_X(z)))\, g_{a_1}(w_X(z))}{D_{2,if}^{BV}(z)} \left(\frac{K_0(z)}{1 - f\, K(z)}\right) \quad (8.2.12)$$

$$BR_1^*(z,0) = \frac{a_1 P_1^*(z,0)(1 - R_1^*(w_X(z)))}{w_X(z)} \quad (8.2.13)$$

$$BR_2^{(i,0)^*}(z,0) = \frac{a_2^{(i,0)} P_2^{(i,0)^*}(z,0)(1 - R_2^{(i,0)^*}(w_X(z)))}{w_X(z)} \quad (8.2.14)$$

$$BR_2^{i^*}(z,0) = \frac{a_2^i P_2^{i^*}(z,0)(1 - R_2^{i^*}(w_X(z)))}{w_X(z)} \quad (8.2.15)$$

The total PGF of the system size distribution is obtained by adding equations (8.2.9) to (8.2.15) and given by

$$P_{if}^{BV}(z) = \frac{\mathrm{PI}\, g_{a_1}(w_X(z))\, S^*(g_{a_1}(w_X(z)))(z-1)(1-f)K_0(z)}{D_{2,if}^{BV}(z)(1 - f\, K(z))} \quad (8.2.16)$$

The constant PI can be calculated by using the normalizing condition

$P_{if}^{BV}(1) = 1$ and found to be $PI = 1 - \rho_{2,if}^{BV}$

where

$$\rho_{2,if}^{BV} = \lambda\, E(X)\, [\frac{(1-S_1^*(a_1))}{a_1 S_1^*(a_1)}(1 + a_1 E(R_1)) + p\, E(V) + \sum_{i=1}^{C} \frac{r_i\,(1+a_2^{(i,0)} E(R_2^{(i,0)}))(1-S_i^*(a_2^{(i,0)}))}{S_i^*(a_2^{(i,0)})\, a_2^{(i,0)}} + \sum_{i=1}^{C} [(\frac{f}{1-f})\frac{r_i\,(1+a_2^i E(R_2^i))(1-S_i^*(a_2^i))}{a_2^i\, S_i^*(a_2^i)}]] \tag{8.2.17}$$

Hence

$$P_{if}^{BV}(z) = \frac{(1-\rho_{2,if}^{BV})\, g_{a_1}(w_X(z))\, S^*(g_{a_1}(w_X(z)))(z-1)(1-f)K_0(z)}{D_{2,if}^{BV}(z)(1-f\,K(z))} \tag{8.2.18}$$

8.2.2 PARTICULAR CASES

If the service times S and S_i $(1 \le i \le C)$ follow exponential distribution with parameters μ and μ_i respectively, then the steady state results of the model in which the service interrupted customers resume services (section 8.1) coincide with that of the model in which the customers repeat service from the beginning (section 8.2) as soon as the server is fixed. The following observations confirm the result :

Let $S^*(\theta) = \frac{\mu}{\mu+\theta}$ and $S_i^*(\theta) = \frac{\mu_i}{\mu_i+\theta}$. $(1 \le i \le C)$

This implies,

$$K(z) = k(z) = \sum_{i=1}^{C} r_i \frac{\mu_i}{\mu_i + h_{a_2^i}(w_X(z))} \text{ and } K_0(z) = k_0(z)$$

Also

$$\rho_{2,if}^{BV} = \rho_{1,IF}^{BV} = \lambda\, E(X)\, \{\frac{f}{1-f}\sum_{i=1}^{C} r_i \frac{1}{\mu_i}(1+a_2^i E(R_2^i)) + \sum_{i=1}^{C} r_i \frac{1}{\mu_i}(1 + a_2^{(i,0)} E(R_2^{(i,0)})) + \frac{1}{\mu}(1 + a_1^0 E(R_1^0)) + p\, E(V)\}$$

By calculation, it is found that,

$$\frac{g_{a_1}(w_X(z))\, S^*(g_{a_1}(w_X(z)))}{D_{2,if}^{BV}(z)(1-f\,K(z))} = \frac{S^*(h_{a_1}(w_X(z)))}{D_{1,IF}^{BV}(z)} = \frac{\mu}{(\mu + h_{a_1}(w_X(z)))\; D_{1,IF}^{BV}(z)}$$

Thus the total PGFs given in equations (8.1.38) and (8.2.18) coincide.

8.3 NUMERICAL ANALYSIS

In this section numerical results are obtained to study the effects of (i) the probability that the server chooses feedback service from first phase (f_1) or from second phase (f_2), (ii) the breakdown rates, (iii) mean repair time

on the expected system size for the model of section 8.1. The different distributions assumed are presented in the following table.

Random variables (Y)		**Distribution F(Y)**	**Mean E(Y)**	**Second order moments $E(Y^2)$**
FPS (S)		Erlang-3 type	$\frac{1}{4}$	$\frac{1}{12}$
SPS (S_1, S_2, S_3)		(Deterministic, Exponential, Gamma-3)	$(\frac{1}{3}, \frac{1}{6}, \frac{3}{5})$	$(\frac{1}{3^2}, \frac{2}{6^2}, \frac{12}{5^2})$
Vacation (V)		Gamma-2 type	$\frac{2}{5}$	$\frac{6}{25}$
Repair time in first phase	Primary (R_1^0)	Erlang-2 type	$\frac{1}{4}$	$\frac{3}{32}$
	Feedback (R_1)	Exponential	$\frac{1}{4}$	$\frac{1}{8}$
Repair time in second phase, i = 1 to 3	Primary $(R_2^{(i,0)})$	Exponential	$\frac{1}{4}$	$\frac{1}{8}$
	Feedback (R_2^i)	Exponential	2	8
Batch size (X)		Geometric (Geo(p_1))	$\frac{1}{1-p_1}$, $p_1 = 0.3$	$\frac{p_1+1}{(1-p_1)^2}$

The parametric values (other than in the table) chosen to construct the Tables 8.1 to 8.3 are listed below.

