7. Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Feedback, Randomized J Vacation and Orbital Search

A single server batch arrival retrial G-queue with multistages of service is discussed. Each stage consists of multi-optional services. The unsatisfied customers are allowed to join the orbit to get re-service. Whenever the system becomes empty, the server goes for vacation and takes at most J vacations repeatedly until at least one customer is recorded in the orbit. At the end of J^{th} vacation, even if the orbit is empty, the server remains idle in the system. On vacation completion, the server searches for the customer in the orbit or remains idle. The orbit size distribution and performance indices are determined using generating function method. Numerical results are also presented.

7.1 Model Description

Consider a single server retrial queueing model with positive and negative customers. The assumptions of the model under consideration are described below.

Arrival process

The positive customers arrive in batches according to a compound Poisson process with rate λ^+ and the negative customers arrive in single according to Poisson process with rate λ^-. Let Y be the random variable denoting the batch size of positive customers with distribution function $P(Y=k) = C_k$, k=1,2,3,..... and probability generating function C(z) having the first two moments m_1 and m_2.

Retrial process

Upon arrival of the batch of positive customers, if the server is not in the system, then all the arriving customers join the orbit and retrial after some random time. If the server is free, then one of the customers from the arriving batch starts the service immediately and others join the orbit. The retrial time follows general distribution with distribution function $A(x)$, Laplace-Stieltjes transform $A^*(s)$ and the hazard rate function $\eta(x)$.

Service process

The server renders two phase service. All the arriving customers benefit the first phase of essential service. The second phase includes M stages of sequential services. In particular, stage i consists of k_i multi-optional services. The essential service time follows general distribution with distribution function $B_0(x)$, Laplace-Stieljes transform $B_0^*(s)$, n^{th} factorial moment $\mu_0^{(n)}$ and the conditional completion rate $\mu_0(x)$. The service time of i^{th} stage, j_i^{th} optional service follows a general distribution with distribution function $B_{i,j_i}(x)$, Laplace-Stieljes transform $B_{i,j_i}^*(s)$ n^{th} factorial moment $\mu_{i,j_i}^{(n)}$ and the conditional completion rate $\mu_{i,j_i}(x)$, $i = 2,3,4,.....,M$, $j_i = 1,2,3,......,k_i$.

Feedback process

After the completion of essential service, the customer may proceed to first stage in phase 2 and opt j_1^{th} (j_1 = 1,2,3,....,k_1) optional service with probability p_{j_1}, join the orbit as a feedback customer with probability δ_0 or leave the system with probability q_0. Likewise, after the completion of i^{th} stage the customer proceeds to $(i+1)^{th}$ stage and opts j_{i+1}^{th} (j_{i+1} = 1,2,3,....,k_{i+1}) optional service with probability $p_{j_{i+1}}$, joins the orbit as a feedback customer with probability δ_i or leaves the system with probability q_i. After the final stage (M^{th} stage) service, the customer either joins the orbit with probability δ_M or leaves the system with probability q_M.

Breakdown and Repair process

The arrival of the negative customer makes the server down and pushes out the customer in service from the system. The repair of the failed server starts instantaneously. The repair time of the server failed during the essential service follows general distribution with distribution function $R_0(x)$, Laplace-Stieltjes transform $R_0^*(s)$, n^{th} factorial moment $\beta_0^{(n)}$, and the conditional completion rate $\beta_0(x)$. The repair time of the server failed during i^{th} stage j_i^{th} optional service follows general distribution with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R_{i,j_i}^*(s)$, n^{th} factorial moment $\beta_{i,j_i}^{(n)}$ and the conditional completion rate $\beta_{i,j_i}(x)$, $i = 1,2,3,.....,M$, $j_i = 1,2,3,.....,k_i$.

Vacation process

Whenever the orbit is empty, the server leaves for a vacation of random length V. At the vacation completion epoch, if the system is still empty, the server either remains idle in the system with probability p or takes another vacation with its complementary probability $\bar{p}$ of same length. Such pattern continues until at least one customer is recorded in the system or the number of vacation reaches J. The vacation time follows general distribution with distribution function $V(x)$, Laplace-Stieltjes transform $V^*(s)$, n^{th} factorial moment γ_n, and the conditional completion rate $\gamma(x)$.

Orbital Search:

On vacation completion, if the orbit is non-empty, the server searches for customers with probability θ or remains idle with the complementary probability $\bar{\theta}=1-\theta$.

The diagrammatic representation of the proposed model is shown in Fig. 7.1

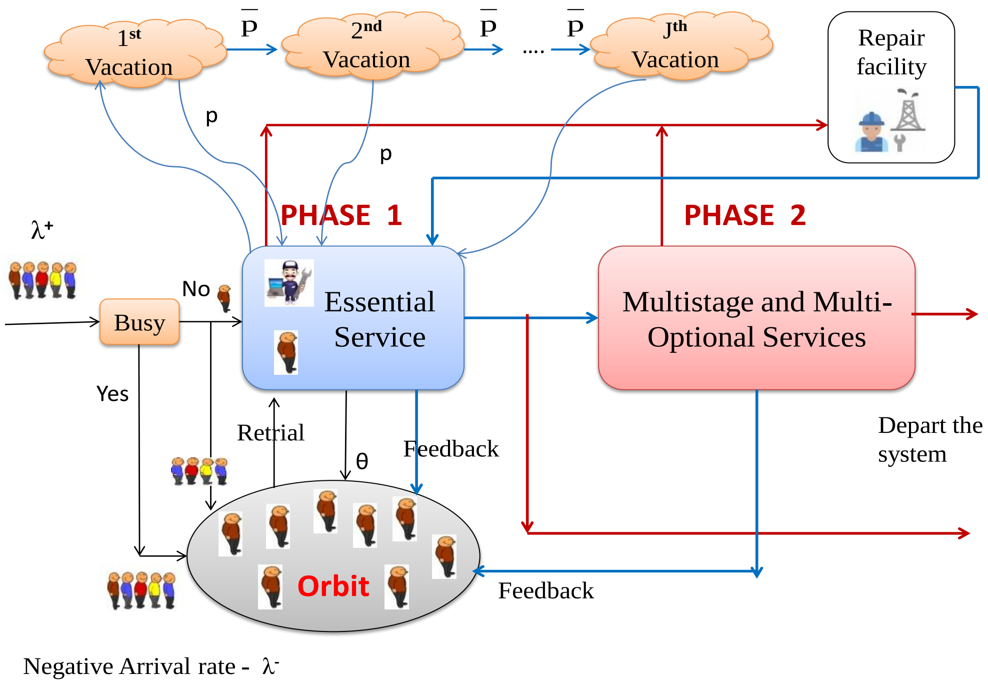

Fig. 7.1 Batch arrival Retrial G-Queue with Multistage and Multi-Optional Services, Feedback, Randomized J Vacation and Orbital Search

7.2 Server State Probability

Define the server state as

$$S(t) = \begin{cases} 0, & \text{server is idle} \\ 1, & \text{server is busy in essential service} \\ 2, & \text{server is busy in } i^{th} \text{ stage } j_i^{th} \text{ optional service,} \\ 3, & \text{server in essential service is under repair} \\ 4, & \text{server in } i^{th} \text{ stage } j_i^{th} \text{ optional service is under repair} \\ 5, & \text{server is on } j^{th} \text{ vacation} \end{cases}$$

and the supplementary variables

ξ_0 – elapsed retrial time, ξ_1 – elapsed service time,

ξ_2 – elapsed repair time, ξ_3 – elapsed vacation time

Let N(t) denote the number of customers in the orbit at time t. For the process, $\{N(t)\ ;\ t \geq 0\}$, the transient probabilities are defined as given below.

