CHAPTER IV

Vertex corona product of Cycle graph with some special graphs

In this chapter, the b-chromatic number of Vertex corona product of two graphs G and H both are $\Delta+1$ b-colorable graphs discussed. Also the b-chromatic number of vertex corona product of cycle with barbell graph, cycle graph with tadpole graph, cycle with fan graph, cycle graph with double fan graph are assessed.

4.1 Introduction

Cycle graph [Jonathan Gross, et al., 2004]

A simple graph of 'n' vertices $(n \geq 3)$ and n edges forming a cycle of length 'n' is called as a **cycle graph.** In a cycle graph, all the vertices are of degree 2.

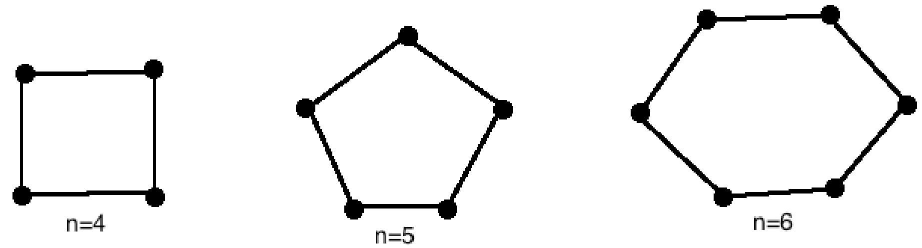

Fig 4.1: Cycle Graph

Cyclic graph

A graph containing at least one cycle in it is called as a **cyclic graph.**

Acyclic graph

A graph not containing any cycle in it is called as an **acyclic graph**.

Path graph

A **path graph** or **linear graph** is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n-1$. Equivalently, a path with at least two vertices is connected and has two terminal vertices (vertices that have degree 1), while all others (if any) have degree 2.

Paths are fundamental concepts of graph theory, described in the introductory sections of most graph theory texts [Bondy, J.A et al., 1976].

Fig 4.2: Path Graph

Connected

A Graph is said to be **connected** if there is path between every pair of vertex. There should be some path to transverse. This is called the Connectivity of a graph.

Disconnected

A Graph with multiple disconnected vertices and edges is said to be **disconnected.**

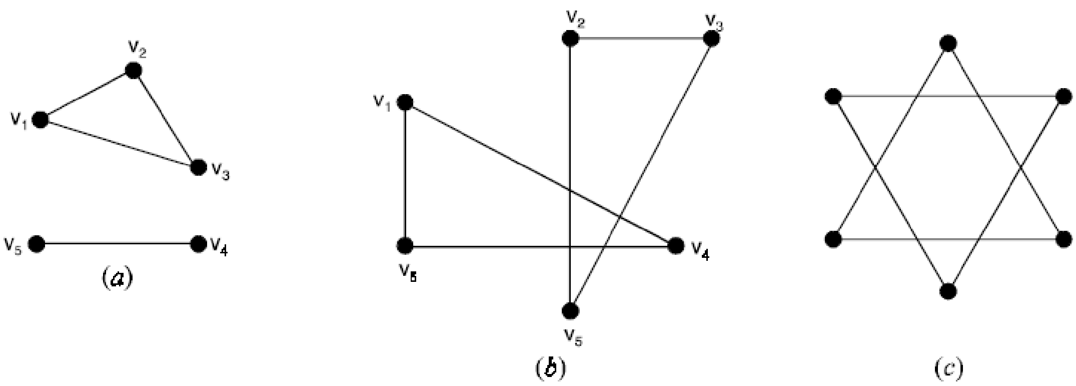

Fig 4.3: Connected and Disconnected Graph

Fan graph

A **fan graph** obtained by joining all vertices of $F_n, n \geq 2$ is a path P_n to a further vertex, called the centre. Thus F_n contains $n+1$ vertices say C, $v_1, v_2, v_3, ..., v_n$ and $(2n-1)$ edges, say cv_i , $1 \leq i \leq n$ and $v_i v_i + 1, 1 \leq i \leq n-1$.

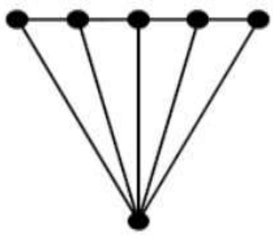

Fig 4.4: Fan Graph F_5

Double fan

The **double fan** DF_n consists of two fan graph that have a common path. In other words $DF_n = P_n + K_2$.

