CHAPTER – I

Preliminaries

Definition 1.1: [Zadeh, 1965] Let X be the Universal set. A ***fuzzy set*** (FS) A in X is represented by

$$A = \{< x, \mu_A(x) > | \, x \in X\},$$

where the function $\mu_A : X \to [0,1]$ is the membership degree of the element x in the fuzzy set A.

Definition 1.2: [Rosenfeld, 1975] A fuzzy set on $X \times X$ is said to be a ***fuzzy relation*** (FR) on X, denoted by

$$B = \{< xy, \mu_B(xy) > | \, xy \in X \times X\},$$

where $\mu_B : X \times X \to [0,1]$ represents the membership grades of A.

Definition 1.3: [Atanassov, 1986] Let P be an ***intuitionistic fuzzy set*** (IFS) in the universe of discourse X, shown as follows:

$$P = \{< x, \mu_P(x), \nu_P(x) > | \, x \in X\},$$

where $\mu_P(x) : X \to [0,1]$ and $\nu_P(x) : X \to [0,1]$ satisfy $0 \le \mu_P(x) + \nu_P(x) \le 1$ for all $x \in X$, $\mu_P(x)$ and $\nu_P(x)$ denote the membership degree and non-membership degree of element x belonging to the IFS P, respectively. Moreover, $\pi_P(x) = 1 - \mu_P(x) - \nu_P(x)$ is called the hesitancy degree of element x belonging to the IFS P.

Definition 1.4: [Shannon and Atanassov, 1994] An intuitionistic fuzzy set on $X \times X$ is said to be an ***intuitionistic fuzzy relation*** (IFR) on X, denoted by

$$B = \{< xy, \mu_B(xy), \nu_B(xy) > | \, x \in X\},$$

where $\mu_B : X \times X \to [0,1]$ and $\nu_B : X \times X \to [0,1]$ represents the membership and non-membership grades of *B*, respectively, such that $0 \le \mu_B(xy) + \nu_B(xy) \le 1$ for all $x, y \in X$.

Definition 1.5: [Yager, 2013] Let *P* be an ***Pythagorean fuzzy set*** (PFS) in the universe of discourse *X*, shown as follows:

$$P = \{< x,\ \mu_P(x),\ \nu_P(x) >|\ x \in X\},$$

where $\mu_P(x) : X \to [0,1]$ and $\nu_P(x) : X \to [0,1]$ satisfy $0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1$ for all $x \in X$, $\mu_p(x)$ and $\nu_P(x)$ denote the membership degree and non-membership degree of element *x* belonging to the PFS *P*, respectively. Moreover, $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \nu_P^2(x)}$ is called the hesitancy degree of element *x* belonging to the PFS *P*. For convenience, we introduce a Pythagorean fuzzy number denoted by $\beta = P(\mu_\beta, \nu_\beta)$, where $\mu_\beta, \nu_\beta \in [0,1]$ and $0 \le (\mu_\beta)^2 + (\nu_\beta)^2 \le 1$.

Definition 1.6: [Naz, Ashraf and Akram, 2018] A Pythagorean fuzzy set on $X \times X$ is said to be a ***Pythagorean fuzzy relation*** (PFR) on *X*, denoted by

$$B = \{< xy, \mu_B(xy), \nu_B(xy) >|\ x \in X\},$$

where $\mu_B : X \times X \to [0,1]$ and $\nu_B : X \times X \to [0,1]$ represents the membership and non-membership grades of *B*, respectively, such that $0 \le \mu_B^2(xy) + \nu_B^2(xy) \le 1$ for all $x, y \in X$.

Definition 1.7: [Smarandache, 1999] Let *X* be a universe. A ***neutrosophic set*** (NS) *A* over *X* is defined by

$$P = \{< x, (T_P(x),\ I_P(x),\ F_P(x)) >|\ x \in X\},$$

where $T_P(x), I_P(x)$ and $F_P(x)$ are called truth-membership function, indeterminacy membership function and falsity-membership function, respectively. They are, respectively, defined by $T_P(x): X \to [^-0,1^+]$, $I_P(x): X \to [^-0,1^+]$, $F_P(x): X \to [^-0,1^+]$ such that $^-0 \le T_P(x) + I_P(x) + F_P(x) \le 3^+$.