$$I_0(t) = P\{S(t)=0,\ N(t)=0\}$$

$$I_n(x,t) = P\{S(t)=0,\ N(t)=n,\ x<\xi_0(t)\leq x+dx\},\quad n\geq 1$$

$$P_{0,n}(x,t)dx = P\{S(t)=1,\ N(t)=n,\ x<\xi_1(t)\leq x+dx\},\quad n\geq 0$$

$$P_{i,j_i,n}(x,t)dx = P\{S(t)=2,\ N(t)=n,\ x<\xi_1(t)\leq x+dx\}, n\geq 0, i=1,2,....,M, j_i=1,2,....,k_i$$

$$R_{0,n}(x,t)dx = P\{S(t)=3,\ N(t)=n,\ x<\xi_2(t)\leq x+dx\},\quad n\geq 0$$

$$R_{i,j_i,n}(x,t)dx = P\{S(t)=4, N(t)=n,\ x<\xi_2(t)\leq x+dx\},\quad n\geq 0,\ i=1,2,....,M, j_i=1,2,....,k_i$$

$$V_j(x,t)dx = P\{S(t)=5,\ N(t)=n,\ x<\xi_3(t)\leq x+dx\},\quad n\geq 0$$

7.3 Steady State Equations Governing the System

The steady state equations governing the model under consideration are given below.

$$\lambda^+ I_0 = \int_0^\infty V_{J,0}(x)\ \gamma(x)dx + p\sum_{j=1}^{J-1}\int_0^\infty V_{j,0}(x)\ \gamma(x)\,dx \tag{7.1}$$

$$\frac{d}{dx} I_n(x) = -(\lambda^+ + \eta(x))\, I_n(x), \quad n \geq 1 \tag{7.2}$$

$$\frac{d}{dx} P_{0,n}(x) = -(\lambda^+ + \lambda^- + \mu_0(x))\, P_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{0,n-k}(x), n \geq 0 \tag{7.3}$$

$$\frac{d}{dx} P_{i,j_i,n}(x) = -(\lambda^+ + \lambda^- + \mu_{i,j_i}(x))\, P_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k P_{i,j_i,n-k}(x), n \geq 0, \quad i = 1,2,\ldots,M, \; j_i = 1,2,\ldots,k_i \tag{7.4}$$

$$\frac{d}{dx} R_{0,n}(x) = -(\lambda^+ + \beta_0(x))\, R_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{0,n-k}(x), \; n \geq 0 \tag{7.5}$$

$$\frac{d}{dx} R_{i,j_i,n}(x) = -(\lambda^+ + \beta_{i,j_i}(x))\, R_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k R_{i,j_i,n-k}(x), n \geq 0, \quad i = 1,2,\ldots,M, \; j_i = 1,2,\ldots,k_i \tag{7.6}$$

$$\frac{d}{dx} V_{j,n}(x) = -(\lambda^+ + \gamma(x))\, V_{j,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} C_k V_{j,n-k}(x), \quad n \geq 0 \tag{7.7}$$

with boundary conditions

$$I_n(0) = \sum_{j=1}^{J} \bar{\theta}\int_0^\infty V_{j,n}(x)\gamma(x)dx + \delta_0\int_0^\infty P_{0,n-1}(x)\mu_0(x)dx + \sum_{i=1}^{M}\delta_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-1}(x)\mu_{i,j_i}(x)dx + q_0\int_0^\infty P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx + \int_0^\infty R_{0,n}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx, \quad n \geq 1 \tag{7.8}$$

$$P_{0,0}(0) = \lambda^+ C_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx + \theta\sum_{j=1}^{J}\int_0^\infty V_{j,1}(x)\gamma(x)\,dx \tag{7.9}$$

$$P_{0,n}(0) = \lambda^+ C_{n+1} I_0 + \lambda^+ \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)dx + \int_0^\infty I_{n+1}(x)\eta(x)dx + \theta\sum_{j=1}^{J}\int_0^\infty V_{j,n+1}(x)\gamma(x)\,dx, n \geq 1 \tag{7.10}$$

$$P_{1,j_1,n}(0) = p_{j_1}\int_0^\infty P_{0,n}(x)\mu_0(x)dx, \; n \geq 0, \quad j_1 = 1,2,\ldots,k_1 \tag{7.11}$$

$$P_{i,j_i,n}(0) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1},n}(x)\mu_{i,j_i}(x)dx, \; n \geq 0, \; i = 2,3,\ldots\ldots,M, \; j_i = 1,2,\ldots,k_i \tag{7.12}$$

$$R_{0,n}(0) = \lambda^-\int_0^\infty P_{0,n}(x)dx, \; n \geq 0 \tag{7.13}$$

$$R_{i,j_i,n}(0) = \lambda^-\int_0^\infty P_{i,j_i,n}(x)dx, \; n \geq 0, \; i = 1,2,3,\ldots\ldots,M, \; j_i = 1,2,\ldots,k_i \tag{7.14}$$

$$V_{1,n}(0) = \begin{cases} q_0 \int_0^{\infty} P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^{\infty} P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx \\ \quad + \int_0^{\infty} R_{0,n}(x)\beta_0(x)dx + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^{\infty} R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx, \ n = 0 \\ 0, \quad n \geq 1 \end{cases} \quad (7.15)$$

$$V_{j,n}(0) = \begin{cases} \bar{p} \int_0^{\infty} V_{j-1,n}(x)\,\gamma(x)\,dx, & n = 0, \ j = 2,3,4,\ldots,J \\ 0, \quad n \geq 1, & j = 2,3,4,\ldots\ldots,J \end{cases} \quad (7.16)$$

Let $I(x,z)$, $R_0(x,z)$, $P_{i,j_i}(x,z)$, $R_0(x,z)$, $R_{i,j_i}(x,z)$ and $V_j(x,z)$ be respectively the generating functions of $I_n(x)$, $P_{0,n}(x)$, $P_{i,j_i,n}(x)$, $R_{0,n}(x)$, $R_{i,j_i,n}(x)$ and $V_{j,n}(x)$.

