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CHAPTER 2

## Chapter 2

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### Generalized $\pi$ closed sets in topological spaces

#### 2.1 Introduction

Generalized closed sets form a stronger tool in the characterization of topological spaces satisfying weak separation axioms. Zaitsev (1968) proposed the concept of  $\pi$ -closed sets. Levine (1970) generalized the concept of topology by introducing generalized open sets. Balachandran et al. (1991) defined a new class of mappings called generalized continuous (briefly  $g$ -continuous) mappings which contains the class of continuous mappings. Dontchev (1996) introduced the notion of contra continuity and strongly  $S$ -closedness in topological spaces. Noiri and Dontchev (2000) introduced the notion of  $\pi$  generalized-closed sets and obtained characterizations of quasi normal spaces. Park and Park (2004) developed the notion of  $\pi$  generalized pre-closed sets and investigated compactness and connectedness. Aslim, Guler and Noiri (2006) introduced the concept of  $\pi$  generalized semi-closed sets and analyzed the basic properties.

In this chapter, we have introduced the concept of generalized  $\pi$ -closed sets, generalized  $\pi$ -open sets and studied the relationships among  $g\pi$ -closed sets, regular closed sets,  $\pi$ -closed sets,  $g$ -closed sets,  $\pi g$ -closed sets,  $\pi gp$ -closed sets,  $\pi gs$ -closed sets,  $\alpha g$ -closed sets,  $gp$ -closed sets and  $gs$ -closed sets. The family of  $g\pi$ -closed sets lies between those of  $\pi$ -closed sets and  $g$ -closed sets. Using the notion of  $g\pi$ -closed sets, the concepts of  $g\pi$ -continuous functions,  $g\pi$ -irresolute functions, contra  $g\pi$ -continuous functions, almost contra  $g\pi$ -continuous functions and  $g\pi T_{1/2}$  spaces are introduced and their properties are discussed.

## 2.2 Generalized $\pi$ closed sets

**Definition 2.2.1** A subset  $A$  of a topological space  $(X, \tau)$  is called a generalized  $\pi$ -closed (briefly  $g\pi$ -closed) set in  $(X, \tau)$  if  $\pi \text{ cl}(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $(X, \tau)$ .

The class of  $g\pi$ -closed subsets of  $(X, \tau)$  is denoted by  $g\pi C(X, \tau)$ .

**Example 2.2.2** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ . The set  $\{c, d\}$  is a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.3.** Every  $\pi$ -closed set is  $g\pi$ -closed.

**Proof:** Let  $A$  be a  $\pi$ -closed set and  $U$  be any open set containing  $A$ . Since  $A$  is  $\pi$ -closed,  $\pi \text{ cl}(A) = A \subseteq U$ . Hence  $A$  is  $g\pi$ -closed.

The converse of the above theorem need not be true by the following example.

In example 2.2.2. The set  $\{c, d\}$  is a  $g\pi$ -closed set but not a  $\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.4** Every  $g\pi$ -closed set is  $g$ -closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an open set containing  $A$ . Since  $A$  is  $g\pi$ -closed,  $\pi \text{ cl}(A) \subseteq U$ , and since  $\text{cl}(A) \subseteq \pi \text{ cl}(A) \subseteq U$ ,  $A$  is  $g$ -closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.5** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{c\}$  is a  $g$ -closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.6** Every  $g\pi$ -closed set is  $\pi g$ -closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an  $\pi$ -open set containing  $A$ . Since every  $\pi$ -open set is open,  $\pi \text{cl}(A) \subseteq U$ , and since  $\text{cl}(A) \subseteq \pi \text{cl}(A) \subseteq U$ ,  $A$  is  $\pi g$ -closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.7** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{a\}$  is a  $\pi g$ -closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.8** Every  $g\pi$ -closed set is  $\pi gp$ -closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an  $\pi$ -open set containing  $A$ . Since every  $\pi$ -open set is open,  $\pi \text{cl}(A) \subseteq U$ , and since  $p\text{cl}(A) \subseteq \pi \text{cl}(A) \subseteq U$ ,  $A$  is  $\pi gp$ -closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.9** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{b\}$  is a  $\pi gp$ -closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.10** Every  $g\pi$ -closed set is  $\pi gs$ -closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an  $\pi$ -open set containing  $A$ . Since every  $\pi$ -open set is open,  $\pi \text{cl}(A) \subseteq U$ , and since  $s\text{cl}(A) \subseteq \pi \text{cl}(A) \subseteq U$ ,  $A$  is  $\pi gs$ -closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.11** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{b\}$  is a  $\pi$ gs-closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.12** Every  $g\pi$ -closed set is gs-closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an open set containing  $A$ . Since  $A$  is  $g\pi$ -closed,  $\pi \text{ cl}(A) \subseteq U$ , and since  $\text{scl}(A) \subseteq \pi \text{ cl}(A) \subseteq U$ ,  $A$  is gs-closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.13** Let  $X = \{a, b, c\}$ , with  $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ . Then the set  $\{a\}$  is a gs-closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.14** Every  $g\pi$ -closed set is ag-closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an open set containing  $A$ . Since  $A$  is  $g\pi$ -closed,  $\pi \text{ cl}(A) \subseteq U$ , and since  $\text{acl}(A) \subseteq \pi \text{ cl}(A) \subseteq U$ ,  $A$  is ag-closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.15** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{c\}$  is a ag-closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.16** Every  $g\pi$ -closed set is gp-closed.

**Proof:** Let  $A$  be a  $g\pi$ -closed set and  $U$  be an open set containing  $A$ . Since  $A$  is  $g\pi$ -closed,  $\pi \text{ cl}(A) \subseteq U$ , and since  $\text{pcl}(A) \subseteq \pi \text{ cl}(A) \subseteq U$ ,  $A$  is gp-closed.

The converse of the above theorem need not be true by the following example.

**Example 2.2.17** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{c\}$  is a gp-closed but not a  $g\pi$ -closed in  $(X, \tau)$ .

**Remark 2.2.18:** closed set and  $g\pi$ -closed set are independent to each other as seen from the following examples.

**Example 2.2.19** Let  $X = \{a, b, c, d\}$ , with

$\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ . Then the set  $\{c\}$  is a closed set but not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Example 2.2.20** Let  $X = \{a, b, c\}$ , with  $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ . Then the set  $\{a, b\}$  is a  $g\pi$ -closed set but not a closed set in  $(X, \tau)$ .

**Theorem 2.2.21** The union of two  $g\pi$ -closed subsets of  $X$  is a  $g\pi$ -closed subset of  $X$ .

**Proof:** Assume that  $A$  and  $B$  are  $g\pi$ -closed sets in  $X$ . Let  $U$  be open in  $X$  such that  $A \cup B \subseteq U$ . Then  $A \subseteq U$  and  $B \subseteq U$ . Since  $A$  and  $B$  are  $g\pi$  closed,  $\pi cl(A) \subseteq U$  and  $\pi cl(B) \subseteq U$ . Hence  $\pi cl(A \cup B) = \pi cl(A) \cup \pi cl(B) \subseteq U$ . That is  $\pi cl(A \cup B) \subseteq U$ . Therefore  $A \cup B$  is an  $g\pi$ -closed set in  $X$ .

**Remark 2.2.22** Intersection of any two  $g\pi$ -closed sets in  $(X, \tau)$  need not be  $g\pi$ -closed.

**Example 2.2.23** Let  $X = \{a, b, c, d\}$ , with  $\tau = \{X, \varphi, \{b\}\}$ . Then  $A = \{a, b\}$  and  $B = \{b, c\}$  are  $g\pi$ -closed set but  $A \cap B = \{b\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.2.24** Let  $A$  be a  $g\pi$ -closed set of  $(X, \tau)$ . Then  $\pi cl(A) - A$  does not contain any non-empty closed set.

**Proof:** Suppose that  $A$  is  $g\pi$ -closed. Let  $F$  be a closed set contained in  $\pi cl(A) - A$ . Then  $F^c$  is an open set of  $(X, \tau)$  such that  $A \subseteq F^c$ . Since  $A$  is

$g\pi$ -closed,  $\pi \text{ cl}(A) \subseteq F^c$ . Thus  $F \subseteq (\pi \text{ cl}(A))^c$ . Also  $F \subseteq \pi \text{ cl}(A) - A$ . Therefore  $F \subseteq (\pi \text{ cl}(A))^c \cap (\pi \text{ cl}(A)) = \varphi$ . Hence  $F = \varphi$ .