Definition 1.8: [Wang, Smarandache, Zhang and Sunderraman, 2010] Let X be a universe. A ***single-valued neutrosophic set*** (SVNS) A over X is defined by

$$P = \{< x, (T_P(x),\ I_P(x),\ F_P(x)) >| x \in X\},$$

where $T_P(x), I_P(x)$ and $F_P(x)$ are called truth-membership function, indeterminacy membership function and falsity-membership function, respectively. They are, respectively, defined by $T_P(x): X \to [0,1]$, $I_P(x): X \to [0,1]$, $F_P(x): X \to [0,1]$ such that $0 \le T_P(x) + I_P(x) + F_P(x) \le 3$.

Definition 1.9: [Deli and Smarandache, 2015] An bipolar ***neutrosophic set*** (BNS) A in X is defined as an object of the form

$$P = \{< x, T_P^+(x),\ I_P^+(x),\ F_P^+(x), T_P^-(x), I_P^-(x),\ F_P^-(x) >| x \in X\},$$

where $T_P^+, I_P^+, F_P^+ : X \to [0,1]$, $T_P^-, I_P^-, F_P^- : X \to [0,1]$.

Definition 1.10: [Jun, Kim and Yang, 2012] Let X be a non-empty set. A ***neutrosophic cubic set*** (NCS) in X is a pair $A = (A, P)$ where $A = \{ < x, A_T(x), A_I(x), A_F(x) >| x \in X \}$ is an interval neutrosophic set in X and $P = \{ < x, \lambda_T(x), \lambda_I(x), \lambda_F(x) >| x \in X \}$ is a neutrosophic set in X.

Definition 1.11: [Kutlu Gündoğdu and Kahraman, 2019] Let X be a universe. Then the set

$$P = \{< x, (T_P(x),\ I_P(x),\ F_P(x)) >| x \in X\},$$

is said to be ***spherical fuzzy set*** (SFS), where $T_P(x): X \to [0,1]$, $I_P(x): X \to [0,1]$ and $F_P(x): X \to [0,1]$ are the degree of positive-membership function of x in X, the degree of neutral-membership function of x in X and the degree of negative-membership function of x in X, respectively. Also T_P, I_P and F_P satisfy the following condition:

$$(\forall x \in X) \quad (0 \leq (T_P(x))^2 + (I_P(x))^2 + (F_P(x))^2 \leq 1).$$

Definition 1.12: [Behzad, 1965] A ***graph*** G is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G, called edges. The vertex set and the edge set of G are respectively denoted by $V(G)$ and $E(G)$ is denoted by $G=(V, E)$.

Definition 1.13: [Behzad, 1965] If $e=(u,v)\in E(G)$, the edge e joins the vertices u and v, then $e=uv$ and u and v are called ***adjacent vertices***. Further, the vertices u and v are said to be ***incident*** with edge e. If two vertices are not joined by an edge, then they are ***non-adjacent***. If distinct edges e_1 and e_2 are incident with a common vertex, then it is said to be ***adjacent edges***.

Definition 1.14: [Behzad, 1965] The cardinality of the vertex set $V(G)$ is said to be the ***order*** of the graph G.

Defintion 1.15: [Behzad, 1965] A graph G is ***simple*** if it has no loops and no two of its edges join the same pair of vertices.

Definition 1.16: [Behzad, 1965] A graph G is ***finite*** if both its vertex set and edge set are finite.

Definition 1.17: [Behzad, 1965] The ***degree*** of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg(v)$ or $d(v)$.

Definition 1.18: [Behzad, 1965] The ***total degree*** of G is the sum of the degrees of all the vertices of G.