Multiplying equations (7.2) to (7.14) by z^n and summing over all possible values of n, we get

$$\left(\frac{d}{dx} + \lambda^+ + \eta(x)\right) I(x,z) = 0 \quad (7.17)$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \lambda^- + \mu_0(x)\right) P_0(x,z) = 0 \quad (7.18)$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \lambda^- + \mu_{i,j_i}(x)\right) P_{i,j_i}(x,z) = 0, \quad i = 1,2,\ldots..,M, \quad j_i = 1,2,\ldots\ldots.,k_i \quad (7.19)$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \beta_0(x)\right) R_0(x,z) = 0 \quad (7.20)$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \beta_{i,j_i}(x)\right) R_{i,j_i}(x,z) = 0, \quad i = 1,2,\ldots..,M, \quad j_i = 1,2,\ldots\ldots..,k_i \quad (7.21)$$

$$\left(\frac{d}{dx} + \lambda^+(1 - C(z)) + \gamma(x)\right) V_j(x,z) = 0 \quad (7.22)$$

$$\begin{aligned} I(0,z) = & \sum_{j=1}^{J} \bar{\theta} \int_0^{\infty} V_j(x,z)\gamma(x)\,dx + (q_0 + \delta_0 z) \int_0^{\infty} P_0(0,z)\,\mu_0(x)dx \\ & + \sum_{i=1}^{M} (q_i + \delta_i z) \sum_{j_i=1}^{k_i} \int_0^{\infty} P_{i,j_i}(0,z)\,\mu_{i,j_i}(x)dx + \int_0^{\infty} R_0(x,z)\beta_0(x)dx \\ & + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^{\infty} R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx - \lambda^+ I_0 - \sum_{j=1}^{J} V_{j,0}(0) \end{aligned} \quad (7.23)$$

$$P_0(0,z) = \frac{1}{z}[\lambda^+ C(z) I_0 + \int_0^{\infty} I(x,z)\eta(x)dx + \lambda^+ C(z) \int_0^{\infty} I(x,z)dx + \theta \sum_{j=1}^{J} \int_0^{\infty} V_j(x,z)\gamma(x)dx] \quad (7.24)$$

$$P_{1,j_1}(0,z) = p_{j_1}\int_0^\infty P_0(x,z)\mu_0(x)dx,\ j_1 = 1,2,......,k_1 \quad (7.25)$$

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx,\ i = 2,3,......,M,\ j_i = 1,2,......,k_i \quad (7.26)$$

$$R_0(0,z) = \lambda^-\int_0^\infty P_0(x,z)dx \quad (7.27)$$

$$R_{i,j_i}(0,z) = \lambda^-\int_0^\infty P_{i,j_i}(x,z)dx,\ i = 1,2,3,......,M,\ j_i = 1,2,......,k_i \quad (7.28)$$

Solving equation (7.7) at n=0, we have

$$V_{j,0}(x) = V_{j,0}(0)e^{-\lambda^+ x}[1 - V(x)],\ j = 1,2,3,......,J \quad (7.29)$$

Multiplying equation (7.29) by $\gamma(x)$ and integrating with respect to x from 0 to ∞, we have

$$\int_0^\infty V_{j,0}(x)\gamma(x)dx = V_{j,0}(0)V^*(\lambda^+) \quad (7.30)$$

Equation (7.16) gives

$$V_{j,0}(0) = \bar{p}\int_0^\infty V_{j-1,0}(x)\gamma(x)dx$$

$$= \bar{p}V_{j-1,0}(0)V^*(\lambda^+) \quad (7.31)$$

From equations (7.15) and (7.16) it is clear that $V_j(0, z) = V_{j,0}(0)$.

Applying the equation (7.31) repeatedly for j = J, J-1,....., we obtain

$$V_{j,0}(0,z) = \frac{V_{J,0}(0)}{\left(\bar{p}V^*(\lambda^+)\right)^{J-j}},\qquad j = 1,2,.......,J-1 \quad (7.32)$$

Substituting equations (7.31) and (7.32) in equation (7.1) and after some algebraic manipulations, we get

$$V_{J,0}(0) = \frac{\lambda^+ I_0(\bar{p}V^*(\lambda^+))^{J-1}(1-\bar{p}V^*(\lambda^+))}{V^*(\lambda^+)[(\bar{p}V^*(\lambda^+))^{J-1}(1-\bar{p}V^*(\lambda^+)) + p(1-(\bar{p}V^*(\lambda^+))^{J-1})]} \quad (7.33)$$

Solving the partial differential equations (7.17) to (7.22), we obtain

$$I(x,z) = I(0,z)\, e^{-\lambda^+ x}(1 - A(x)) \tag{7.34}$$

$$P_0(x,z) = P_0(0,z)\, e^{-(\lambda^+ + \lambda^- - \lambda^+ C(z))x}(1 - B_0(x)) \tag{7.35}$$

$$P_{i,j_i}(x,z) = P_{i,j_i}(0,z)\, e^{-(\lambda^+ + \lambda^- - \lambda^+ C(z))x}(1 - B_{i,j_i}(x)),\ i = 1,2,......,M,\ j_i = 1,2,.....,k_i \tag{7.36}$$

$$R_0(x,z) = R_0(0,z)\, e^{-(\lambda^+ - \lambda^+ C(z))x}(1 - R_0(x)) \tag{7.37}$$

$$R_{i,j_i}(x,z) = R_{i,j_i}(0,z)\, e^{-(\lambda^+ - \lambda^+ C(z))x}(1 - R_{i,j_i}(x)),\ i = 1,2,......,M,\ j_i = 1,2,.....,k_i \tag{7.38}$$

$$V_j(x,z) = V_j(0,z)\, e^{-(\lambda^+ - \lambda^+ C(z))x}(1 - V(x)) \tag{7.39}$$

Using the expressions in equations (7.34) to (7.39) in the equations (7.23) to (7.28) and solving, we obtain

$$I(0,z) = \lambda^+ I_0 \left\{ T_1(z)[C(z) + K\theta V^*(h(z))] + z[K\bar{\theta} V^*(h(z)) - K - 1] \right\} \Big/ D(z) \tag{7.40}$$

$$P_0(0,z) = \lambda^+ I_0\, T_2(z) \Big/ D(z) \tag{7.41}$$

$$P_{1,j_1}(0,z) = p_{j_1} \lambda^+ I_0\, T_2(z)\, B_0^*(g(z)) \Big/ D(z), \qquad j_1 = 1,2,......,k_1 \tag{7.42}$$

$$P_{i,j_i}(0,z) = p_{j_i} \lambda^+ I_0\, T_2(z)\, \Lambda_{i-1}^*(g(z))\, B_0^*(g(z)) \Big/ D(z),\ i = 2,3,......,M,\ j_i = 1,2,......,k_i \tag{7.43}$$

$$R_0(0,z) = \lambda^- \lambda^+ I_0\, T_2(z)\, ((1 - B_0^*(g(z)))/g(z)) \Big/ D(z) \tag{7.44}$$

$$R_{i,j_i}(0,z) = p_{j_i} \lambda^- \lambda^+ I_0\, T_2(z)\, \Lambda_{i-1}^*(g(z))\, ((1 - B_{i,j_i}^*(g(z)))/g(z)) B_0^*(g(z)) \Big/ D(z),\quad i = 1,2,3,......,M,\ j_i = 1,2,......,k_i \tag{7.45}$$

where

$$D(z) = z - [A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))]T_1(z)$$

$$\begin{aligned} T_1(z) = {} & (q_0 + \delta_0 z)\, B_0^*(g(z)) + \sum_{i=1}^{M} (q_i + \delta_i z)\, \Lambda_i^*(g(z)) B_0^*(g(z)) \\ & + \lambda^- ((1 - B_0^*(g(z)))/g(z)) R_0^*(h(z)) \\ & + \lambda^- \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))((1 - B_{i,j_i}^*(g(z)))/g(z)) R_{i,j_i}^*(h(z)) B_0^*(g(z)) \end{aligned}$$

$$T_2(z) = C(z) + K\theta V^*(h(z)) + (A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+)))[K\bar{\theta} V^*(h(z)) - K - 1]$$

$$K = \frac{1 - (\bar{p}\, V^*(\lambda^+))^J}{V^*(\lambda^+)\, [(\bar{p}\, V^*(\lambda^+))^{J-1}(1 - \bar{p}\, V^*(\lambda^+)) + p(1 - (\bar{p}\, V^*(\lambda^+))^{J-1})]}$$

$$\Lambda_0^*(g(z)) = 1, \qquad \Lambda_i^* = \prod_{l=1}^{i} \sum_{j_l=1}^{k_l} p_{j_l} B_{l,j_l}^*(g(z))$$

$$g(z) = \lambda^+(1 - C(z)) + \lambda^- \qquad \text{and} \qquad h(z) = \lambda^+(1 - C(z))$$