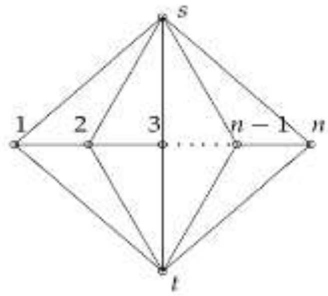

Fig 4.5: Double Fan Graph (DF_n)

Tadpole graph

A **Tadpole graph** is denoted by $T_{m,n}$ we mean the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge

Example: $T_{4,5}$ a typical Tadpole graph is in Figure 4.6**.**

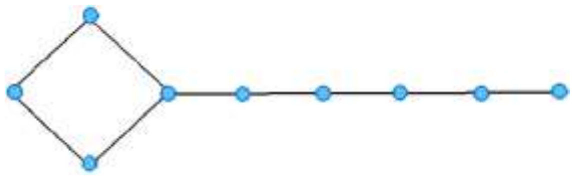

Fig 4.6: Tadpole graph $G = T_{4,5}$

4.2 The vertex corona graph of Cycle graph with Barbell Graph

Algorithm 4.2.1: The b-coloring of vertex corona product of cycle with barbell graph

Input: $C_n \text{ o} B(K_n, K_n), n \geq 3.$

$V \leftarrow \{a_1, a_2, ...a_n, x^1{}_1, x^1{}_2, ...x^n{}_n, y^1{}_1, y^2{}_2 ..., y^n{}_n\}$.

for $i = 1$ to n

$a_i \leftarrow i;$

end for

for $i = 1$ to n, $j = 1$ to $n - 1$, $k = 1$ to n

$x^i{}_j \leftarrow k;$

end for

for j=n

$x^i{}_j \leftarrow n+1;$

end for

for $i = 1$ to n, $j = 1$ to $n - 1$, $k = 1$ to n

$y^i{}_j \leftarrow k;$

end for

for j=n

$y^i{}_j \leftarrow n+2;$

end for

end procedure

output: vertex colored $C_n \circ B(K_n, K_n)$.

Theorem 4.2.1: For any cycle graph C_n and barbell graph $B(K_n, K_n)$, the b-chromatic number is $\varphi[C_n \circ B(K_n, K_n)] = n+2, n \geq 3.$

Proof:

Let $V(C_n) = \{a_i : 1 \leq i \leq n\}$ and $V[B(K_n, K_n)] = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$

By the definition of corona product each vertex of C_n is adjacent to every vertex of number of copies of $B(K_n, K_n)$, then the vertex set of the corona product $C_n \circ B(K_n, K_n)$

$V[C_n \circ B(K_n, K_n)] = \{a_i : 1 \leq i \leq n\} \cup \{x^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup$
$\{y^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\}$

Assign the colors as per the algorithm.

By this coloring procedure, we have that $\varphi[C_n \circ B(K_n, K_n)] \geq n+2$

To prove the lower-bound, let us assume that, b-chromatic number of corona product of cycle graph C_n with barbell graph $B(K_n, K_n)$ is greater than n+2.

That is the b-chromatic number of $[C_n \, o B(K_n, K_n)] = n + 2$

We can assign n+3 colors only if the graph having n+3 vertices assigned with $n+3$ distinct colors which are adjacent to each other.

Here the color class c_{n+3} is not adjacent to the color class c_{n+2} , c_{n+1} & c_n which is the contradiction.

Therefore assigning n+3 colors is not possible.