$(a_1^0, a_1, r_1, r_2, r_3, p, c, \lambda)$ = (3, 0.03, 0.5, 0.4, 0.1, 0.6, 3, 0.1) and

$(a_{2,1}^0, a_{2,1}^1, a_{2,2}^0, a_{2,2}^1, a_{2,3}^0, a_{2,3}^1)$ = (2, 3, 5, 3, 3, 3). The change of parameter values are given in the corresponding tables.

Table 8.1 shows that the mean system size (L) for the infinite feedback queueing model (section 8.1) increases with f_1 as well as f_2. Table 8.2 shows that L increase with breakdown rate (a_1^0) and mean repair time $E(R_1^0)$. The graphical representations of 8.1 and 8.2 are given in Figures 8.1 and 8.2 respectively.

Table 8.1. The Mean System Length L Vs Feedback Probabilities f_1 and f_2 with $p_1 = 0.4$

f_1 \ f_2	0.5	0.55	0.58	0.6
0.05	2.663	4.082	5.80	7.94
0.06	2.882	4.563	6.76	9.75
0.07	3.134	5.157	8.06	12.56
0.08	3.426	5.911	9.93	17.52
0.09	3.771	6.898	12.84	28.54
0.1	4.183	8.246	18.04	74.56

Table 8.2 Mean System Length L Vs. Breakdown Rate (a_1^0) for Different Values of Repair time E(R_1^0) with $f_1 = 0.01$, $f_2 = 0.6$, $\lambda = 0.17$ and $p = 0.03$

a_1^0 \ E(R_1^0)	1/20	1/15	1/10	1/5
1	22.441	22.733	23.341	25.374
2	23.341	23.981	25.371	30.683
3	24.314	25.370	27.777	38.724
4	25.369	26.926	30.676	52.337
5	26.519	28.680	34.235	80.391
6	27.775	30.673	38.710	171.863

Figure 8.1 Expected System Size Vs. Feedback Probabilities

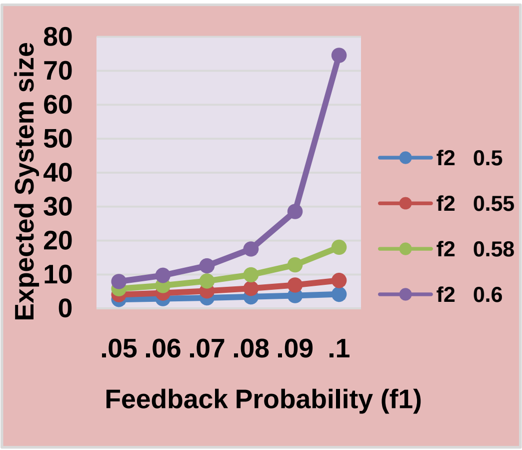
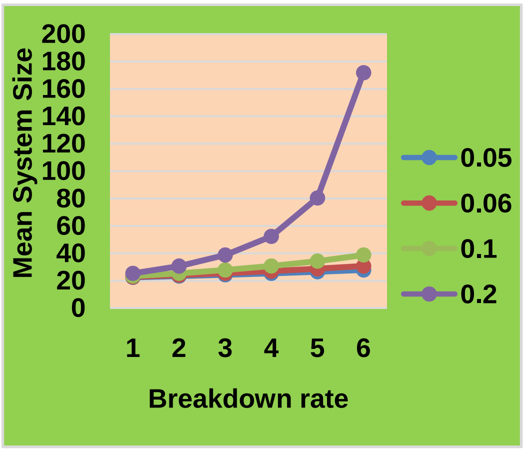


Figure 8.2 Expected System Size Vs. Breakdown Rate (a_1^0) for Different Values of Repair Time E(R_1^0)



In Table 8.3, L for different values of m (the number of feedbacks) are calculated for exponential repair times of $(R_2^{i,j})$ (i = 1 to 3, j = 0 to m–1) with $E(R_2^{(i,0)}) = \frac{1}{4}, E(R_2^{(i,j)}) = 2$ (j = 1 to m–1) for the finite feedback queueing model of Section 7.1. The last row gives the mean system size of the infinite feedback model of Section 8.1. It is found from the table values that as $m \rightarrow \infty$, L for finite feedback model (section 7.1) approach the corresponding value of infinite feedback model (section 8.1).

Table 8.3. Expected System Size Vs. m for Different Values of Arrival Rate (λ) with ($a_1 = 0$, $f_1 = 0$, $f_2 = 0.6$ and $p = 0.03$)

m \ λ	**0.07**	**0.09**	**0.11**	**0.13**	**015**	**0.17**
2	0.367	0.514	0.689	0.898	1.152	1.466
5	0.807	1.247	1.888	2.893	4.662	8.526
10	0.982	1.579	2.524	4.200	7.873	21.743
24	1.003	1.621	2.607	4.378	8.355	24.638
L_{IF}^{BV}	1.003	1.621	2.607	4.378	8.356	24.641