Using equation (7.33), equation (7.32) yields

$$V_j(0,z) = V_{j,0}(0) = \frac{\lambda^+ I_0 (\bar{p}\, V^*(\lambda^+))^{J-1}(1-\bar{p}\, V^*(\lambda^+))}{(\bar{p}\, V^*(\lambda^+))^{J-j} V^*(\lambda^+)[(\bar{p}\, V^*(\lambda^+))^{J-1}(1-\bar{p}\, V^*(\lambda^+)) + p(1-(\bar{p}\, V^*(\lambda^+))^{J-1})]} \quad (7.46)$$

- The probability generating function of the orbit size when the server is idle in the non-empty system is given by

$$I(z) = \int_0^\infty I(x,z)dx$$

$$= I_0(1-A^*(\lambda^+))\left\{T_1(z)[C(z)+K\theta V^*(h(z))] + z[K\bar{\theta} V^*(h(z)) - K - 1]\right\} / D(z) \quad (7.47)$$

- The probability generating function of the orbit size when the server is busy in providing first phase service is given by

$$P_0(z) = \int_0^\infty P_0(x,z)dx$$

$$= \lambda^+ I_0 T_2(z)((1-B_0^*(g(z))/g(z)) / D(z) \quad (7.48)$$

- The probability generating function of the orbit size when the server is busy in providing second phase service is given by

$$P(z) = \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(z)$$

$$= \int_0^\infty \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(x,z)dx$$

$$= \lambda^+ I_0 A^*(\lambda^+) T_2(z) B_0^*(g(z)) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z))/g(z)) \Big/ D(z) \quad (7.49)$$

- The probability generating function of the orbit size when the server is under repair in first phase is given by

$$R_0(z) = \int_0^\infty R_0(x,z)dx$$

$$= \lambda^+ \lambda^- I_0 T_2(z)((1-B_0^*(g(z)))/g(z))(1-R_0^*(h(z))) / h(z) D(z) \quad (7.50)$$

- The probability generating function of the orbit size when the server is under repair in second phase is given by

$$R(z) = \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \int_0^\infty \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} R_{i,j_i}(x,z)dx$$

$$= \lambda^+ \lambda^- I_0 T_2(z) p_{j_i} B_0^*(g(z)) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))(1-R_{i,j_i}^*(h(z))) \Big/ h(z)D(z) \tag{7.51}$$

- The probability generating function of the orbit size when the server is on j[th] vacation

$$V_j(z) = \int_0^\infty V_j(x,z)dx$$

$$= \frac{\lambda^+ I_0 (\bar{p}\ V^*(\lambda^+))^{J-1} (1-\bar{p}\ V^*(\lambda^+))\ ((1-V^*(h(z)))/h(z))}{(\bar{p}\ V^*(\lambda^+))^{J-j} V^*(\lambda^+)[(\bar{p}\ V^*(\lambda^+))^{J-1}(1-\bar{p}\ V^*(\lambda^+)) + p(1-(\bar{p}\ V^*(\lambda^+))^{J-1})]} \tag{7.52}$$

Using the normalizing condition, I_0 can be obtained as

$$I_0 = \frac{1-m_1(1-A^*(\lambda^+)) - T_1'(1)}{[T_1'(1)-1][(1-A^*(\lambda^+))K\theta - A^*(\lambda^+) - \lambda^+\gamma_1 K] + \lambda^+ T_2'(1)[N_1+N_2]} \tag{7.53}$$

where

$$T_1'(1) = \sum_{i=0}^{M} \delta_i \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) - \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} + \sum_{i=1}^{M} (\delta_i + q_i)[M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) f_0^{(1)}]$$

$$+ \lambda^+ m_1 (N_1+N_2) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)})(1-B_{i,j_i}^*(\lambda^-))$$

$$- \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} B_0^*(\lambda^-)$$

$$T_2'(1) = m_1 + \lambda^+ m_1 K \gamma_1 - m_1(1-A^*(\lambda^+))[K\theta + 1]$$

$$f_0^{(1)} = \lambda^+ m_1 \int_0^\infty x e^{-\lambda^- x} b_0(x)dx, \qquad f_{i,j_i}^{(1)} = \lambda^+ m_1 \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

$$M_i^{(1)} = \lim_{z\to 1} \Lambda_i^{*\prime}(g(z)), \qquad M_i^{(2)} = \lim_{z\to 1} \Lambda_i^{*\prime\prime}(g(z)),$$

$$N_1 = ((1-B_0^*(\lambda^-))/\lambda^-) + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)((1-B_{i,j_i}^*(\lambda^-))/\lambda^-)$$

$$N_2 = (1-B_0^*(\lambda^-))\beta_0^{(1)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}$$

The probability generating function of the orbit size is given by

$$P_q(z) = I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + \sum_{j=1}^{J} V_j(z)$$

$$= \frac{\begin{array}{c} I_0\{[z-T_1(z)][A^*(\lambda^+))\, h(z)(2-K\bar{\theta}V^*(h(z))) - h(z)K\theta V^*(h(z)) \\ +\lambda^+ C(z)K(1-V^*(h(z)))] + \lambda^+ h(z)T_2(z)[T_3(z)+\lambda^- T_4(z)]\} \end{array}}{h(z)D(z)} \qquad (7.54)$$

The probability generating function of the system size is given by

$$P_s(z) = I_0 + I(z) + zP_0(z) + z\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + \sum_{j=1}^{J} V_j(z)$$

$$= \frac{\begin{array}{c} I_0\{[z-T_1(z)][A^*(\lambda^+))\, h(z)(2-K\bar{\theta}V^*(h(z))) - h(z)K\theta V^*(h(z)) \\ +\lambda^+ C(z)K(1-V^*(h(z)))] + \lambda^+ h(z)T_4(z)[zT_3(z)+\lambda^- T_4(z)]\} \end{array}}{h(z)D(z)} \qquad (7.55)$$

where

$$T_3(z) = ((1-B_0^*(g(z)))/g(z)) + B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))$$

$$T_4(z) = ((1-B_0^*(g(z)))/g(z))(1-R_0^*(h(z)))$$
$$+ B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(g(z))((1-B_{i,j_i}^*(g(z)))/g(z))(1-R_{i,j_i}^*(h(z)))$$

7.4 Stability Condition

The necessary and sufficient condition for the system to be stable is

$$m_1(1-A^*(\lambda^+)) + T_1'(1) < 1$$

7.5 Performance Measures

Let $N_I(z), N_{P_0}(z), N_P(z), N_{R_0}(z)$ and $N_R(z)$ denotes the numerators of $I(z)$, $P_0(z), P(z), R_0(z)$ and $R(z)$ respectively.

