Chapter VI

CHAPTER VI

LINEAR OPERATORS PRESERVING RANK AND PERIMETER OF RANK-1 MATRICES OVER SEMIFIELDS

Let $\mathbf{M}_{m,n}(\mathbb{F}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{F}_+$, the set of nonnegative part of any field $\mathbb{F}_+$. Throughout this paper, we shall adopt the convention that $m \leq n$ unless otherwise specified.

Definition : 6.1

If $A=[a_{i,j}]$ is any matrix in $\mathbf{M}_{m,n}(\mathbb{F}_+)$, we define $A^*=[a_{i,j}^*]$ to be the $m \times n$ **Boolean matrix** whose $(i,j)^{th}$ entry is 1 if and only if $a_{i,j} \neq 0$. Then * maps $\mathbf{M}_{m,n}(\mathbb{F}_+)$ onto $\mathbf{M}_{m,n}(\mathbb{B})$, and preserves matrix addition, product and multiplication by scalars. That is * is a homomorphism. It follows that

$$(A+B)^* = A^* + B^* \quad \text{and} \quad (BC)^* = B^* C^* \qquad \text{--------------------(1)}$$

for all $A,B \in \mathbf{M}_{m,n}(\mathbb{F}_+)$ and all $C \in \mathbf{M}_{m,n}(\mathbb{F}_+)$.

Definition : 6.2

An $n \times n$ matrix A over $\mathbb{F}_+$ is said to be **invertible** if there exist an $n \times n$ matrix B over $\mathbb{F}_+$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

Note : 6.3

A square matrix A over $\mathbb{F}_+$ is invertible if and only if some permutation of its rows is a diagonal matrix all of whose diagonal entries are nonzero in $\mathbb{F}_+$.

Notation : 6.4

Lowercase boldface letters will represent column vectors, all vectors **u** are column vectors ($\mathbf{u}^t$ is a row vector) for $\mathbf{u} \in \mathbb{F}_+^m$ $[=\mathbf{M}_{m,1}(\mathbb{F}_+)]$.

Definition : 6.5

The rank of $A \in \mathbf{M}_{m,n}(\mathbb{F}_+)$ is 1 if and only if there exist nonzero vectors $\mathbf{a} \in \mathbf{M}_{m,1}(\mathbb{F}_+)$ and $\mathbf{b} \in \mathbf{M}_{n,1}(\mathbb{F}_+)$ such that $A=\mathbf{ab}^t$. We call **a** the **left factor**, and **b** the **right factor** of A . But these vectors **a** and **b** are not uniquely determined by A as shown in the following example.

Example : 6.6

$$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1/2\\ 1\end{bmatrix}\begin{bmatrix}2 & 4\end{bmatrix} = \dots\dots\dots\dots$$

Lemma : 6.7

For any factorization $\mathbf{ab}^t$ of an $m \times n$ rank-1 matrix A over $\mathbb{F}_+$, $|\mathbf{a}|$ and $|\mathbf{b}|$ are uniquely determined by A .

Note : 6.8

Even though the factorization of A are not unique, Lemma 6.6 shows that the perimeter of A is unique, and that $P(A)=P(A^*)$.

Theorem : 6.9

If A, B and A+B are rank-1 matrices in $\mathbf{M}_{m,n}(\mathbb{F}_+)$, then $P(A+B)<P(A)+P(B)$.

Proof :

Since $P(A)=P(A^*)$, it is sufficient to consider A, B, $A+B \in \mathbf{M}_{m,n}(\mathbb{B})$. Let $A=\mathbf{ax}^t$, $B=\mathbf{by}^t$ and $A+B=\mathbf{cz}^t$ be any factorizations of A, B and A+B. Then we have for all i,j

$$a_i\mathbf{x} + b_i\mathbf{y} = c_i\mathbf{z} \qquad \text{---------------------(2)}$$

and $$x_j\mathbf{a} + y_j\mathbf{b} = z_j\mathbf{c}. \qquad \text{---------------------(3)}$$

If $B \le A$, then we have A+B=A. Thus we obtain that

$$P(A+B) = P(A) < P(B) + P(B)$$

because $P(B) \neq 0$, as required.

Similar argument shows that $A \le B$, then $P(A+B) < P(A) < P(A) + P(B)$. So we can assume that $A \le B$ and $B \le A$. We consider three cases :

Case 1

$\mathbf{a} \le \mathbf{b}$ and $\mathbf{b} \le \mathbf{a}$. The equation (2) implies that $a_i\mathbf{x} = c_i\mathbf{z}$ and $b_j\mathbf{y} = c_j\mathbf{z}$ for some nonzero a_i , c_i , b_j , $c_j \in \mathbb{B}$ so that $\mathbf{x} = \mathbf{y} = \mathbf{z}$. Thus we have the following $P(A+B) = P((\mathbf{a}+\mathbf{b})\mathbf{z}^t) = |\mathbf{a}+\mathbf{b}|+|\mathbf{z}| < (|\mathbf{a}|+|\mathbf{z}|)+(|\mathbf{b}|+|\mathbf{z}|) = P(A)+P(B)$ as required.