$\therefore \varphi[C_n \, o B(K_n, K_n)] \le n + 2$

Hence $\varphi[C_n \, o B(K_n, K_n)] = n + 2$

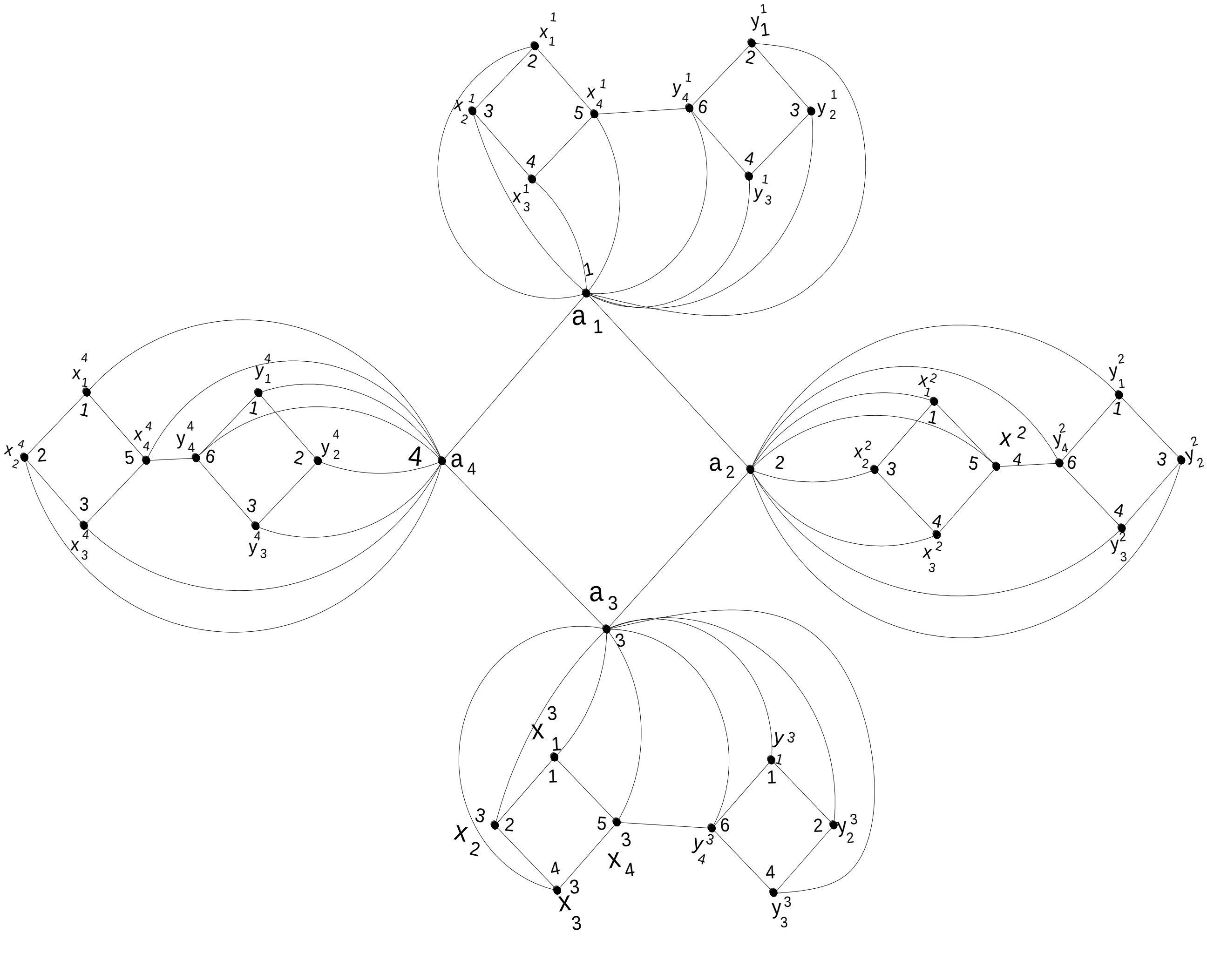

Fig 4.7 : $C_4 \, o B(K_4, K_4) = 6$

4.3 The vertex corona graph of Cycle graph with Tadpole graph

Theorem 4.3.1: For any positive number n, the b-chromatic number of the corona graph of cycle graph with tadpole graph is

$$\varphi[C_n \circ T_{4,n}] = \begin{cases} n+2, & n=3 \\ n+1, & n=4 \\ n, & n \geq 5 \end{cases}$$

Proof:

Let $V(C_n) = \{x_i : 1 \leq i \leq n\}$ and $V(T_{4,n}) = \{y_i : 1 \leq i \leq n\}$

By the definition of corona product each vertex of C_n is adjacent to every vertex of number of copies of $T_{4,n}$.

i.e., every vertex $x_i \in V(C_n)$ is adjacent to every vertex from the set

$\{y^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n+3\} \in V(C_n \circ T_{4,n})$.

Let $V(C_n \circ T_{4,n}) = \{x_i : 1 \leq i \leq n\} \cup \{y^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n+3\}$

We prove the results by the following three cases.