**Theorem 2.2.25** If  $A$  is an open and  $g\pi$ -closed subset of  $(X, \tau)$ , then  $A$  is a  $\pi$ -closed subset of  $(X, \tau)$ .

**Proof:** Since  $A$  is open and  $g\pi$ -closed,  $\pi \text{ cl}(A) \subseteq A$ . Hence  $A$  is  $\pi$ -closed.

**Theorem 2.2.26** The intersection of a  $g\pi$ -closed set and  $\pi$ -closed set is a  $g\pi$ -closed set.

**Proof:** Let  $A$  be  $g\pi$ -closed and let  $F$  be  $\pi$ -closed. If  $U$  is an open set such that  $A \cap F \subseteq U$ , then  $A \subseteq U \cup F^c$  and so  $\pi \text{ cl}(A) \subseteq U \cup F^c$ . Now  $\pi \text{ cl}(A \cap F) \subseteq \pi \text{ cl}(A) \cap F \subseteq U$ . Hence  $A \cap F$  is  $g\pi$ -closed.

**Theorem 2.2.27** If  $A$  is a  $g\pi$ -closed set in a space  $(X, \tau)$  and  $A \subseteq B \subseteq \pi \text{ cl}(A)$ , then  $B$  is also a  $g\pi$ -closed set.

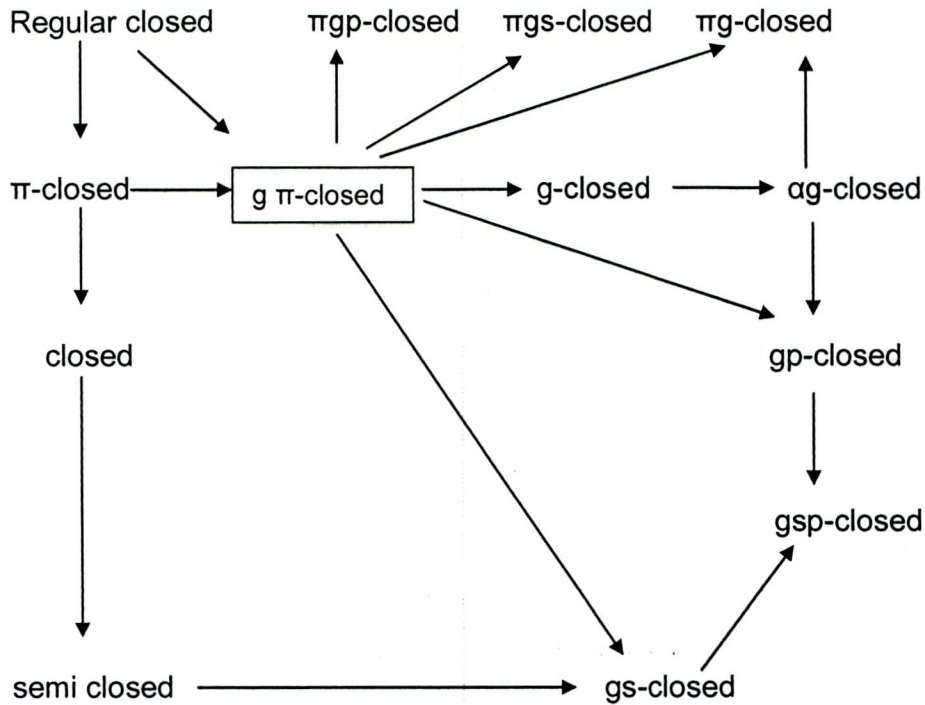
**Proof:** Let  $U$  be an open set of  $(X, \tau)$  such that  $B \subseteq U$ . Then  $A \subseteq U$ . Since  $A$  is a  $g\pi$ -closed set,  $\pi \text{ cl}(A) \subseteq U$ . Since  $B \subseteq \pi \text{ cl}(A)$ ,  $\pi \text{ cl}(B) \subseteq \pi \text{ cl}(A)$ . Therefore  $\pi \text{ cl}(B) \subseteq U$  and hence  $B$  is  $g\pi$ -closed.

**Theorem 2.2.28** Let  $A$  be a  $g\pi$ -closed set of  $(X, \tau)$ . Then  $A$  is  $\pi$ -closed if and only if  $\pi \text{ cl}(A) - A$  is closed.

**Proof: Necessity:** Let  $A$  be a  $\pi$ -closed subset of  $X$ . Then  $\pi \text{ cl}(A) = A$  and so  $\pi \text{ cl}(A) - A = \varphi$  which is closed.

**Sufficiency:** Since  $A$  is  $g\pi$ -closed, by theorem 2.2.24  $\pi \text{ cl}(A) - A$  does not contain any non empty closed set. Therefore  $\pi \text{ cl}(A) - A = \varphi$ . That is  $\pi \text{ cl}(A) = A$ . Hence  $A$  is  $\pi$ -closed.

**Remark 2.2.29** The following diagram shows the relationships of  $g\pi$ -closed sets with other sets.



where  $A \rightarrow B$  represents  $A$  implies  $B$  but not conversely

### 2.3 Generalized $\pi$ open sets

**Definition 2.3.1** A subset  $A$  of a topological space  $X$  is called a generalized  $\pi$ -open (briefly  $g\pi$ -open) set if its complement  $A^c$  is  $g\pi$ -closed.

**Theorem 2.3.3** A subset  $A$  of  $(X, \tau)$  is  $g\pi$ -open if and only if  $F \subseteq \pi \text{int}(A)$  whenever  $F$  is closed and  $F \subseteq A$ .

**Proof:** Assume that  $A$  is  $g\pi$ -open in  $X$ . Let  $F$  be closed and  $F \subseteq A$ . This implies  $F^c$  is open and  $A^c \subseteq F^c$  since  $A^c$  is  $g\pi$ -closed,  $\pi \text{cl}(A^c) \subseteq F^c$  since  $\pi \text{cl}(A^c) = (\pi \text{int}(A))^c$ ,  $(\pi \text{int}(A))^c \subseteq F^c$ . Therefore  $F \subseteq \pi \text{int}(A)$ .

Conversely assume that  $F \subseteq \pi \text{int}(A)$  whenever  $F$  is closed, and  $F \subseteq A$ . Let  $U$  be a open set in  $X$  containing  $A^c$ . Therefore  $U^c$  is a closed set contained

in  $A$ . By hypothesis  $U^c \subseteq \pi \text{int}(A)$ ,  $U \supseteq \pi \text{cl}(A^c)$ . Therefore  $A^c$  is  $g\pi$ -closed in  $X$ . Hence  $A$  is  $g\pi$ -open in  $X$ .

**Theorem 2.3.4** If  $A$  is a  $g\pi$ -open subset of  $(X, \tau)$  and  $\pi \text{int}(A) \subseteq B \subseteq A$ , then  $B$  is  $g\pi$ -open.

**Proof:** Let  $F \subseteq B$  and  $F$  be a closed subset of  $X$ . Since  $A$  is  $g\pi$ -open and  $F \subseteq A$ ,  $F \subseteq \pi \text{int}(A)$  and then  $F \subseteq \pi \text{int}(B)$ . Hence  $B$  is  $g\pi$ -open.

**Theorem 2.3.5** If a subset  $A$  of  $(X, \tau)$  is  $g\pi$ -closed, then  $\pi \text{cl}(A) - A$  is  $g\pi$ -open.

**Proof:** Let  $F \subseteq \pi \text{cl}(A) - A$ , where  $F$  be closed in  $X$ . Then by theorem 2.2.24,  $F = \emptyset$  and so  $F \subseteq \pi \text{int}(\pi \text{cl}(A) - A)$ . This shows that  $\pi \text{cl}(A) - A$  is  $g\pi$ -open.

## 2.4 Applications

**Definition 2.4.1** A topological space  $(X, \tau)$  is called a  $g\pi T_{1/2}$  space if every  $g\pi$ -closed set is  $\pi$  closed

**Example 2.4.2** Let  $X = \{a, b, c\}$ , with  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, \{a, b\}\}$ . Then  $(X, \tau)$  is a  $g\pi T_{1/2}$ -space

**Definition 2.4.3** A topological space  $X$  is said to be a  $T_{g\pi}$ -space if every  $g\pi$ -closed subset of  $X$  is a closed subset of  $X$ .