Definition 1.19: [Behzad, 1965] A graph *G* is said to be ***regular*** of degree *r* if every vertex of *G* has degree *r*. Such graphs are called ***r-regular graphs***. Any 3-regular graph is called a ***cubic graph***.

Definition 1.20: [Behzad, 1965] An ***undirected graph*** is a graph in which the edges do not point in any direction (ie. the edges are bidirectional).

Definition 1.21: [Behzad, 1965] A ***connected graph*** is a graph in which there is always a path from a vertex to any other vertex.

Defintion 1.22: [Behzad, 1965] An ***acyclic graph*** is one that contains no cycles. A ***tree*** is a connected acyclic graph.

Definition 1.23: [Behzad, 1965] A ***spanning tree*** is a sub-graph of an undirected connected graph, which includes all the vertices of the graph with a minimum possible number of edges.

Definition 1.24: [Behzad, 1965] A ***minimum spanning tree*** (MST) of a graph *G* is a spanning tree whose weight is minimum among all spanning trees of the graph *G*.

Defintion 1.25: [Behzad, 1965] The ***complement*** $\overline{G}$ of a graph *G* is the graph with vertex set *V*(*G*) such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in *G*.

Definition 1.26: [Behzad, 1965] A ***proper vertex coloring*** of a graph *G* is an assignment of colors to the vertices of *G*, one color to each vertex, so that adjacent vertices are colored differently. A ***k-coloring*** of a graph *G* is a function $c : V(G) \rightarrow \{1,2,...,k\}$. The labels are colors; the vertices with color i form a color class.

Definition 1.27: [Behzad, 1965] A graph G is ***k-colorable*** if it has a proper k-coloring. The ***chromatic number*** $\chi(G)$ is the minimum k such that G is k-colorable. If $\chi(G) = k$, then G is said to be ***k-chromatic***.

Defintion 1.28: [Rosenfeld, 1975] A ***fuzzy graph*** (FG) on a non-empty set X is a pair G=(A, B) with A a fuzzy set on X and B a fuzzy relation on X such that

$$\mu_B(st) \leq \mu_A(s) \wedge \mu_A(t) \text{ for all } s, t \in X,$$

where $A : X \rightarrow [0,1]$ and $A : X \times X \rightarrow [0,1]$.

Definition 1.29: [Shannon and Atanassov, 1994] An ***intuitionistic fuzzy graph*** (IFG) on a non-empty set X is a pair G=(A, B) with A a intuitionistic fuzzy set on X and B a intuitionistic fuzzy relation on X such that

$$\mu_B(st) \leq \mu_A(s) \wedge \mu_A(t),$$

$$\nu_B(st) \geq \nu_A(s) \vee \nu_A(t)$$

and $0 \leq \mu_B(st) + \nu_B(st) \leq 1$ for all $s, t \in X$, where $\mu_B : X \times X \rightarrow [0,1]$ and $\nu_B : X \times X \rightarrow [0,1]$ represent the membership and non-membership grades of B, respectively.

Definition 1.30: [Akram, 2011] An bipolar ***fuzzy graph*** (BFG) on a non-empty V is a pair G=(X,Y) where, $X = \left(\tau_X^P, \tau_X^N\right)$ is an bipolar fuzzy set on V and $Y = \left(\tau_Y^P, \tau_Y^N\right)$ is an bipolar fuzzy relation on V such that $\tau_Y^P(xy) \leq \min\left(\tau_X^P(x), \tau_X^P(y)\right)$ and $\tau_Y^N(xy) \geq \max\left(\tau_X^N(x), \tau_X^N(y)\right)$, for all $x, y \in V$. y is called relations on x.

Definition 1.31: [Naz, Ashraf and Akram, 2018] A ***Pythagorean fuzzy graph*** (PFG) on a non-empty set *X* is a pair *G*=(*A*, *B*) with *A* a Pythagorean fuzzy set on *X* and *B* a Pythagorean fuzzy relation on *X* such that

$$\mu_B(st) \le \mu_A(s) \wedge \mu_A(t),$$

$$\nu_B(st) \ge \nu_A(s) \vee \nu_A(t)$$

and $0 \le \mu_B^2(st) + \nu_B^2(st) \le 1$ for all $s, t \in X$, where $\mu_B : X \times X \to [0,1]$ and $\nu_B : X \times X \to [0,1]$ represent the membership and non-membership grades of *B*, respectively.