Case 2

$\mathbf{a} \le \mathbf{b}$. Then $\mathbf{x} \le \mathbf{y}$. Thus (3) implies that $x_j\mathbf{a} = z_j\mathbf{c}$ for some nonzero x_j , $z_j \in \mathbb{B}$ and so that $\mathbf{b} \le \mathbf{c}$. Therefore $\mathbf{a} = \mathbf{b} = \mathbf{c}$ and we have $P(A+B) = P(\mathbf{c}(\mathbf{x}+\mathbf{y})^t) = |\mathbf{c}|+|\mathbf{x}+\mathbf{y}| < (|\mathbf{c}|+|\mathbf{x}|)+(|\mathbf{c}|+|\mathbf{y}|) = P(A)+P(B)$ as required.

Case 3

$\mathbf{b} \le \mathbf{a}$. It is similar to the case 2.

Proposition : 6.10

If T is a (U,V)-operator on $\mathbf{M}_{m,m}(\mathbb{F}_+)$, then T preserves both rank and perimeter of rank-1 matrices.

Proof :

Since T is a (U,V)-operator, there exist invertible matrices $U \in \mathbf{M}_{m,m}(\mathbb{F}_+)$ and $V \in \mathbf{M}_{n,n}(\mathbb{F}_+)$ such that either $T(A) = UAV$, or $m = n$ and $T(A) = UA^tV$ for all A in $\mathbf{M}_{m,n}(\mathbb{F}_+)$. Let A be a matrix in $\mathbf{M}_{m,n}(\mathbb{F}_+)$ with $r(A) = 1$ and $A = \mathbf{a}\mathbf{b}^t$ be any factorization of A with $P(A) = |\mathbf{a}|+|\mathbf{b}|$. For the case $T(A) = UAV$,

$$T(A) = UAV = (U\mathbf{a})(\mathbf{b}^tV) = (U\mathbf{a})(V^t\mathbf{b})^t.$$

Thus we have

$$r(T(A)) = r((U\mathbf{a})(V^t\mathbf{b})^t) = 1,$$

and

$P(T(A)) = |U\mathbf{a}|+|V^t\mathbf{b}| = |\mathbf{a}|+|\mathbf{b}|$ by the similar method as above.

For the case $T(A) = UA^tV$, we can show that $r(T(A)) = 1$ and $P(T(A)) = |\mathbf{a}|+|\mathbf{b}|$ by the similar method as above. Therefore a (U,V)-operator preserves the rank and the perimeter of rank-1 matrices over $U \in \mathbb{F}_+$.

Definition : 6.11

We say that A is a **row (column) matrix** if A has a nonzero entries only in one row (column).

Lemma : 6.12

Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbb{F}_+)$. If T preserves rank and perimeter 2 of rank-1 matrices, then the following statements hold :

(1) T maps a cell into a nonzero scalar multiple of a cell.

(2) T maps a row (or a column) of a matrix into a row or a column (if m=n) with scalar multiplication.

Definition :6.13

Let $R_i = \{E_{i,j} | 1 \le j \le n\}$, $C_j = \{E_{i,j} | 1 \le i \le m\}$, $\mathfrak{R} = \{R_i | 1 \le i \le m\}$ and $\mathcal{C} = \{C_j | 1 \le j \le n\}$. For a linear operator T on $\mathbf{M}_{m,n}(\mathbb{F}_+)$, define $T^*(A) = [T(A)]^*$ for all A in $\mathbf{M}_{m,n}(\mathbb{F}_+)$. Let $T^*(R_i) = \{T^*(E_{i,j}) | 1 \le j \le n\}$, for all i = 1,2,........,m and $T^*(C_j) = \{T^*(E_{i,j}) | 1 \le i \le m\}$ for all j = 1,2,........,n.

Lemma :6.14

Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbb{F}_+)$. Suppose that T preserves rank and perimeters 2 and p (for some $p \ge 3$) of every rank-1 matrix. Then

(1) T maps two distinct cells in a row (or a column) into a nonzero scalar multiple of two distinct cells in a row or in a column ;

(2) In the case of m=n, if T maps some R_i into a row (or column) matrix then T maps every row matrix into a row (or column) matrix, and if T maps some C_j into a row (column) matrix then T maps every column matrix into a row (column) matrix.

Lemma : 6.15

Let T be a linear operator defined by $T(A)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{i,j}u_{i,j}\left(E_{T'(i,j)}\right)$ for some function $T':\Delta_{m,n}\rightarrow\Delta_{m,n}$ and for some nonzero scalar $u_{i,j}\in\mathbb{F}_+$, i = 1,2,……,m, j = 1,2,……,n. If T preserves both rank and perimeters 2 and k (for some $k\geq 4$, $k\neq n+1$) of rank-1 matrices,then the corresponding map T' is a bijection on $\Delta_{m,n}$.

Theorem : 6.16

Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbb{F}_+)$. Then the following are equivalent:

(1) T is a (U,V)-operator ;

(2) T preserves both rank and perimeter of rank-1 matrices ;

(3) T preserves both rank and perimeters 2 and k (for some $k\geq 4, k\neq n+1$) of rank-1 matrices.