**Example 2.4.4** Let  $X = \{a, b, c\}$ , with  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$ . Then  $(X, \tau)$  is a  $T_{g\pi}$ -space

**Remark 2.4.5**  $g\pi T_{1/2}$  space  $\longrightarrow T_{g\pi}$ -space

The converse of the above remark need not be true. In example 2.4.4, the space  $(X, \tau)$  is a  $T_{g\pi}$ -space but not  $g\pi T_{1/2}$ -space, since  $\{c\}$  is  $g\pi$ -closed but not  $\pi$ -closed.

**Theorem 2.4.5** A topological space  $(X, \tau)$  is  $g\pi T_{1/2}$  if and only if each singleton of  $X$  is either closed or  $\pi$ -open.

**Proof: Necessity:** Let  $x \in X$  and assume that  $\{x\}$  is not closed. Then clearly  $X - \{x\}$  is not open and  $X - \{x\}$  is trivially  $g\pi$ -closed. Since  $(X, \tau)$  is a  $g\pi T_{1/2}$  space, it is  $\pi$ -closed. Therefore  $\{x\}$  is  $\pi$ -open.

**Sufficiency:** Let  $A \subset X$  be  $g\pi$ -closed. Let  $x \in \pi cl(A)$ . We will show that  $x \in A$ . Consider the following two cases.

Case(a) : The set  $\{x\}$  is closed . Then if  $x \notin A$ , then  $A \subset X - \{x\}$ . Since  $A$  is  $g\pi$ -closed and  $X - \{x\}$  is open,  $\pi cl(A) \subset X - \{x\}$  and hence  $x \notin \pi cl(A)$ . This is a contradiction. Therefore  $x \in A$

Case (b): The set  $\{x\}$  is  $\pi$  open. Since  $x \in \pi cl(A)$ ,  $\{x\} \cap A \neq \emptyset$  . Thus  $x \in A$ . So, in both cases,  $x \in A$ . This shows that  $A$  is  $\pi$ -closed.

## 2.5 On Generalized $\pi$ -Continuous and Generalized $\pi$ -Irresolute Functions

**Definition 2.5.1** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is said to be  $g\pi$ -continuous if the inverse image of every open set in  $(Y, \sigma)$  is  $g\pi$ -open in  $(X, \tau)$ .

**Example 2.5.2** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \emptyset, \{a\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $g\pi$ -continuous

**Theorem.2.5.3** Every  $\pi$ -continuous function is  $g\pi$ -continuous

**Proof:** Assume  $f$  is a  $\pi$ -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $\pi$  -open in  $(X, \tau)$ . Since every  $\pi$ -open set is  $g\pi$ -open,  $f^{-1}(V)$  is  $g\pi$ -open in  $(X, \tau)$ . Therefore  $f$  is  $g\pi$ -continuous.

The converse of the above theorem need not be true. In example 2.5.2  $f$  is  $g\pi$ -continuous but not  $\pi$  continuous, since  $f^{-1}(\{b, c, d\})=\{b, c, d\}$  is not a  $\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.4** Every  $g\pi$  -continuous function is  $g$ -continuous

**Proof:** Assume  $f$  is a  $g\pi$  -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$  -open in  $(X, \tau)$ . Since every  $g\pi$  -open set is  $g$ -open,  $f^{-1}(V)$  is  $g$ -open in  $(X, \tau)$ . Therefore  $f$  is  $g$ -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.5** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varphi, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $g$  continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.6** Every  $g\pi$  -continuous function is  $\pi g$  -continuous

**Proof:** Assume  $f$  is a  $g\pi$ -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$  -open in  $(X, \tau)$ . Since every  $g\pi$ -open set is  $\pi g$ -open,  $f^{-1}(V)$  is  $\pi g$ -open in  $(X, \tau)$ . Therefore  $f$  is  $\pi g$  -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.7** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varphi, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $\pi g$ -continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.8** Every  $g\pi$ -continuous function is  $\pi gp$ -continuous

**Proof:** Assume  $f$  is a  $g\pi$ -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$ -open in  $(X, \tau)$ . Since every  $g\pi$ -open set is  $\pi gp$ -open,  $f^{-1}(V)$  is  $\pi gp$ -open in  $(X, \tau)$ . Therefore  $f$  is  $\pi gp$ -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.9** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varnothing, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $\pi gp$ -continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.10** Every  $g\pi$ -continuous function is  $\pi gs$ -continuous

**Proof:** Assume  $f$  is a  $g\pi$ -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$ -open in  $(X, \tau)$ . Since every  $g\pi$ -open set is  $\pi gs$ -open,  $f^{-1}(V)$  is  $\pi gs$ -open in  $(X, \tau)$ . Therefore  $f$  is  $\pi gs$ -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.11** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varnothing, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $\pi gs$ -continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.12** Every  $g\pi$  -continuous function is  $gs$  -continuous

**Proof:** Assume  $f$  is a  $g\pi$  -continuous function .Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$ -open in  $(X, \tau)$ . Since every  $g\pi$  -open set is  $gs$  -open,  $f^{-1}(V)$  is  $gs$  -open in  $(X, \tau)$ . Therefore  $f$  is  $gs$  -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.13** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varphi, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $gs$ -continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.14** Every  $g\pi$  -continuous function is  $ag$  -continuous

**Proof:** Assume  $f$  is a  $g\pi$  -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$  -open in  $(X, \tau)$ . Since every  $g\pi$  -open set is  $ag$  open,  $f^{-1}(V)$  is  $ag$  -open in  $(X, \tau)$ . Therefore  $f$  is  $ag$  -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.15** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varphi, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $ag$  -continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Theorem 2.5.16** Every  $g\pi$  -continuous function is  $gp$  -continuous

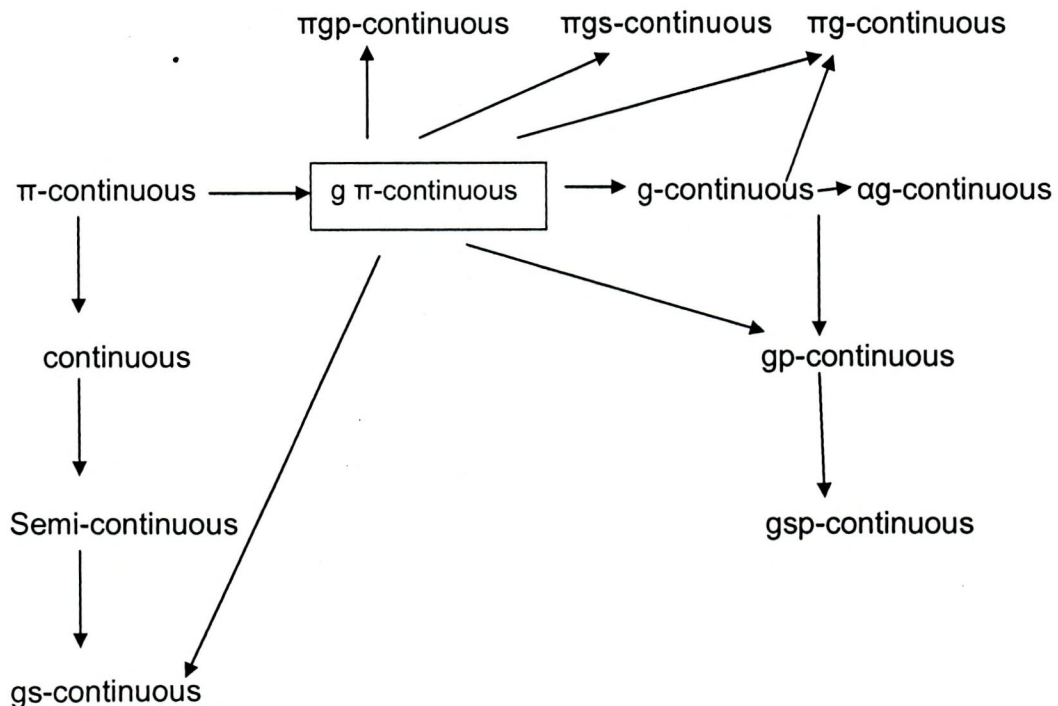
**Proof:** Assume  $f$  is  $g\pi$  -continuous function. Let  $V$  be any open set in  $(Y, \sigma)$ . Then  $f^{-1}(V)$  is  $g\pi$ -open in  $(X, \tau)$ . Since every  $g\pi$ -open set is  $gp$ -open,  $f^{-1}(V)$  is  $gp$  -open in  $(X, \tau)$ . Therefore  $f$  is  $gp$  -continuous.