Case 1: For $n = 3$

Assign the colors as follows, by using the proper coloring procedure

- For $x_i : 1 \leq i \leq 3$, assign the color c_i
- For $y^i{}_j : 1 \leq i \leq 3, 1 \leq j \leq 5$, assign the color c_k , $1 \leq k \leq 5$

For the graph $C_3 \circ T_{4,3}$ the maximization of b-coloring depends the vertex set of $T_{4,3}$. But $T_{4,3}$ has 5 vertices of degree 4, then we can assign 5 colors to get b-coloring

Therefore $\varphi[C_3 \circ T_{4,3}] = 5$

Case 2: For n=4

Assign the following 5 colors as b-chromatic number for $C_n \circ T_{4,n}$

- For x_1, assign the color c_1
- For x_2, assign the color c_2

- For x_3, assign the color c_3
- For x_4, assign the color c_4
- For $y^i{}_j$, assign the color c_k , $1 \le i \le 4,\ 1 \le j \le 8,\ 1 \le k \le 5$
- Assign the color c_5 to the vertex $y^i{}_j : 1 \le i \le 4,\ j = 5.$

By considering the proper coloring procedure, $\varphi\lfloor C_4 \circ T_{4,4} \rfloor = 5$

Case 3: For $n \ge 5$

Assign the following n colors as b-chromatic number for $C_n \circ T_{4,n}$

- For $x_i : 1 \le i \le n$ assign the color c_i
- For $y^i{}_j : 1 \le i \le n, 1 \le j \le n+3$ assign the color c_k , $1 \le k \le n$

By this coloring procedure, we have that $\varphi\lfloor C_n \circ T_{4,n} \rfloor \ge n$

To prove the lower bound, let us assume that b-chromatic number of corona product of cycle graph c_n with tadpole graph $T_{4,n}$ is greater than n.

That is the b-chromatic number of $C_n \circ T_{4,n} = n+1$

We can assign n+1 colors, the graph should have n+1distinct vertices with degree n. But the graph $C_n \circ T_{4,n}$ have only n vertices with maximum degree $\ge n+1$. which is the contradiction.

Therefore assigning n+1colors is not possible.

$$\therefore\ \varphi\lfloor C_n \circ T_{4,n} \rfloor \le n$$

Hence $\varphi\lfloor C_n \circ T_{4,n} \rfloor = n$.