- The probability that the server is idle in the non-empty system

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{I_0(1-A^*(\lambda^+))\left(T_1'(1)(1+K\theta)+m_1+\lambda^+ m_1 K\gamma_1 - K\theta - 1\right)}{1-m_1(1-A^*(\lambda^+))-T_1'(1)} \tag{7.56}$$

- Mean number of customers in the orbit when the server is idle in the non-empty system is

$$L_I = \lim_{z\to 1}\frac{d}{dz} I(z) = \frac{D'N_I'' - N_I'D''}{3D'^2} \tag{7.57}$$

where

$$N_I' = I_0(1-A^*(\lambda^+))\left(T_1'(1)(1+K\theta)+m_1+\lambda^+ m_1 K\gamma_1 - K\theta - 1\right)$$

$$N_I'' = I_0(1-A^*(\lambda^+))\{T_1''(1)(1+K\theta)+2T_1'(1)[m_1+2\lambda^+ m_1 K\theta\gamma_1] \\ + K\bar{\theta}[2\lambda^+ m_1\gamma_1+(\lambda^+ m_1)^2\gamma_2+\lambda^+ m_2\gamma_1]\}$$

$$D' = 1-m_1(1-A^*(\lambda^+))-T_1'(1)$$

$$D'' = -[T_1''(1)+m_2(1-A^*(\lambda^+))+2m_1(1-A^*(\lambda^+))T_1'(1)]$$

$$T_1''(1) = 2[\sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-)f_0^{(1)}+\sum_{i=1}^{M}\delta_i M_i^{(1)}B_0^*(\lambda^-)] - \sum_{j_1=1}^{k_1}p_{j_1}f_0^{(2)}+\sum_{i=1}^{M}(\delta_i+q_i)[M_i^{(2)}B_0^*(\lambda^-) \\ + M_i^{(1)}f_0^{(1)}+\Lambda_i^*(\lambda^-)f_0^{(2)}] + 2(\lambda^+ m_1/\lambda^-)[-f_0^{(1)}+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}(M_{i-1}^{(1)}B_0^*(\lambda^-) \\ + \Lambda_{i-1}^*(\lambda^-)f_0^{(1)})((1-B_{i,j_i}^*(\lambda^-)) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)f_{i,j_i}^{(1)}B_0^*(\lambda^-))] - 2\lambda^+ m_1[-f_0^{(1)}\beta_0^{(1)} \\ + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)}B_0^*(\lambda^-) \\ + (1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}f_0^{(1)}) + 2((\lambda^+ m_1)^2/\lambda^-)\left(N_1+N_2\right)+\lambda^+ m_2 N_1+(\lambda^+ m_1)^2 N_3 \\ + \lambda^+ m_2 N_2 + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}M_{i-1}^{(2)}(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)[(1-B_{i,j_i}^*(\lambda^-)) \\ f_0^{(2)} - f_{i,j_i}^{(2)}B_0^*(\lambda^-) - 2f_{i,j_i}^{(1)}f_0^{(1)}] + 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}M_{i-1}^{(1)}[(1-B_{i,j_i}^*(\lambda^-))f_0^{(1)} - f_{i,j_i}^{(1)}B_0^*(\lambda^-)]$$

$$N_3 = (1-B_0^*(\lambda^-)))\beta_0^{(2)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(2)}$$

$$f_0^{(2)} = (\lambda^+ m_1)^2 \int_0^\infty x^2 e^{-\lambda^- x} b_0(x)dx + \lambda^+ m_2 \int_0^\infty x e^{-\lambda^- x} b_0(x)dx \quad ,$$

$$f_{i,j_i}^{(2)} = (\lambda^+ m_1)^2 \int_0^\infty x^2 e^{-\lambda^- x} b_{i,j_i}(x)dx + \lambda^+ m_2 \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

- The probability that the server is busy in providing essential service

$$P_0 = \lim_{z\to 1} P_0(z)$$

$$= \frac{I_0(\lambda^+/\lambda^-)\, T_2'(1)\, (1 - B_0^*(\lambda^-))}{1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)} \tag{7.58}$$

- Mean number of customers in the orbit when the server busy in providing essential service is

$$L_{P_0} = \lim_{z\to 1} \frac{d}{dz} P_0(z)$$

$$= \frac{D'\, N_{P_0}'' - N_{P_0}' D''}{3D'^2} \tag{7.59}$$

where

$$N_{P_0}' = I_0(\lambda^+/\lambda^-)\, T_2'(1)\, (1 - B_0^*(\lambda^-))$$

$$N_{P_0}'' = I_0(\lambda^+/\lambda^-)\{T_2''(1)(1 - B_0^*(\lambda^-)) - 2T_2'(1)[f_0^{(1)} - 2(\lambda^+ m_1/\lambda^-)\,(1 - B_0^*(\lambda^-))]\}$$

$$T_2''(1) = m_2 + K[(\lambda^+ m_1)^2 \gamma_2 + \lambda^+ m_2 \gamma_1] - m_2(1 - A^*(\lambda^+))[K\theta + 1] + 2\lambda^+ m_1^2 (1 - A^*(\lambda^+))K\bar{\theta}\gamma_1$$

- The probability that the server is busy in providing optional services

$$P = \lim_{z\to 1} \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(z)$$

$$= \frac{I_0(\lambda^+/\lambda^-)\, T_2'(1)\, B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))}{1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)} \tag{7.60}$$

- Mean number of customers in the orbit when the server busy in providing optional services is

$$L_P = \lim_{z\to 1} \frac{d}{dz} P(z)$$

$$= \frac{D' N_P'' - N_P' D''}{3D'^2} \tag{7.61}$$

where

$$N_P' = I_0 (\lambda^+/\lambda^-) T_2'(1) B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))$$

$$N_P'' = I_0 (\lambda^+/\lambda^-) \{ T_2''(1)(1 + 2(\lambda^+ m_1/\lambda^-)) B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))$$

$$+ 2T_2'(1)[\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)})(1 - B_{i,j_i}^*(\lambda^-))$$

$$- B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)}]\}$$

- The probability that the server is under repair during essential service

$$R_0 = \lim_{z \to 1} R_0(z)$$

$$= \frac{I_0 \lambda^+ T_2'(1)(1 - B_0^*(\lambda^-)) \beta_0^{(1)}}{1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)} \tag{7.62}$$

- Mean number of customers in the orbit when the server is under repair during essential service is

$$L_{R_0} = \lim_{z \to 1} \frac{d}{dz} R_0(z)$$

$$= \frac{D_1'' N_{R_0}''' - N_{R_0}'' D_1'''}{3D_1''^2} \tag{7.63}$$

where

$$N_{R_0}'' = -2\lambda^+ m_1 I_0 \lambda^+ T_2'(1)(1 - B_0^*(\lambda^-)) \beta_0^{(1)}$$

$$N_{R_0}''' = -3\lambda^+ I_0 \{ \lambda^+ m_1 T_2''(1)(1 - B_0^*(\lambda^-)) \beta_0^{(1)} + T_2'(1)[(1 - B_0^*(\lambda^-))((\lambda^+ m_1)^2 \beta_0^{(2)}$$

$$+ \lambda^+ m_2 \beta_0^{(1)}) + \lambda^+ m_1 f_0^{(1)} \beta_0^{(1)} + ((\lambda^+ m_1)^2/\lambda^-)(1 - B_0^*(\lambda^-)) \beta_0^{(1)}]\}$$