The converse of the above theorem need not be true as seen from the following example.

**Example 2.5.17** Let  $X = \{a, b, c, d\} = Y$

Let  $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$  and  $\sigma = \{Y, \varphi, \{a, b, d\}\}$ . Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c, f(d)=d$ . Then  $f$  is  $gp$  -continuous but not  $g\pi$ -continuous, since  $f^{-1}(\{c\})=\{c\}$  is not a  $g\pi$ -closed set in  $(X, \tau)$ .

**Remark 2.5.18** The following diagram shows the relationships between  $g\pi$  -continuous functions with various continuous functions.



where  $A \rightarrow B$  represents  $A$  implies  $B$  but not conversely.

**Theorem 2.5.19** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is  $g\pi$ -continuous if and only if the inverse image of every closed set in  $Y$  is  $g\pi$ -closed in  $X$ .

**Proof:** Assume that  $f$  is  $g\pi$ -continuous. Let  $F$  be a closed set in  $Y$ . Then  $F^c$  is open in  $Y$ . Since  $f$  is  $g\pi$ -continuous,  $f^{-1}(F^c) = X - f^{-1}(F)$  is  $g\pi$ -open in  $(X, \tau)$ . Hence  $f^{-1}(F)$  is  $g\pi$ -closed in  $(X, \tau)$ .

Conversely assume that the inverse image of every closed set in  $Y$  is  $g\pi$ -closed in  $X$ . Let  $V$  be an open set in  $Y$ , then  $V^c$  is closed in  $Y$ . By assumption  $f^{-1}(V^c)$  is  $g\pi$ -closed in  $X$ . But  $f^{-1}(V^c) = X - f^{-1}(V)$  and so  $f^{-1}(V)$  is  $g\pi$ -open in  $X$ . Thus  $f$  is  $g\pi$ -continuous.

**Definition 2.5.20** A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is called  $g\pi$ -irresolute if  $f^{-1}(V)$  is  $g\pi$ -closed in  $(X, \tau)$  for every  $g\pi$  closed set  $V$  of  $(Y, \sigma)$ .

**Example: 2.5.20(a)** Let  $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}\} = \sigma$ . Let  $f : (X, \tau) \rightarrow (X, \sigma)$  be defined by  $f(a)=a, f(b)=b, f(c)=c$ . Then  $f$  is  $g\pi$ -irresolute.

**Definition 2.5.21** Let  $(X, \tau)$  be a topological space and  $B \subseteq X$ . We define the  $g\pi$ -closure of  $B$  (briefly  $g\pi Cl(B)$ ) to be the intersection of all  $g\pi$ -closed sets containing  $B$  which is denoted by

$g\pi Cl(B) = \bigcap \{A : B \subseteq A \text{ and } A \in g\pi C(X, \tau)\}$  where  $g\pi C(X, \tau)$  is set of all  $g\pi$ -closed subsets of  $X$ .

**Theorem 2.5.22** Let  $A$  be a subset of  $(X, \tau)$  and  $x \in X$ . Then  $x \in g\pi Cl(A)$  if and only if  $V \cap A \neq \emptyset$  for every  $g\pi$ -open set  $V$  containing  $x$ .

**Proof:** Let  $A$  be a subset of  $(X, \tau)$  and  $x \in g\pi Cl(A)$ . Suppose that there exists a  $g\pi$ -open set  $V$  containing  $x$  such that  $V \cap A = \emptyset$ . Then  $A \subseteq X - V$ ,  $g\pi Cl(A) \subseteq X - V$  and then  $x \notin g\pi Cl(A)$ , a contradiction.

Conversely, suppose that  $x \notin g\pi Cl(A)$ . Then there exists a  $g\pi$ -closed set  $F$  contains  $A$  such that  $x \notin F$ . Since  $x \in X-F$  and  $X-F$  is  $g\pi$ -open,  $(X-F) \cap A = \emptyset$ , a contradiction.

**Theorem 2.5.23**

a) The following statements are equivalent

i)  $f$  is  $g\pi$ -continuous

ii) The inverse image of every open set in  $Y$  is  $g\pi$ -open in  $X$ .

b) If  $f : (X, \tau) \rightarrow (Y, \sigma)$  is  $g\pi$ -continuous, then  $f(g\pi Cl(A)) \subset cl(f(A))$  for every subset  $A$  of  $X$

c) The following statements are equivalent

i) For each  $x \in X$  and each open set  $V$  containing  $f(x)$  there exist a  $g\pi$ -open set  $U$  containing  $x$  such that  $f(U) \subset V$

ii) For every subset  $A$  of  $X$ ,  $f(g\pi Cl(A)) \subset cl(f(A))$

**Proof:** (i)  $\Leftrightarrow$  (ii) is obvious.

b) Let  $A \subset X$ . Since  $f$  is  $g\pi$ -continuous and  $A \subset f^{-1}(cl(f(A)))$ ,  $g\pi cl(A) \subset f^{-1}(cl(f(A)))$  and hence  $f(g\pi cl(A)) \subset cl(f(A))$

c) i)  $\Rightarrow$  ii) Let  $y \in f(g\pi Cl(A))$  and let  $V$  be any open neighbourhood of  $y$ . Then there exist a  $x \in X$  and a  $g\pi$ -open set  $U$  such that  $f(x) = y$ ,  $x \in U$ ,  $x \in g\pi Cl(A)$  and  $f(U) \subset V$ . By theorem 2.5.22,  $U \cap A \neq \emptyset$  and hence  $f(A) \cap V \neq \emptyset$ . Hence  $y = f(x) \in cl(f(A))$ .

ii)  $\Rightarrow$  i) Let  $x \in X$  and  $V$  be any open set containing  $f(x)$ . Let  $A = f^{-1}(Y-V)$ . Since  $f(g\pi Cl(A)) \subset cl(f(A)) \subset Y-V$ ,  $g\pi Cl(A) = A$ . Since  $x \notin g\pi Cl(A)$ , there exists a  $g\pi$ -open set  $U$  containing  $x$  such that  $U \cap A = \emptyset$  and hence  $f(U) \subset f(X-A) \subset V$ .

**Remark 2.5.24** The composition of two  $g\pi$ -continuous functions need not be a  $g\pi$ -continuous function.

**Example 2.5.25** Let  $X = \{a, b, c, d\}$ ,

$\tau = \{X, \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ ,  $\sigma = \{X, \varphi, \{b\}\}$ , and  $\mu = \{X, \varphi, \{a, b, d\}\}$ . Define  $f : (X, \tau) \rightarrow (X, \sigma)$  by  $f(a) = a, f(b) = b, f(c) = c, f(d) = d$  and  $g : (X, \sigma) \rightarrow (X, \mu)$  by  $g(a) = a, g(b) = b, g(c) = c, g(d) = d$ . Then  $f$  and  $g$  are  $g\pi$ -continuous. The set  $\{a, b, d\}$  is open in  $(X, \mu)$  and  $(g \circ f)^{-1}(\{a, b, d\}) = f^{-1}(g^{-1}(\{a, b, d\})) = f^{-1}(\{a, b, d\}) = \{a, b, d\}$  is not  $g\pi$ -open in  $(X, \tau)$ . Hence  $g \circ f$  is not  $g\pi$ -continuous.

**Theorem 2.5.26** Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  and  $g : (Y, \sigma) \rightarrow (Z, \mu)$  be any two functions. Then

- a)  $g \circ f$  is  $g\pi$ -continuous, if  $g$  is continuous and  $f$  is  $g\pi$ -continuous.
- b)  $g \circ f$  is  $g\pi$ -irresolute, if  $g$  is  $g\pi$ -irresolute and  $f$  is  $g\pi$ -irresolute.
- c)  $g \circ f$  is  $g\pi$ -continuous, if  $g$  is  $g\pi$ -continuous and  $f$  is  $g\pi$ -irresolute.