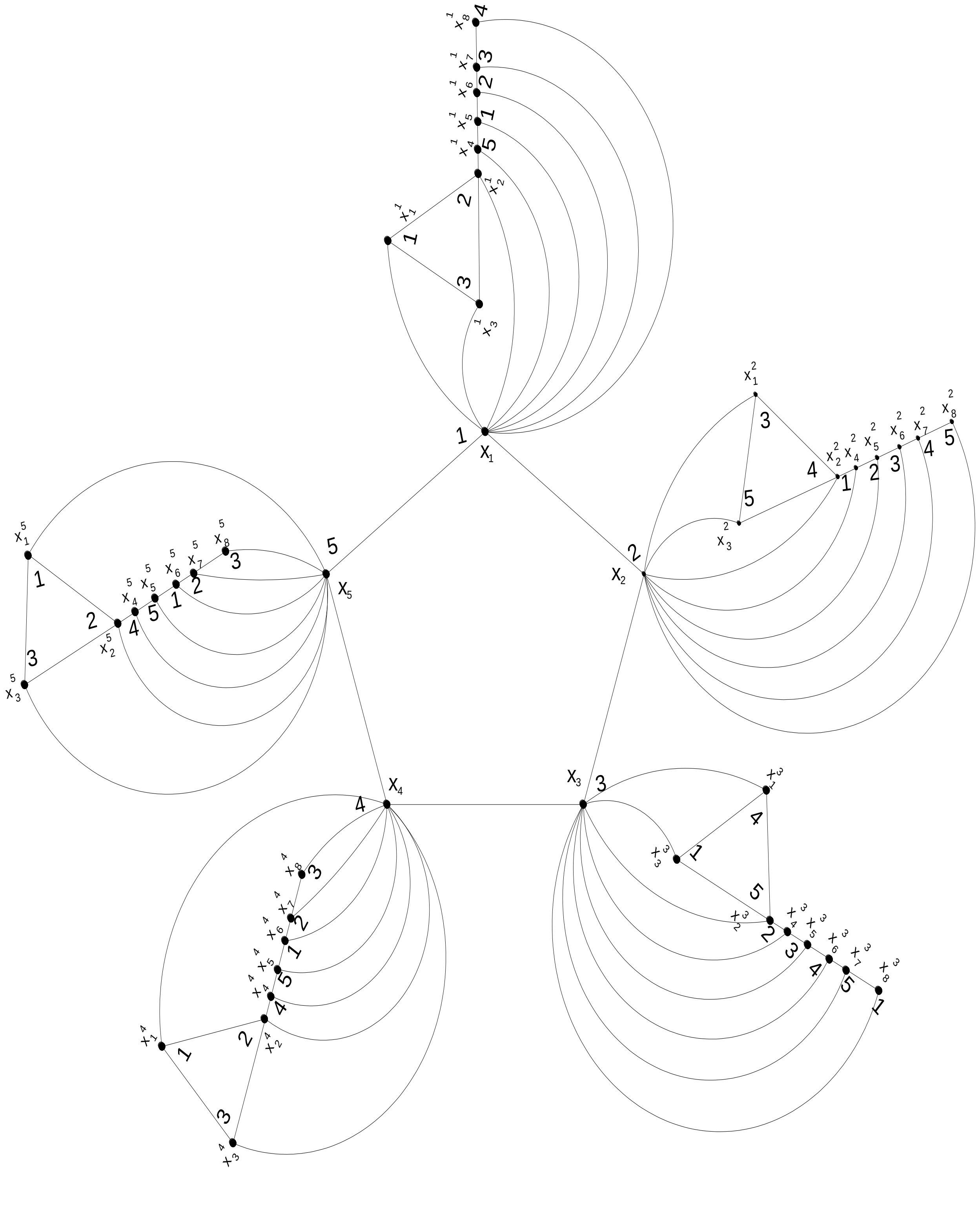

Fig 4.8: $\varphi[C_5 \circ T_{4,5}] = 5$

4.4 The Vertex corona graph of Cycle graph with Fan graph

Theorem 4.4.1: For any positive number n, the b-chromatic number of the corona product of cycle graph with fan graph is

$$\varphi[C_n \circ F_{1,n}] = \begin{cases} n+2, & n=3 \\ n+1, & n \geq 4 \end{cases}$$

Proof:

Let $V(C_n) = \{u_i : 1 \leq i \leq n\}$ and $V(F_{1,n}) = \{w\} \cup \{w^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\}$

By the definition of corona graph, each vertex of C_n is adjacent to every vertex of number of copies of $F_{1,n}$.

i.e., every vertex $w_i \in V(C_n)$ is adjacent to every vertex from the set

$\{w^i{}_j : 1 \leq i \leq n, 1 \leq j \geq n\}$ and $w \in V(F_{1,n})$ is adjacent to every vertex from the set $\{w_i : 1 \leq i \leq n\}$.

Let $V(C_n \circ F_{1,n}) = \{u_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \cup \{w^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n\}$

Consider the following cases.

Case 1: For n=3

Assign the following n+2 colors as b-chromatic number for $C_3 \circ F_{1,3}$

- For u_1 ,assign the color c_1
- For u_2 ,assign the color c_2
- For u_3, assign the color c_3
- For w_1, assign the color c_4
- For w_2, assign the color c_4
- For w_3, assign the color c_4
- For $w_1^1, w^1{}_2, w^1{}_3$ assign the color $c_k, 1 \leq k \leq n+2$, except c_1 & c_4

The above shows that this coloring is a b-coloring.

Case 2: For $n \geq 4$.

Assign the following n+1colors as b-chromatic number for $C_n \text{o} F_{1,n}$

- For $u_i : 1 \leq i \leq n$, assign the color c_i
- For $w_i : 1 \leq i \leq n$, assign the color c_{n+1}
- For $w^i{}_j$, assign the colors $c_i : 1 \leq i \leq n+1$, except the colors of vertices w_i & u_i , $1 \leq i \leq n$.

It shows that this coloring is a b-coloring.

The b-coloring procedure depends the number of vertices of the cycle graph C_n which has degree n. So that, we can assign maximum n+1 colors to get b-coloring.

Hence the proof.