- The probability that the server is under repair during optional services

$$R = \lim_{z \to 1} \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \frac{I_0 \lambda^+ T_2'(1) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \beta_{i,j_i}^{(1)}}{1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)} \quad (7.64)$$

- Mean number of customers in the orbit when the server is under repair during optional services is

$$L_R = \lim_{z \to 1} \frac{d}{dz} \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \frac{D_1'' N_R''' - N_R'' D_1'''}{3 D_1''^2} \quad (7.65)$$

where

$$N_R'' = -2\lambda^+ m_1 I_0 \lambda^+ T_2'(1) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \beta_{i,j_i}^{(1)}$$

$$\begin{aligned} N_R''' = & -3\lambda^+ I_0 \{ \lambda^+ m_1 T_2''(1) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \beta_{i,j_i}^{(1)} \\ & + T_2'(1) [\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)((\lambda^+ m_1)^2 B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(2)} \\ & + \lambda^+ m_2 B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)} + \lambda^+ m_1 f_0^{(1)}(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)} \\ & - \lambda^+ m_1 B_0^*(\lambda^-) f_{i,j_i}^{(1)} \beta_{i,j_i}^{(1)} + ((\lambda^+ m_1)^2 / \lambda^-) B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)}) \\ & + \lambda^+ m_1 \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)}] \} \end{aligned}$$

- The probability that the server is on vacation

$$V = \lim_{z \to 1} \sum_{j=1}^{J} V_j(z)$$

$$= I_0 \lambda^+ \gamma_1 K, \quad 1 \le j \le J \quad (7.66)$$

- Mean number of customers in the orbit when the server is on vacation is

$$L_V = \lim_{z \to 1} \frac{d}{dz} \sum_{j=1}^{J} V_j(z)$$

$$= \frac{I_0 \lambda^{+2} m_1 \gamma_2 K}{2} \quad (7.67)$$

Let $N_q(z)$ be the numerator of $P_q(z)$.

- Mean number of customers in the orbit is

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)$$
$$= \frac{D_1'' \, N_q''' - N_q'' D_1'''}{3 D_1''^{\,2}} \qquad (7.68)$$

where

$$D_1''(1) = -2\lambda^+ m_1 [1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)]$$
$$D_1'''(1) = 3[-\lambda^+ m_2 \left(1 - m_1(1 - A^*(\lambda^+)) - T_1'(1)\right) + \lambda^+ m_1 T_5]$$

$$T_5 = T_1''(1) + m_2(1 - A^*(\lambda^+)) + m_1(1 - A^*(\lambda^+)) T_1'(1)$$

$$\begin{aligned} N_q''(1) = \; & I_0 \{ 2(1 - T_1'(1))[K\theta\lambda^+ m_1 + \lambda^{+2} m_1 \gamma_1 K - A^*(\lambda^+)\lambda^+ m_1 (2 - K\bar{\theta})] \\ & - 2\lambda^+ m_1 T_2'(1)(N_1 + N_2)\} \end{aligned}$$

$$\begin{aligned} N_q'''(1) = \; & 3I_0 \{ T_1''(1)[\lambda^+ m_1 A^*(\lambda^+)(2 - K\bar{\theta}) - K\theta\lambda^+ m_1 + K\lambda^{+2} m_1 \gamma_1] + (1 - T_1'(1)) \\ & [2A^*(\lambda^+)(\lambda^+ m_1)^2 K\bar{\theta}\gamma_1 + \lambda^+ m_2 K\theta - A^*(\lambda^+)\lambda^+ m_2 (2 - K\bar{\theta}) + 2(\lambda^+ m_1)^2 K\theta\gamma_1 \\ & - 2\lambda^+ m_1^2 K\gamma_1 - \lambda^+ K((\lambda^+ m_1)^2 \gamma_2 + \lambda^+ m_2 \gamma_1)] - \lambda^+ (N_1 + N_2)(\lambda^+ m_2 T_2'(1) \\ & + \lambda^+ m_1 T_2''(1)) - 2\lambda^{+2} m_1 T_2'(1)(T_3'(1) + \lambda^- T_4'(1))\} \end{aligned}$$

$$\begin{aligned} T_3'(1) = \; & (1/\lambda^-)[\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)})(1 - B_{i,j_i}^*(\lambda^-)) - f_0^{(1)} \\ & - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) f_{i,j_i}^{(1)}] + (\lambda^+ m_1 / \lambda^-) N_1 \end{aligned}$$

$$\begin{aligned} T_3''(1) = \; & (1/\lambda^-)[-2\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)})(1 - B_{i,j_i}^*(\lambda^-)) - f_0^{(2)} \\ & - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) f_{i,j_i}^{(2)}] - 2(\lambda^+ m_1 / (\lambda^-)^2)[f_0^{(1)} + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) \\ & B_0^*(\lambda^-) f_{i,j_i}^{(1)} - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) f_0^{(1)})(1 - B_{i,j_i}^*(\lambda^-))] \\ & + N_1 [\lambda^+ m_2 + 2((\lambda^+ m_1)^2 / \lambda^-)] \end{aligned}$$

$$T_4'(1) = -(\lambda^+ m_1 / \lambda^-) N_2$$

$$T_4''(1) = 2(\lambda^+ m_1/\lambda^-)[f_0^{(1)}\beta_0^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(M_{i-1}^{(1)}B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-)f_0^{(1)})(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}$$

$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-)\, f_{i,j_i}^{(1)}\,] - 2(\lambda^+ m_1/\lambda^-)^2 N_2$$

- Mean number of customers in the system is

$$L_S = \lim_{z\to 1}\frac{d}{dz}P_S(z)$$

$$= L_q + P_0 + P \qquad (7.69)$$

7.6 Reliability Indices

Let $\mathcal{A}$ be the availability of the server and $\mathcal{F}$ be the failure frequency of the server in the system under steady state condition.

The steady state availability ($\mathcal{A}$) of the server is given by

$$\mathcal{A} = I_0 + \lim_{z\to 1}\left(I(z) + P_0(z) + P(z)\right)$$

$$= \frac{1 - T_1'(1) + (1-A^*(\lambda^+))\,[T_1'(1)\,(1+K\,\theta) + \lambda^+ m_1 K\gamma_1 - K\theta - 1] + (\lambda^+/\lambda^-)T_2'(1)N_1}{[T_1'(1)-1][(1-A^*(\lambda^+))K\theta - A^*(\lambda^+) - \lambda^+\gamma_1 K] + \lambda^+ T_2'(1)[N_1+N_2]}$$

The failure frequency ($\mathcal{F}$) of the server is given by

$$\mathcal{F} = \lambda^- \lim_{z\to 1}\left(P_0(z) + P(z)\right)$$

$$= \frac{\lambda^+ T_2'(1)T_5}{[T_1'(1)-1][(1-A^*(\lambda^+))K\theta - A^*(\lambda^+) - \lambda^+\gamma_1 K] + \lambda^+ T_2'(1)[N_1+N_2]}$$

7.7 Special Cases

Case (i) If $\lambda^- = 0$, M = 0, C(z) = z, θ = 0, p = 1, then the system reduces to M/G/1 retrial queueing system with Bernoulli feedback and modified vacation. In this case

$$I(z) = \frac{I_0(1-A^*(\lambda^+))\,z\,\{N(z) - 1 + (\delta_0 z + q_0)\,B_0^*(\lambda^+(1-z))\}}{z - (\delta_0 z + q_0)[A^*(\lambda^+) + z(1-A^*(\lambda^+))]B_0^*(\lambda^+(1-z))}$$

$$P_0(z)=\frac{\lambda^+ I_0[z+(N(z)-1)+[A^*(\lambda^+)+z(1-A^*(\lambda^+))](1-B_0^*(\lambda^+(1-z)))}{\lambda^+(1-z)\{z-(\delta_0 z+q_0)[A^*(\lambda^+)+z(1-A^*(\lambda^+))]\}}$$

$$V_j(z)=\frac{I_0[1-V^*(\lambda^+(1-z))]}{[V^*(\lambda^+)]^{J-j+1}(1-z)}, \qquad j=1,2,\ldots,J$$

where $N(z)=\dfrac{1-[V^*(\lambda^+)]^J}{[V^*(\lambda^+)]^J[1-V^*(\lambda^+)]}[V^*(\lambda^+)-1]$

The above results coincide with the results of Pankaj Sharma (2018).