Proof :

(1) implies (2) by Proposition 6.10. It is obvious that (2) implies (3). We now show that (3) implies (1).

Assume (3). Then the corresponding mapping $T':\Delta_{m,n}\rightarrow\Delta_{m,n}$ is a bijection by Lemma 6.15.

By Lemma 6.14, there are two cases ;

(a) T^* maps $\mathfrak{R}$ onto $\mathfrak{R}$ and maps $\mathfrak{C}$ onto $\mathfrak{C}$ or

(b) T^* maps $\mathfrak{R}$ onto $\mathfrak{C}$ and $\mathfrak{C}$ onto $\mathfrak{R}$.

Case (a)

We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i,j, where σ and τ are permutations of {1,2,……,m} and {1,2,…...,n}, respectively. Let P and Q be the permutation matrices corresponding to σ and τ, respectively. Then for any $E_{i,j} \in \mathbb{E}_{m,n}$, we can write $T(E_{i,j}) = b_{i,j}\, E_{\sigma(i)\tau(i)}$ for some nonzero scalar $b_{i,j} \in \mathbb{F}_+$. Now we claim that for all i, $l \in$ {1,2,……,m} and all j,r∈{1,2,……,n},

$$\frac{b_{i,j}}{b_{i,r}} = \frac{b_{l,j}}{b_{l,r}}$$

Consider a matrix $E = E_{i,j} + E_{i,r} + E_{l,j} + E_{l,r}$ with rank 1. Then we have

$$T(E) = b_{i,j}\, E_{\sigma(i)\tau(j)} + b_{i,r}\, E_{\sigma(i)\tau(r)} + b_{l,j}\, E_{\sigma(l)\tau(j)} + b_{l,r}\, E_{\sigma(l)\tau(r)} \,.$$

Since T(E) has rank 1, it follows that $\frac{b_{i,j}}{b_{i,r}} = \frac{b_{l,j}}{b_{l,r}}$. Let $C \in \mathbf{M}_{m,m}(\mathbb{F}_+)$ and $D \in \mathbf{M}_{n,n}(\mathbb{F}_+)$ be diagonal matrices such that $c_{1,1} = 1, d_{1,1} = 1, c_{i,i} = \frac{b_{i,1}}{b_{1,1}}$ and $d_{j,j} = b_{1,j}$ for all i = 1,2,……,m and j = 1,2,……,n. Then $b_{i,j} = c_{j,j} d_{j,j}$ for all i∈{1,2,……,m} and j∈{1,2,……,n}.

Let A = $[a_{i,j}]$ be any $m \times n$ matrix in $\mathbf{M}_{m,n}(\mathbb{F}_+)$. Then we have

$$T(A) = T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} E_{i,j}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} T(E_{i,j})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} b_{i,j} E_{\sigma(i)\tau(j)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,i} a_{i,j} E_{\sigma(i)\tau(j)} d_{j,j}$$

= CPAQD.

Since CP = U is an $m \times m$ invertible matrix and QD = V is an invertible matrix, it follows that T is a (U,V)-operator.

Case (b)

Then m = n and $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all i,j, where σ and τ are permutations of {1,2,……,m}. By an argument similar to case (a), we obtain that T(A) is of the form $T(A) = CPA^tQD$. Thus T is a (U,V)-operator.

Definition : 6.17

We say that a linear operator T on $\mathbf{M}_{m,n}(\mathbb{F}_+)$ strongly preserves perimeter k of rank-1 matrices if P(T(A))=k if and only if P(A)=k.

Example : 6.18

Consider a linear operator T on $\mathbf{M}_{2,2}(\mathbb{F}_+)$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c+d)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then T preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2, since $T\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$ with $P\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = 4$ but $P\left(\begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}\right) = 2$.

Theorem : 6.19

Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbb{F}_+)$. Then T preserves both rank and perimeter of rank-1 matrices if and only if it strongly preserves perimeter 2 and preserves perimeter 3 of rank-1 matrices.

Proof :

Suppose T strongly preserves perimeter 2 and preserves perimeter 3 of rank-1 matrices. Then T maps each row of a matrix into a nonzero scalar multiple of a row or a column (if m=n). Since T strongly preserves perimeter 2, T maps each cell onto a nonzero scalar multiple of a cell. This means that the corresponding mapping T' is a bijection. Thus T preserves both rank and perimeter of rank-1 matrices by a similar method as in the proof of Theorem 6.16. The converse is immediate.

Theorem : 6.20

Let T be a linear operator on $\mathbf{M}_{m,n}(\mathbb{F}_+)$ that preserves the rank-1 matrices. Then T preserves the perimeter of rank-1 matrices if and only if it strongly preserves perimeter 2 of rank-1 matrices.

Proof :

Suppose T strongly preserves perimeter 2 of rank-1 matrices. Then T maps each cell onto a nonzero scalar multiple of a cell. Thus T' is a bijection. Since T preserves rank 1, it maps a row of a matrix into a row or a column (if m=n). Thus T preserves both rank and perimeter of rank-1 matrices by similar methods to the proof of Theorem 6.16. The converse is immediate.