**Proof** a) Let  $V$  be closed in  $(Z, \mu)$ . Then  $g^{-1}(V)$  is closed in  $(Y, \sigma)$ , since  $g$  is continuous.  $g\pi$ -continuity of  $f$  implies that  $f^{-1}(g^{-1}(V))$  is  $g\pi$ -closed in  $(X, \tau)$ . Hence  $g \circ f$  is  $g\pi$ -continuous.

b) Let  $V$  be  $g\pi$ -closed in  $(Z, \mu)$ . Since  $g$  is  $g\pi$ -irresolute. Then  $g^{-1}(V)$  is  $g\pi$ -closed in  $(Y, \sigma)$ . Since  $f$  is  $g\pi$ -irresolute,  $f^{-1}(g^{-1}(V))$  is  $g\pi$ -closed in  $(X, \tau)$ . Hence  $g \circ f$  is  $g\pi$ -irresolute.

c) Let  $V$  be closed in  $(Z, \mu)$ . Since  $g$  is  $g\pi$ -continuous,  $g^{-1}(V)$  is  $g\pi$ -closed in  $(Y, \sigma)$ . As  $f$  is  $g\pi$ -irresolute  $f^{-1}(g^{-1}(V))$  is  $g\pi$ -closed in  $(X, \tau)$ . Hence  $g \circ f$  is  $g\pi$ -continuous.

## 2.6 Contra generalized $\pi$ -continuous functions

**Definition 2.6.1.** A function  $f : X \rightarrow Y$  is called contra  $g\pi$ -continuous if  $f^{-1}(V)$  is  $g\pi$ -closed in  $X$  for every open set  $V$  of  $Y$ .

**Example 2.6.2** Let  $X = \{a, b, c\}$ ,  $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$  and  $\sigma = \{X, \varphi, \{a, c\}\}$ . Define  $f : (X, \tau) \rightarrow (X, \sigma)$  by  $f(a) = b$ ,  $f(b) = c$ ,  $f(c) = a$ . Then  $f$  is contra  $g\pi$ -continuous.

**Theorem 2.6.3** The following are equivalent for a function  $f : X \rightarrow Y$

- (1)  $f$  is contra  $g\pi$ -continuous,
- (2) the inverse image of every closed set of  $Y$  is  $g\pi$ -open in  $X$ .

**Proof:** Let  $U$  be any closed set of  $Y$ . Since  $Y-U$  is open, then by (1), it follows that  $f^{-1}(Y-U) = X - f^{-1}(U)$  is  $g\pi$ -closed. This shows that  $f^{-1}(U)$  is  $g\pi$ -open in  $X$ .

Converse is similar.

**Theorem 2.6.4.** The following are equivalent for a function  $f : X \rightarrow Y$

- (1)  $f$  is contra- $g\pi$  continuous
- (2) for every closed subset  $F$  of  $Y$ ,  $f^{-1}(F) \in g\pi O(X)$
- (3) for each  $x \in X$  and each  $F \in C(Y, f(x))$ , there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq F$
- (4)  $f(g\pi Cl(A)) \subseteq \ker(f(A))$  for every subset  $A$  of  $X$
- (5)  $g\pi Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$  for every subset of  $B$  of  $Y$ .

**Proof:** The implications (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2): Let  $F$  be any closed set of  $Y$  and  $x \in f^{-1}(F)$ . Then  $f(x) \in F$  and there exists  $U_x \in \text{g}\pi\text{O}(X, x)$  such that  $f(U_x) \subseteq F$ . Therefore,  $f^{-1}(F) = \cup \{U_x \mid x \in f^{-1}(F)\} \in \text{g}\pi\text{O}(X)$ .

(2) $\Rightarrow$ (4): Let  $A$  be any subset of  $X$ . Suppose that  $y \notin \ker(f(A))$ . Then by Lemma 1.6, there exists  $F \in \mathcal{C}(Y, f(x))$  such that  $f(A) \cap F = \emptyset$ . Thus,  $A \cap f^{-1}(F) = \emptyset$  and since  $f^{-1}(F)$  is  $\text{g}\pi$ -open, then  $\text{g}\pi\text{Cl}(A) \cap f^{-1}(F) = \emptyset$ . Therefore, we obtain  $f(\text{g}\pi\text{Cl}(A)) \cap F = \emptyset$  and  $y \notin f(\text{g}\pi\text{Cl}(A))$ . This implies that  $f(\text{g}\pi\text{Cl}(A)) \subseteq \ker(f(A))$ .

(4) $\Rightarrow$ (5): Let  $B$  be any subset of  $Y$ . By (4) and Lemma 1.6,

$$f(\text{g}\pi\text{Cl}(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B) \text{ and } \text{g}\pi\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B)).$$

(5) $\Rightarrow$ (1): Let  $V$  be any open set of  $Y$ . Then, by Lemma 1.6,

$$\text{g}\pi\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V) \text{ and } \text{g}\pi\text{Cl}(f^{-1}(V)) = f^{-1}(V). \text{ This shows that } f^{-1}(V) \text{ is } \text{g}\pi \text{ closed in } X.$$

**Theorem 2.6.5.** If a function  $f : X \rightarrow Y$  is contra- $\text{g}\pi$ -continuous and  $Y$  is regular, then  $f$  is  $\text{g}\pi$ -continuous.

**Proof:** Let  $x$  be an arbitrary point of  $X$  and let  $V$  be an open set of  $Y$  containing  $f(x)$ . Since  $Y$  is regular, there exists an open set  $W$  in  $Y$  containing  $f(x)$  such that  $\text{Cl}(W) \subseteq V$ . Since  $f$  is contra- $\text{g}\pi$ -continuous, so by Theorem 2.6.4(3) there exists  $U \in \text{g}\pi\text{O}(X, x)$  such that  $f(U) \subseteq \text{Cl}(W)$ . Then  $f(U) \subseteq \text{Cl}(W) \subseteq V$ . Hence,  $f$  is  $\text{g}\pi$ -continuous.

**Definition 2.6.6.** A space  $(X, \tau)$  is said to be a locally  $\text{g}\pi$ -indiscrete if every  $\text{g}\pi$ -open set is closed in  $X$ .

**Theorem 2.6.7** A contra-  $\text{g}\pi$  -continuous function  $f : X \rightarrow Y$  is continuous when  $X$  is locally  $\text{g}\pi$  -indiscrete.

**Proof:** Obvious

**Theorem 2.6.8.** If a function  $f : X \rightarrow Y$  is contra- $g\pi$ -continuous and  $X$  is a  $g\pi$ -space, then  $f$  is contra-continuous.

**Definition 2.6.9.** A function  $f : X \rightarrow Y$  is called almost- $g\pi$ -continuous if for each  $x \in X$  and each open set  $V$  of  $Y$  containing  $f(x)$ , there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq g\pi\text{-int}(cl(V))$ .

**Definition 2.6.10.** A function  $f : X \rightarrow Y$  is said to be pre- $g\pi$ -open if the image of each  $g\pi$ -open set is  $g\pi$ -open.

**Theorem 2.6.11.** If a function  $f : X \rightarrow Y$  is a pre- $g\pi$ -open and contra  $g\pi$ -continuous, then  $f$  is almost  $g\pi$ -continuous.

**Proof:** Let  $x$  be any arbitrary point of  $X$  and  $V$  be an open set containing  $f(x)$ . Since  $f$  is contra- $g\pi$ -continuous, then by Theorem 2.6.3(3) there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq cl(V)$ . Since  $f$  is pre- $g\pi$ -open,  $f(U)$  is  $g\pi$ -open in  $Y$ . Therefore,  $f(U) = g\pi\text{-Int}(U) \subseteq g\pi\text{-Int}(cl(f(U))) \subseteq g\pi\text{-Int}(cl(V))$ . So  $f$  is almost  $g\pi$ -continuous.

**Definition 2.6.12.** A function  $f : X \rightarrow Y$  is said to be almost weakly  $g\pi$ -continuous if for each  $x \in X$  and each open set  $V$  of  $f(x)$  there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq cl(V)$ .

**Theorem 2.6.13.** If a function  $f : X \rightarrow Y$  is contra- $g\pi$ -continuous, then  $f$  is almost weakly  $g\pi$ -continuous.