Case (ii) If $\lambda^-=0$, $\theta=0$, $q_0=\delta_0=0$, $k_i=1$, then the model reduces to batch arrival multistage retrial queue with variant vacation policy. In this case

$$I(z)=\frac{I_0(1-A^*(\lambda^+))\{C(z)\sum_{i=1}^{M}(\delta_i z+q_i)\Lambda_{i-1}^*(\lambda^+(1-C(z)))B_i^*(\lambda^+(1-C(z)))+z(N(z)-1)\}}{z-[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]\sum_{i=1}^{M}(\delta_i z+q_i)\Lambda_{i-1}^*(\lambda^+(1-C(z)))B_i^*(\lambda^+(1-C(z)))}$$

$$P_i(z)=\frac{I_0\Lambda_{i-1}^*(\lambda^+(1-C(z)))(1-B_i^*(\lambda^+(1-C(z))))\{C(z)+(N(z)-1)[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]\}}{(1-C(z))\{z-[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]\sum_{i=1}^{M}(\delta_i z+q_i)\Lambda_{i-1}^*(\lambda^+(1-C(z)))B_i^*(\lambda^+(1-C(z)))\}}$$

$$V_j(z)=\frac{I_0[1-V^*(\lambda^+(1-C(z)))]}{[V^*(\lambda^+)]^{J-j+1}(1-C(z))}, \qquad j=1,2,\ldots,J$$

where

$$N(z)=\frac{1-[V^*(\lambda^+)]^J}{[V^*(\lambda^+)]^J[1-V^*(\lambda^+)]}[V^*(\lambda^+)-1]$$

$$\Lambda_0^*=1,\ \Lambda_i^*=\prod_{l=1}^{i}p_l B_l^*,\ i=1,2,\ldots,M$$

The above results agree with the results of Radha et al. (2017c) with no random breakdown.

7.8 Practical Justification of the Model

Our proposed model has its potential application in e-commerce. Online Shopping is a form of electronic commerce which allows consumers (positive customers) to directly buy goods (services) from a seller over Internet using a web browser or a mobile app (Essential service). If the number of customers requesting service at a point of time is high, then a delay is experience by the customers which is indicated by messages such as loading or please wait. This case is referred as orbit in retrial queueing theory.

Consumers find a product of interest by visiting any one of the retailers such as Amazon.com, eBay, Flipkart, etc., (Multi-Optional Stage 1). Once a particular retailer website has been chosen, the consumer has the options to choose the product of his wish from the list of products like books, mobile phones, clothes, electronics, etc., (Multi-Optional Stage 2). Next to this, a checkout process follows in which payment and delivery information is collected. Payment option includes cash on delivery, credit card, debit card, Gpay, etc., (Multi-Optional Stage 3). The consumer often receives an email confirmation once the transaction is complete. The customer who faces the network failure at any stage of service will be treated as the feedback customer.

The server experiences a failure due to software or hardware issue (Negative arrival) at any time during the working period and therefore need to be repaired.

The system may undergo one or more maintenance activities (Vacation) such as hardware upgradation, software updates, increasing the capacity of the server and modernization. After maintenance the server searches for the customers request which are in queue (Orbital Search).

7.9 Numerical Results

In this section, some performance measures of the system under consideration are numerically calculated by assuming that the retrial times, first phase service time, second phase service time, repair time at first phase, repair at second phase and vacation time are exponentially distributed with the respective parameters $\eta, \mu_0, \mu_{i,j_i}, \beta_0, \beta_{i,j_i}$ and γ. MATLAB software has been adopted to illustrate the results numerically. The input parameters are $\lambda^+ = 1.4$, $\lambda^- = 0.1$, $\eta = 50$, M=3, $C_1 = C_2 = 0.5$, $k_1 = 2$, $k_2 = 3$, $k_3 = 2$, J = 4, p = 0.3, p_{j_1} = [0.4 0.3], p_{j_2} = [0.2 0.3 0.1], p_{j_3} = [0.4 0.2], $\delta_0 = 0.2$, $q_0 = 0.1$, δ = [0.2 0.3 0.6], q = [0.2 0.1 0.4], $\mu_0 = 30$, $\beta_0 = 2$, $\gamma = 10$, $\theta = 0.4$, μ_1 = [70 40], μ_2 =[62 42 52], μ_3 =[85 73], β_1=[1 3], β_2=[5 7 10], β_3 =[12 14].

Table 7.1 depicts the impact of the retrial rate (η) on the performance measures I_0 – the probability that the server is idle in empty system, I – the probability that the server is idle in non-empty system, P_0 – the probability that the server is busy at first phase, P – the probability that the server is busy at second phase, R_0 – the probability that the server is under repair during first phase service, R – the probability that the server is under repair during second phase service, V – the probability that the server is on vacation, L_q – the mean queue length, and L_s – the expected system size.

It is observed that

- Increase in η, increases I_0 and V
- Increase in η, decreases I, L_q and L_s
- P_0 , P, R_0 and R are independent of η

Table 4.2 depicts the influence of p on the system measures I_0+I, P_0, P, R_0, R, V, $\mathcal{A}$, $\mathcal{F}$ and L_s. It is observed that increase in p increases I_0+I and $\mathcal{A}$ slightly, decreases V and L_s and has no effect on P_0, P, R_0, R and $\mathcal{F}$.

Fig. 7.2 shows the effect of the positive arrival rate (λ^+) on the expected number of customers in the system (L_s) with and without orbital search (θ). It can be seen that the expected system size (L_s) increases with increase in λ^+. Further, the expected system size (L_s) is higher in the case of the system including orbital search (θ) as compared to the system without orbital search.

Fig. 7.3 to 7.7 reveal the combined effect of γ and J on the system measures L_I, L_P , L_R , L_V and L_s.