Definition 1.32: [Antony Crispin Sweety, Vaiyomathi and Nirmala Irudayam, 2020] Let $G^* = (V, E)$ be a graph and $G = (P, Q)$ is an bipolar neutrosophic cubic graph of G^*, if

$$P = (A, \lambda) = \left(\left(T_A^+, T_\lambda^+\right), \left(I_A^+, I_\lambda^+\right), \left(F_A^+, F_\lambda^+\right), \left(T_A^-, T_\lambda^-\right), \left(I_A^-, I_\lambda^-\right), \left(F_A^-, F_\lambda^-\right)\right)$$

is the bipolar neutrosophic cubic set representation of vertex set V and

$$Q = (B, \mu) = \left(\left(T_B^+, T_\mu^+\right), \left(I_B^+, I_\mu^+\right), \left(F_B^+, F_\mu^+\right), \left(T_B^-, T_\mu^-\right), \left(I_B^-, I_\mu^-\right), \left(F_B^-, F_\mu^-\right)\right)$$

is the bipolar neutrosophic cubic set representation of the edge set E such that

1. $$\begin{aligned} &T_B^+(u_i v_i) \le r\min\{T_A^+(u_i), T_A^+(v_i)\},\ T_\mu^+(u_i v_i) \ge r\max\{T_\lambda^+(u_i), T_\lambda^+(v_i)\} \\ &T_B^-(u_i v_i) \ge r\max\{T_A^-(u_i), T_A^-(v_i)\},\ T_\mu^-(u_i v_i) \ge r\min\{T_\lambda^-(u_i), T_\lambda^-(v_i)\} \end{aligned}$$

2. $$\begin{aligned} &I_B^+(u_i v_i) \le r\min\{I_A^+(u_i), I_A^+(v_i)\},\ I_\mu^+(u_i v_i) \ge r\max\{I_\lambda^+(u_i), I_\lambda^+(v_i)\} \\ &I_B^-(u_i v_i) \ge r\max\{I_A^-(u_i), I_A^-(v_i)\},\ I_\mu^-(u_i v_i) \ge r\min\{I_\lambda^-(u_i), I_\lambda^-(v_i)\} \end{aligned}$$

3. $$\begin{aligned} &F_B^+(u_i v_i) \le r\max\{F_A^+(u_i), F_A^+(v_i)\},\ F_\mu^+(u_i v_i) \ge r\min\{F_\lambda^+(u_i), F_\lambda^+(v_i)\} \\ &F_B^-(u_i v_i) \ge r\min\{F_A^-(u_i), F_A^-(v_i)\},\ F_\mu^-(u_i v_i) \ge r\max\{F_\lambda^-(u_i), F_\lambda^-(v_i)\} \end{aligned}$$

Let $G^* = (V, E)$ be a graph and $G(P, Q)$ is an bipolar neutrosophic cubic graph of G^*, if

$$P = (A, \lambda) = \left(\left(T_A^+, T_\lambda^+\right), \left(I_A^+, I_\lambda^+\right), \left(F_A^+, F_\lambda^+\right), \left(T_A^-, T_\lambda^-\right), \left(I_A^-, I_\lambda^-\right), \left(F_A^-, F_\lambda^-\right)\right)$$

is the bipolar neutrosophic cubic graph representation of vertex set *V* and

$$P = (B, \mu) = \left(\left(T_B^+, T_\mu^+\right), \left(I_B^+, I_\mu^+\right), \left(F_B^+, F_\mu^+\right), \left(T_B^-, T_\mu^-\right), \left(I_B^-, I_\mu^-\right), \left(F_B^-, F_\mu^-\right)\right)$$

is the bipolar neutrosophic cubic graph representation of edge set E and λ and μ are bipolar neutrosophic cubic sets.