**Proof:** Let  $V$  be any open set of  $Y$ . Then  $f^{-1}(Cl(V))$  is  $g\pi$ -open in  $X$ . Let  $U = f^{-1}(cl(V))$ , then by Theorem 2.6.3(3)  $f(U) \subseteq cl(V)$ . Hence  $f$  is almost weakly  $g\pi$ -continuous.

**Definition 2.6.14** The  $g\pi$ -frontier of a subset  $A$  of a space  $X$ , denoted by  $g\pi\text{-Fr}(A)$ , is defined as  $g\pi\text{-Fr}(A) = g\pi Cl(A) \cap g\pi Cl(X - A)$ .

**Theorem 2.6.15.** The set of all points  $x$  of  $X$  at which  $f : X \rightarrow Y$  is not contra- $g\pi$ -continuous is identical with the union of the  $g\pi$ -frontier of the inverse images of closed sets of  $Y$  containing  $f(x)$ .

**Proof: Necessity:** Suppose  $f$  is not contra- $g\pi$ -continuous at  $x \in X$ . There exists  $F \in C(Y, f(x))$  such that  $f(U) \cap (Y - F) \neq \emptyset$  for every  $U \in g\pi O(X, x)$  by Theorem 2.6.3. This implies that  $U \cap f^{-1}(Y - F) \neq \emptyset$ . Therefore,  $x \in g\pi Cl(f^{-1}(Y - F)) = g\pi Cl(X - f^{-1}(F))$ . Since  $x \in f^{-1}(F) \subseteq g\pi Cl(f^{-1}(F))$ ,  $x \in g\pi Cl(f^{-1}(F)) \cap g\pi Cl(f^{-1}(Y - F))$ . Therefore,  $x \in g\pi Fr(f^{-1}(F))$ .

**Sufficiency:** Suppose that  $x \in g\pi Fr(f^{-1}(F))$  for some  $F \in C(Y, f(x))$ , and  $f$  is contra- $g\pi$ -continuous at  $x$ . Then there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq F$ . Therefore,  $x \in U \subseteq f^{-1}(F)$  and hence  $x \in g\pi Int(f^{-1}(F)) \subseteq X - g\pi Fr(f^{-1}(F))$ . This is a contradiction. Therefore  $f$  is not contra- $g\pi$ -continuous.

**Definition 2.6.16** A topological space  $(X, \tau)$  is said to be  $g\pi$ -normal if each pair of non-empty disjoint closed sets can be separated by disjoint  $g\pi$ -open sets.

**Theorem 2.6.17** If  $f : X \rightarrow Y$  is contra- $g\pi$ -continuous, closed injection and  $Y$  is ultranormal, then  $X$  is  $g\pi$ -normal.

**Proof:** Let  $F_1$  and  $F_2$  be disjoint closed subsets of  $X$ . Since  $f$  is  $g\pi$ -closed injection,  $f(F_1)$  and  $f(F_2)$  are disjoint closed subsets of  $Y$ . Again, since  $Y$  is ultranormal  $f(F_1)$  and  $f(F_2)$  are separated by disjoint clopen sets  $P$  and  $Q$  respectively. Therefore,  $f(F_1) \subseteq P$  and  $f(F_2) \subseteq Q$  i.e.  $F_1 \subseteq f^{-1}(P)$  and  $F_2 \subseteq f^{-1}(Q)$ , where  $f^{-1}(P)$  and  $f^{-1}(Q)$  are disjoint  $g\pi$ -open sets of  $X$ , since  $f$  is contra- $g\pi$ -continuous. Therefore  $X$  is  $g\pi$ -normal.

**Definition 2.6.18.** A space  $X$  is said to be  $g\pi$ - $T_2$  if for each pair of distinct points  $x$  and  $y$  in  $X$ , there exist  $U \in g\pi O(X, x)$  and  $V \in g\pi O(X, y)$  such that  $U \cap V = \emptyset$ .

**Example 2.6.18 (a)** Let  $X=\{a, b, c, d\}$ ,  $\tau = \{X, \varphi, \{b\}\}$ . Then  $(X, \tau)$  is a  $g\pi-T_2$  space.

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**Theorem 2.6.19.** Let  $X$  and  $Y$  be topological spaces. If

(1) for each pair of distinct points  $x$  and  $y$  in  $X$  there exists a function  $f$  of  $X$  into  $Y$  such that  $f(x) \neq f(y)$ ,

(2)  $Y$  is an Urysohn space and

(3)  $f$  is contra  $g\pi$  -continuous at  $x$  and  $y$ , then  $X$  is  $g\pi -T_2$ .

**Proof:** Let  $x$  and  $y$  be any distinct points in  $X$  and a function  $f : X \rightarrow Y$  such that  $f(x) \neq f(y)$  and  $f$  is contra  $g\pi$ -continuous at  $x$  and  $y$ . Let  $z = f(x)$  and  $v = f(y)$ . Then  $z \neq v$ . Since  $Y$  is Urysohn, there exist open sets  $V$  and  $W$  containing  $z$  and  $v$ , respectively such that  $cl(V) \cap cl(W) = \varphi$ . Since  $f$  is contra  $g\pi$ -continuous at  $x$  and  $y$ , there exist  $g\pi$  -open sets  $A$  and  $B$  containing  $x$  and  $y$ , respectively such that  $f(A) \subseteq cl(V)$  and  $f(B) \subseteq cl(W)$ . Then  $f(A) \cap f(B) = \varphi$  and so  $A \cap B = \varphi$ , since  $cl(V) \cap cl(W) = \varphi$ . Hence,  $X$  is  $g\pi -T_2$ .

**Theorem.2.6.20** If  $f$  is a contra  $g\pi$  -continuous injection of a topological space  $X$  into a Urysohn space  $Y$ , then  $X$  is  $g\pi-T_2$

**Proof:** Let  $x_1$  and  $x_2$  be two distinct points in  $X$ . Since  $f$  is a injection contra  $g\pi$  -continuous function of  $X$  into a Urysohn space  $Y$ ,  $f(x_1) \neq f(x_2)$ . Hence by Theorem 2.6.19,  $X$  is  $g\pi-T_2$ .

**Theorem.2.6.21** If  $f$  is a contra  $g\pi$  -continuous injection of a topological space  $X$  into a Ultra Hausdorff space  $Y$ , then  $X$  is  $g\pi-T_2$

**Proof:** Let  $x_1$  and  $x_2$  be any distinct points in  $X$ . Then since  $f$  is injective and  $Y$  is Ultra Hausdorff  $f(x_1) \neq f(x_2)$  and there exist  $V_1, V_2 \in CO(Y)$  such that  $f(x_1) \in V_1$ ,  $f(x_2) \in V_2$  and  $V_1 \cap V_2 = \varphi$ . Then  $x_i \in f^{-1}(V_i) \in g\pi Cl(X)$  for  $i = 1, 2$  and  $f^{-1}(V_1) \cap f^{-1}(V_2) = \varphi$ . Thus  $X$  is  $g\pi-T_2$ .

**Definition 2.6.22** A subset  $A$  of a topological space  $X$  is said to be a  $g\pi$ -dense in  $X$  if  $g\pi Cl(A) = X$ .

**Example 2.6.22 (a)** Let  $X = \{a, b, c\}$ ,  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ . Then the set  $A = \{a, b\}$  is  $g\pi$ -dense as  $g\pi Cl(A) = X$ .

**Theorem 2.6.23** If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are contra  $g\pi$ -continuous and  $Y$  is Urysohn, then  $E = \{x \in X : f(x) = g(x)\}$  is  $g\pi$ -closed in  $X$ .

**Proof:** Let  $x \in X - E$ . Then  $f(x) \neq g(x)$ . Since  $Y$  is Urysohn, there exist open sets  $V$  and  $W$  such that  $f(x) \in V$ ,  $g(x) \in W$ , and  $cl(V) \cap cl(W) = \emptyset$ . Since  $f$  and  $g$  are contra  $g\pi$ -continuous,  $f^{-1}(cl(V)) \in g\pi O(X)$  and  $g^{-1}(cl(W)) \in g\pi O(X)$ . Let  $U = f^{-1}(cl(V))$  and  $G = g^{-1}(cl(W))$ . Then  $U$  and  $G$  are  $g\pi$ -open sets containing  $x$ . Let  $A = U \cap G$ , thus  $A$  is  $g\pi$ -open in  $X$ . Hence,  $f(A) \cap g(A) = f(U \cap G) \cap g(U \cap G) \subseteq f(U) \cap g(G) = cl(V) \cap cl(W) = \emptyset$ , therefore,  $A \cap E = \emptyset$ , and  $x \notin g\pi Cl(E)$ . Hence,  $E$  is  $g\pi$ -closed in  $X$ .