It is observed that

- Increase in number of vacations, increases L_I , L_P , L_R , L_V and L_s
- Increase in vacation completion rate, decreases L_I , L_P , L_R , L_V and L_s

Table 7.1 Impact of Retrial Rate (η) on the System Measures

η	I_0	I	P_0	P	R_0	R	V	L_q	L_s
15	0.2286	0.2377	0.1520	0.2727	0.0076	0.0125	0.0889	6.8443	7.2689
20	0.2730	0.1760	0.1520	0.2727	0.0076	0.0125	0.1061	6.6462	7.0708
25	0.2992	0.1398	0.1520	0.2727	0.0076	0.0125	0.1163	6.5257	6.9504
30	0.3164	0.1159	0.1520	0.2727	0.0076	0.0125	0.1230	6.4452	6.8699
35	0.3285	0.0990	0.1520	0.2727	0.0076	0.0125	0.1277	6.3877	6.8124
40	0.3376	0.0864	0.1520	0.2727	0.0076	0.0125	0.1312	6.3446	6.7693

Table 7.2 Impact of p on the System Measures

p	I_0+I	P_0	P	R_0	R	V	$\mathcal{A}$	$\mathcal{F}$	Ls
0.1	0.3883	0.1520	0.2727	0.0076	0.0125	0.1669	0.8130	0.0425	9.5361
0.3	0.4323	0.1520	0.2727	0.0076	0.0125	0.1230	0.8569	0.0425	6.8699
0.5	0.4624	0.1520	0.2727	0.0076	0.0125	0.0928	0.8871	0.0425	5.0412
0.7	0.4830	0.1520	0.2727	0.0076	0.0125	0.0723	0.9076	0.0425	3.7940
0.9	0.4969	0.1520	0.2727	0.0076	0.0125	0.0584	0.9215	0.0425	2.9509

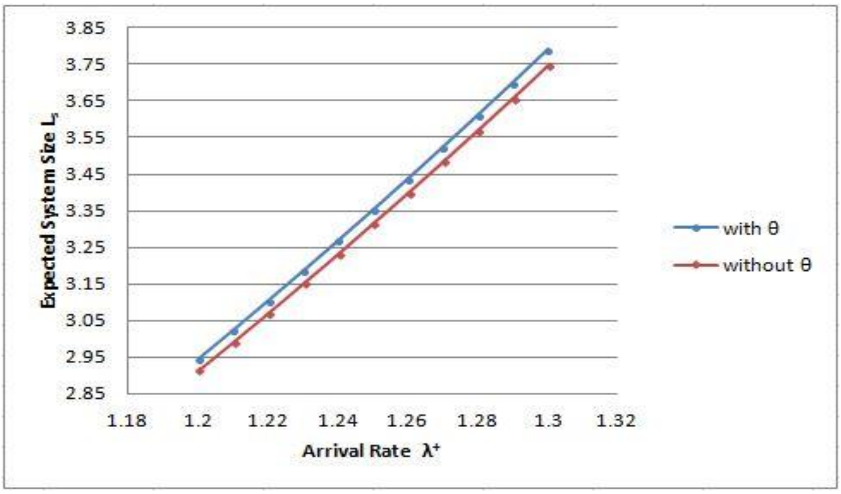

Fig. 7.2 Effect of λ^+ on L_s with and without orbital search θ

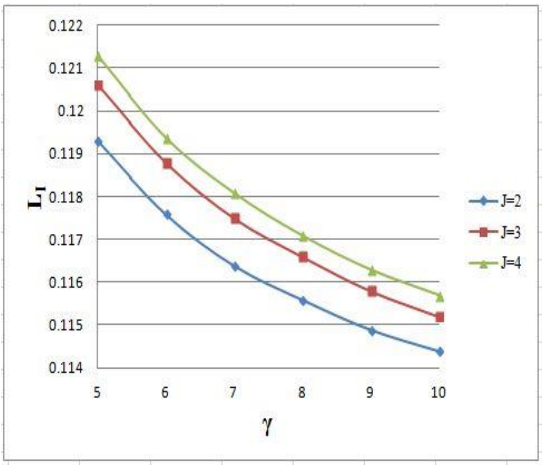

Fig. 7.3 Effect of (γ , J) on L_I

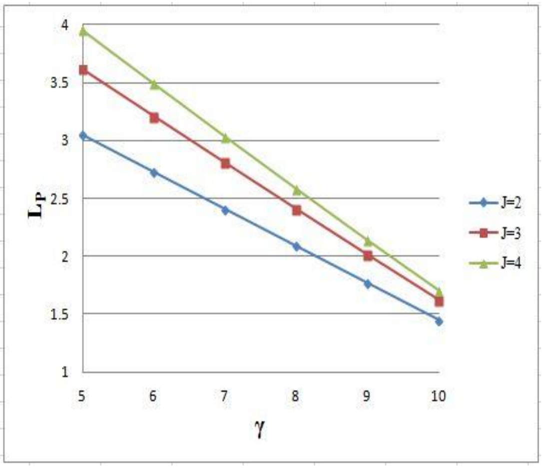

Fig. 7.4 Effect of (γ , J) on L_P

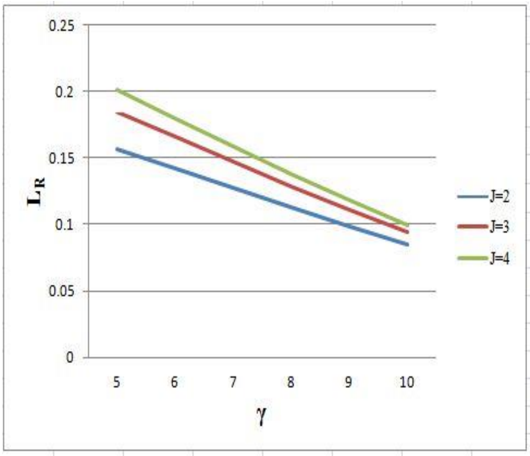

Fig. 7.5 Effect of (γ , J) on L_R

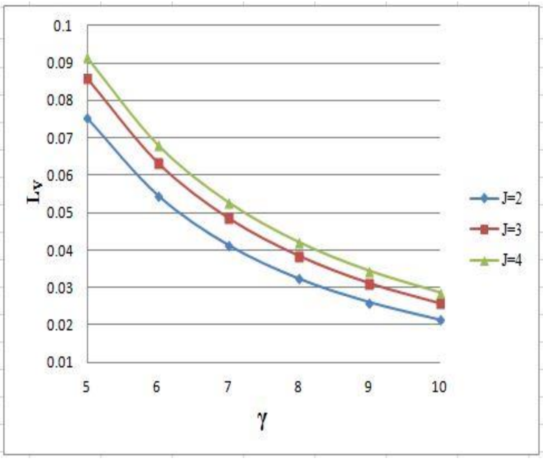

Fig. 7.6 Effect of (γ , J) on L_V

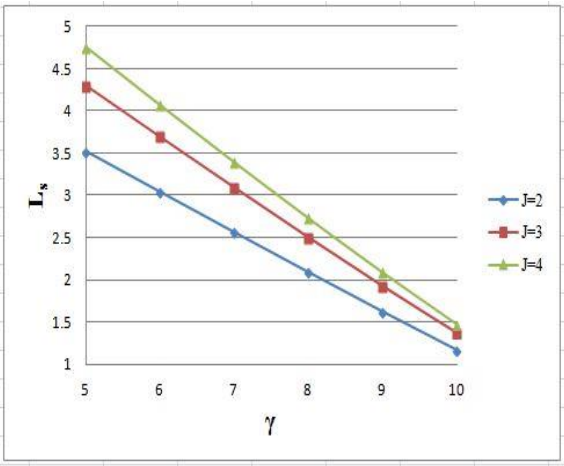

Fig. 7.7 Effect of (γ , J) on L_s