Definition 1.33: [Akram, Saleem and Al-Hawary, 2020] A ***spherical fuzzy graph*** (SFG) on an underlying set V is a pair $G=(A,B)$ where A is a spherical fuzzy set in V and B is a spherical fuzzy relation on $V \times V$ such that

$T_B(x,y) \le \min\ (T_A(x), T_A(y)),$

$I_B(x,y) \le \min\ (I_A(x), I_A(y)),$

$F_B(x,y) \le \max\ (F_A(x), F_A(y))$

where T_B denote the truth membership function, I_B denote the indeterminacy membership function, F_B denote the falsity membership function and fulfils the following condition: $0 \le \left((T_B)^2 + (I_B)^2 + (F_B)^2 \right) \le 1$ and where A is a spherical fuzzy vertex set and B is a spherical fuzzy edge set of G.

Definition 1.34: [Rifayathali, Prasanna and Ismail Mohideen, 2018] The intuitionistic fuzzy graph $\hat{G}=(A,B)$ consisting of a family $\Gamma=\{\alpha_1,\alpha_2,...,\alpha_k\}$ of Pythagorean fuzzy set A is called a k-vertex coloring if:

a) $\max(\alpha_i(x)) = A$ for all $x \in A$,

b) $\max(\alpha_i, \alpha_j) = 0$ *and*

c) For every strong edge

$\min\{\gamma_i(\mu_1(x)), \gamma_i(\mu_1(y))\} = 0$ and $\max\{\gamma_i(\nu_1(x)), \gamma_i(\nu_1(y))\} = 1,\ (1 \le \mathrm{i} \le \mathrm{k})$.

The least value of k for which the $\hat{G}$ has a k-vertex color is called the ***coloring number of the intuitionistic fuzzy graph*** $\hat{G}$ or the intuitionistic fuzzy chromatic number and it is denoted by $\chi(\hat{G})$.

Definition 1.35: [Yamuna, Arun Prakash and Indra Kumar, 2020] The Pythagorean fuzzy graph $\hat{G}=(V,E)$ consisting of a family $\Gamma=\{\gamma_1,\gamma_2,...,\gamma_k\}$ of Pythagorean fuzzy set V is called a k-vertex coloring if:

a) $\vee(\gamma_i(x))=V$ for all $x\in V$,

b) $\gamma_i \wedge \gamma_j = 0$ *and*

c) For every strong edge xy of $\hat{G}$, $\min\{\gamma_i(\mu_1(x)),\gamma_i(\mu_1(y))\}=0$ and $\max\{\gamma_i(\nu_1(x)),\gamma_i(\nu_1(y))\}=1$, $(1\le i\le k)$.

The least value of k for which the $\hat{G}$ has a k-vertex color is called the ***coloring number of the Pythagorean fuzzy graph*** $\hat{G}$ or the Pythagorean fuzzy chromatic number and it is denoted by $\chi(\hat{G})$.

Definition 1.36: [Rohini, Venkatachalam, Broumi and Smarandache, 2019] A family $\Gamma=\{\gamma_1,\gamma_2,...,\gamma_k\}$ of single-valued neutrosophic fuzzy set is called a ***k-single-valued neutrosophic vertex coloring*** of a single-valued neutrosophic graph $G=(X,Y)$ if

a) $\vee(\gamma_i(x))=X$ for all $x\in X$,

b) $\gamma_i \wedge \gamma_j = 0$ *and*

c) For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)),\gamma_i(m_1(y))\}=0$, $\min\{\gamma_i(i_1(x)),\gamma_i(i_1(y))\}=0$, $\max\{\gamma_i(n_1(x)),\gamma_i(n_1(y))\}=1$, $(1\le i\le k)$.

This k-single-valued neutrosophic vertex coloring of G is denoted by $\chi_\gamma(G)$, is called the single-valued neutrosophic chromatic number of the single-valued neutrosophic graph G.