**Theorem 2.6.24.** Let  $f : X \rightarrow Y$  be a function and let  $g : X \rightarrow X \times Y$  be the graph function of  $f$  defined by  $g(x) = (x, f(x))$  for every  $x \in X$ . If  $g$  is contra  $g\pi$ -continuous, then  $f$  is contra- $g\pi$ -continuous.

**Proof:** Let  $U$  be an open set in  $Y$ , then  $X \times U$  is an open set in  $X \times Y$ . Since  $g$  is contra- $g\pi$ -continuous,  $g^{-1}(X \times U) = f^{-1}(U)$  and therefore  $f^{-1}(U)$  is  $g\pi$ -closed in  $X$ . Thus,  $f$  is contra- $g\pi$ -continuous.

**Theorem 2.6.25** Let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  be functions. If  $Y$  is Urysohn,  $f$  and  $g$  are contra- $g\pi$ -continuous and  $f = g$  on  $g\pi$ -dense set  $A \subseteq X$ , then  $f = g$  on  $X$ .

**Proof:** Since  $f$  and  $g$  are contra  $g\pi$ -continuous and  $Y$  is Urysohn, by theorem 2.6.23,  $E = \{x \in X : f(x) = g(x)\}$  is  $g\pi$ -closed in  $X$ . By assumption,  $f = g$  on  $g\pi$ -dense set  $A \subseteq X$ . Since  $A \subseteq E$  and  $A$  is  $g\pi$ -dense set in  $X$ , then  $X = g\pi Cl(A) \subseteq g\pi Cl(E) = E$ . Hence,  $f = g$  on  $X$ .

**Definition 2.6.26.** The graph  $G(f)$  of a function  $f : X \rightarrow Y$  is said to be contra- $g\pi$ -closed if for each  $(x, y) \in (X \times Y) - G(f)$ , there exist  $U \in g\pi O(X, x)$  and  $V \in C(Y, y)$  such that  $(U \times V) \cap G(f) = \emptyset$ .

**Lemma 2.6.27** The graph  $G(f)$  of  $f : X \rightarrow Y$  is contra- $g\pi$ -closed in  $X \times Y$  if and only if for each  $(x, y) \in (X \times Y) - G(f)$ , there exist  $U \in g\pi O(X, x)$  and  $V \in C(Y, y)$  such that  $f(U) \cap V = \emptyset$ .

**Proof:** We shall prove that  $f(U) \cap V = \emptyset, \Leftrightarrow (U \times V) \cap G(f) = \emptyset$ . Let  $(U \times V) \cap G(f) \neq \emptyset$ . Then there exist  $(x, y) \in (U \times V)$  and  $(x, y) \in G(f)$ . This implies that  $x \in U, y \in V$  and  $y = f(x) \in V$ . Therefore,  $f(U) \cap V \neq \emptyset$ . Hence the result follows.

**Theorem 2.6.28** If  $f : X \rightarrow Y$  is contra- $g\pi$ -continuous and  $Y$  is Urysohn, then  $G(f)$  is contra- $g\pi$ -closed in  $X \times Y$ .

**Proof:** Let  $(x, y) \in (X \times Y) - G(f)$ . Then  $y \neq f(x)$  and since  $Y$  is Urysohn, there exists open sets  $V, W$  in  $Y$  such that  $f(x) \in V, y \in W$ , and  $cl(V) \cap cl(W) = \emptyset$ . Since  $f$  is contra- $g\pi$ -continuous, there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq cl(V)$ . Hence by lemma 2.6.27,  $G(f)$  is contra- $g\pi$ -closed in  $X \times Y$ .

**Theorem 2.6.29** If  $f : X \rightarrow Y$  is  $g\pi$ -continuous and  $Y$  is  $T_1$ , then  $G(f)$  is contra- $g\pi$ -closed in  $X \times Y$ .

**Proof:** Let  $(x, y) \in (X \times Y) - G(f)$ . Then  $y \neq f(x)$  and there exists an open set  $V$  of  $Y$ , such that  $f(x) \in V, y \notin V$ . Since  $f$  is  $g\pi$ -continuous, there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq V$ . Therefore,  $f(U) \cap (Y - V) = \emptyset$  and  $Y - V \in C(Y, y)$ . Hence by lemma 2.6.27,  $G(f)$  is contra- $g\pi$ -closed in  $X \times Y$ .

**Definition 2.6.30.** The graph  $G(f)$  of a function  $f : X \rightarrow Y$  is said to be strongly contra- $g\pi$ -closed if for each  $(x, y) \in (X \times Y) - G(f)$ , there exist  $U \in g\pi O(X, x)$  and  $V \in RC(Y, y)$  such that  $(U \times V) \cap G(f) = \emptyset$ .

**Lemma 2.6.31** The graph  $G(f)$  of  $f : X \rightarrow Y$  is strongly contra- $g\pi$ -closed graph in  $X \times Y$  if and only if for each  $(x, y) \in (X \times Y) - G(f)$ , there exist  $U \in g\pi O(X, x)$  and  $V \in RC(Y, y)$  such that  $f(U) \cap V = \emptyset$ .

**Theorem 2.6.32** If  $f : X \rightarrow Y$  is almost weakly- $g\pi$ -continuous and  $Y$  is Urysohn, then  $G(f)$  is strongly contra- $g\pi$ -closed in  $X \times Y$ .

**Proof:** Suppose that  $(x, y) \in (X \times Y) - G(f)$ . Then  $y \neq f(x)$ . Since  $Y$  is Urysohn, there exist open sets  $V$  and  $W$  in  $Y$  containing  $y$  and  $f(x)$ , respectively, such that  $cl(V) \cap cl(W) = \emptyset$ . Since  $f$  is almost weakly  $g\pi$ -continuous, there exists  $U \in g\pi O(X, x)$  such that  $f(U) \subseteq cl(W)$ . This shows that  $f(U) \cap cl(V) = f(U) \cap cl(Int(V)) = \emptyset$ , where  $cl(Int(V)) \in RC(Y)$  and hence by Lemma 2.6.31  $G(f)$  is strongly contra- $g\pi$ -closed.

## 2.7. Almost contra generalized $\pi$ -continuous

**Definition 2.7.1.** A function  $f : X \rightarrow Y$  is said to be almost contra- $g\pi$ -continuous  $f^{-1}(V) \in g\pi Cl(X)$  for every  $V \in RO(Y)$ .

**Example 2.7.2** Let  $X = \{a, b, c\}$ ,  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$  and  $\sigma = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$ . Then the identity function  $f : (X, \tau) \rightarrow (X, \sigma)$  is almost contra  $g\pi$ -continuous.

**Theorem 2.7.3.** The following are equivalent for a function  $f : X \rightarrow Y$ :

- (1)  $f$  is almost contra- $g\pi$ -continuous
- (2)  $f^{-1}(F) \in g\pi O(X, x)$  for every  $F \in RC(Y)$
- (3) for each  $x \in X$  and each regular closed set  $F$  in  $Y$  containing  $f(x)$ , there exists an  $g\pi$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subseteq F$
- (4) for each  $x \in X$  and each regular open set  $V$  in  $Y$  non containing  $f(x)$ , there exists an  $g\pi$ -closed set  $K$  in  $X$  noncontaining  $x$  such that  $f^{-1}(V) \subseteq K$ .

**Proof:** (1) $\Leftrightarrow$ (2). Let  $F$  be any regular closed set of  $Y$ . Then  $Y - F$  is regular open. By (1),  $f^{-1}(Y - F) = X - f^{-1}(F) \in g\pi Cl(X)$ . Hence  $f^{-1}(F) \in g\pi O(X)$ . The converse is obvious.

(2) $\Rightarrow$ (3). Let  $F$  be any regular closed set in  $Y$  containing  $f(x)$ . Then by (2)  $f^{-1}(F) \in g\pi O(X)$  and  $x \in f^{-1}(F)$ . Take  $U = f^{-1}(F)$ . Then  $f(U) \subseteq F$ .

(3) $\Rightarrow$ (2). Let  $F$  be any regular closed set in  $Y$  and  $x \in f^{-1}(F)$ . From (3) there exists an  $g\pi$ -open set  $U_x$  in  $X$  containing  $x$  such that  $f(U_x) \subseteq F$ , thus  $U_x \subseteq f^{-1}(F)$ ,  $f^{-1}(F) \subseteq \bigcup_{x \in f^{-1}(F)} U_x$ . This implies that  $f^{-1}(F)$  is  $g\pi$ -open.

(3) $\Leftrightarrow$ (4). Let  $V$  be any regular open set in  $Y$  noncontaining  $f(x)$ . Then  $Y - V$  is a regular closed set containing  $f(x)$ . By (3), there exists an  $g\pi$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subseteq Y - V$ . Hence,  $U \subseteq f^{-1}(Y - V) \subseteq X - f^{-1}(V)$  and then  $f^{-1}(V) \subseteq X - U$ . Take  $H = X - U$ . Hence  $H$  is an  $g\pi$ -closed set in  $X$  noncontaining  $x$ . Similarly we can prove (4)  $\Rightarrow$  (3).

**Theorem 2.7.4.** Every regular set connected is almost contra  $g\pi$ -continuous.

**Proof:** Let  $V$  be any regular open set in  $Y$ . Since  $f : X \rightarrow Y$  is regular set connected,  $f^{-1}(V)$  is clopen in  $X$ . Hence  $f^{-1}(V)$  is  $g\pi$ -closed and  $g\pi$ -open. Therefore  $f$  is almost contra- $g\pi$ -continuous.

The converse of the above theorem need not be true. In example 2.7.2  $f$  is almost contra  $g\pi$ -continuous, but it is not regular set connected.

**Remark 2.7.5**

Regular set connected  $\longrightarrow$  almost contra  $g\pi$ -continuous

**Definition 2.7.6** A space  $X$  is said to be  $g\pi$ - $T_1$  if for each pair of distinct points  $x$  and  $y$  of  $X$ , there exists  $g\pi$ -open sets  $U$  and  $V$  containing  $x$  and  $y$ , respectively, such that  $y \notin U$  and  $x \notin V$ .

**Theorem 2.7.7.** If  $f : X \rightarrow Y$  is an almost contra- $g\pi$ -continuous injection and  $Y$  is weakly Hausdorff, then  $X$  is  $g\pi$ - $T_1$ .

**Proof:** Suppose that  $Y$  is weakly Hausdorff. For any two distinct points  $x$  and  $y$  in  $X$ , there exist regular closed sets  $V$  and  $W$  in  $Y$  such that  $f(x) \in V$ ,  $f(y) \notin V$ ,  $f(x) \notin W$ , and  $f(y) \in W$ . Since  $f$  is almost contra- $g\pi$ -continuous, then  $f^{-1}(V)$  and  $f^{-1}(W)$  are  $g\pi$ -open subsets of  $X$  such that  $x \in f^{-1}(V)$ ,  $y \notin f^{-1}(V)$ ,  $x \notin f^{-1}(W)$ , and  $y \in f^{-1}(W)$ . This show that  $X$  is  $g\pi$ - $T_1$ .

**Corollary 2.7.8** If  $f : X \rightarrow Y$  is a contra- $g\pi$ -continuous injection and  $Y$  is weakly Hausdorff, then  $X$  is  $g\pi$ - $T_1$ .

**Theorem 2.7.9** If  $f : X \rightarrow Y$  is almost contra- $g\pi$ -continuous , closed injection and  $Y$  ultra normal, then  $X$  is  $g\pi$ -normal.

**Proof:** Let  $F_1$  and  $F_2$  be disjoint closed subsets of  $X$ . Since  $f$  is closed injective,  $f(F_1)$  and  $f(F_2)$  are disjoint closed subsets of  $Y$ . Since  $Y$  is ultra normal,  $f(F_1)$  and  $f(F_2)$  are separated by disjoint clopen sets  $V_1$  and  $V_2$  respectively. Hence  $F_i \subset f^{-1}(V_i)$ ,  $f^{-1}(V_i) \in g\pi Cl(X)$  for  $i=1,2$  and  $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ . Thus  $X$  is  $g\pi$ -normal.

**Theorem 2.7.10.** Let  $X$  and  $Y$  be topological spaces. If

(1) for each pair of distinct points  $x$  and  $y$  in  $X$  there exists a function  $f$  of  $X$  into  $Y$  such that  $f(x) \neq f(y)$ ,

(2)  $Y$  is an Urysohn space,

(3)  $f$  is almost contra- $g\pi$ -continuous at  $x$  and  $y$ , then  $X$  is  $g\pi$ - $T_2$ .

**Proof:** Let  $x$  and  $y$  be any distinct points in  $X$ , and  $f : X \rightarrow Y$  be a function such that  $f(x) \neq f(y)$  and almost contra- $g\pi$ -continuous at  $x$  and  $y$ . Let  $a = f(x)$  and  $b = f(y)$ . Then  $a \neq b$ . Since  $Y$  is Urysohn space, there exist open sets  $V$  and  $W$  containing  $a$  and  $b$ , respectively, such that  $cl(V) \cap cl(W) = \emptyset$ . Hence  $cl(int(V)) \cap cl(int(W)) = \emptyset$ . Since  $f$  is almost contra- $g\pi$ -continuous at  $x$  and  $y$ ,

there exist  $g\pi$ -open sets  $A$  and  $B$  containing  $a$  and  $b$ , respectively, such that  $f(A) \subseteq \text{cl}(\text{int}(V))$  and  $f(B) \subseteq \text{cl}(\text{int}(W))$ . Then  $f(A) \cap f(B) = \emptyset$ , so  $A \cap B = \emptyset$ . Hence,  $X$  is  $g\pi$ - $T_2$ .

**Theorem 2.7.11** Let  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  be functions. Then the following properties hold.

(1) If  $f$  is almost contra  $g\pi$ -continuous and  $g$  is regular set connected, then  $g \circ f : X \rightarrow Z$  is almost contra  $g\pi$ -continuous and almost  $g\pi$ -continuous

(2) If  $f$  is almost contra  $g\pi$ -continuous and  $g$  is perfectly continuous, then  $g \circ f : X \rightarrow Z$  is  $g\pi$ -continuous and contra  $g\pi$ -continuous.

(3) If  $f$  is contra  $g\pi$ -continuous and  $g$  is regular set connected, then  $g \circ f : X \rightarrow Z$  is almost contra  $g\pi$ -continuous and almost  $g\pi$ -continuous.

**Proof:** (1) Let  $V$  be any regular open set in  $Z$ . Since  $g$  is regular set connected,  $g^{-1}(V)$  is clopen. Since  $f$  is almost contra  $g\pi$ -continuous,  $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$  is  $g\pi$ -open and  $g\pi$ -closed. Therefore,  $g \circ f$  is almost contra  $g\pi$ -continuous and almost  $g\pi$ -continuous.

(2) Let  $V$  be any open set in  $Z$ . Since  $g$  is perfectly continuous,  $g^{-1}(V)$  is clopen. Since  $f$  is almost contra  $g\pi$ -continuous,  $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$  is  $g\pi$ -open and  $g\pi$ -closed. Therefore,  $g \circ f$  is  $g\pi$ -continuous and contra  $g\pi$ -continuous.

(3) Let  $V$  be any regular open set in  $Z$ . Since  $g$  is regular set connected,  $g^{-1}(V)$  is clopen. Since  $f$  is contra  $g\pi$ -continuous,  $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$  is  $g\pi$ -open and  $g\pi$ -closed. Therefore,  $g \circ f$  is almost contra  $g\pi$ -continuous and almost  $g\pi$ -continuous.

**Theorem 2.7.12** Let  $f : X \rightarrow Y$  is a surjective  $g\pi$ -open and  $g : Y \rightarrow Z$  is a function such that  $g \circ f : X \rightarrow Z$  is almost contra  $g\pi$ -continuous then  $g$  is almost contra-  $g\pi$ -continuous.

**Proof :** Let  $V$  be any regular closed set in  $Z$ . Since  $g \circ f$  is almost contra  $g\pi$ -continuous,  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  is  $g\pi$ -open. Since  $f$  is surjective and  $g\pi$ -open,  $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$  is  $g\pi$ -open. Therefore,  $g$  is almost contra  $g\pi$ -continuous.