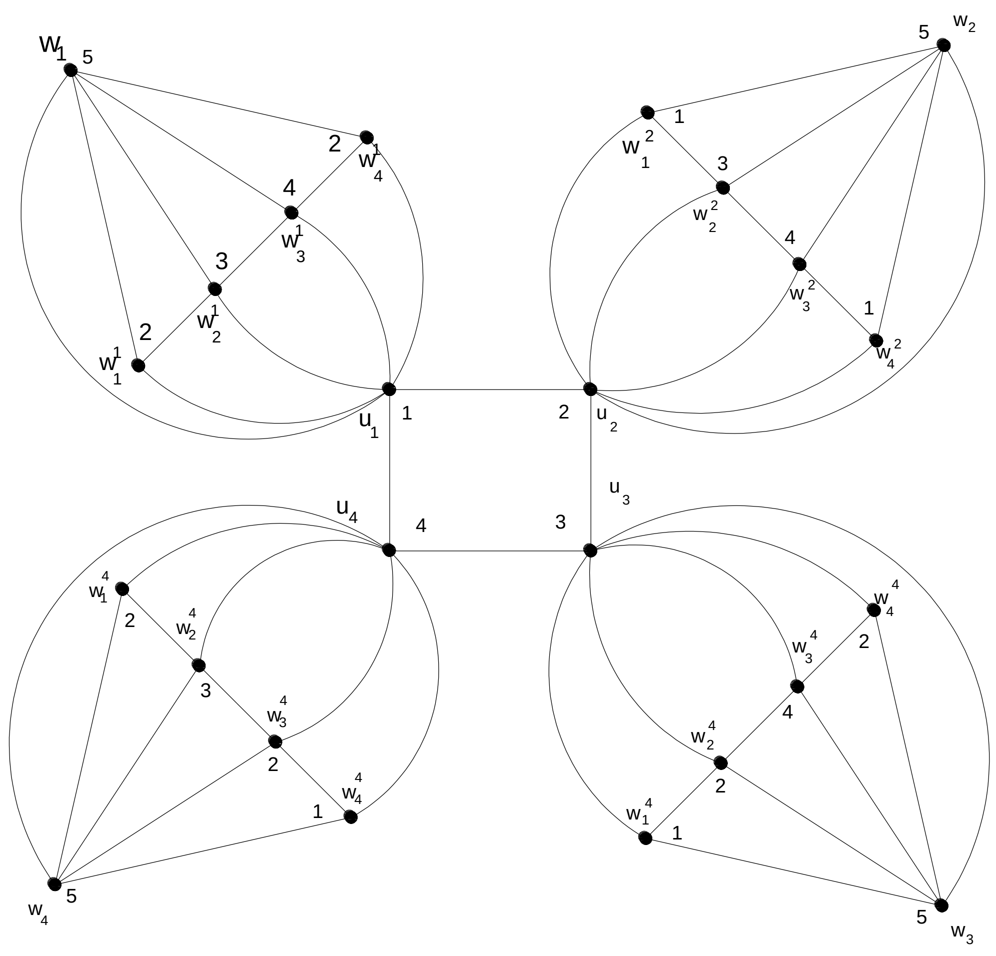

Fig 4.9: $\varphi(C_4 \text{o} F_{1,4}) = 5$

4.5 The Vertex corona graph of Cycle graph with Double Fan graph

Theorem 4.5.1: For any cycle graph C_n and a double fan $F_{2,n}$ then

b- Chromatic number is $\varphi(C_n \circ F_{2,n}) = \begin{cases} n+3, & n=3 \\ n+2, & n \geq 4 \end{cases}$

Proof

we divide the proof into two cases (for $n = 3, n \geq 4$)

let $V(C_n) = \{a_i : 1 \leq i \leq n\}$ and $V(F_{2,n}) = \{b_i : 1 \leq i \leq n+2\}$

By the definition of corona product each vertex of C_n is adjacent to every vertex of number of copies of $F_{2,n}$ then the vertex set of the corona product $C_n \circ F_{2,n}$ is ,

$V(C_n \circ F_{2,n}) = \{a_i : 1 \leq i \leq n\} \cup \{b^i{}_j : 1 \leq i \leq n, 1 \leq j \leq n+2\}$.

Case 1: For $n = 3$

Assign the colors to $v(C_n \circ F_{2,n})$ as follows.

- For $a_i : 1 \leq i \leq n$, assign the color c_i
- For $b^i{}_j : 1 \leq i \leq n,\ 1 \leq j \leq n+2, j \neq n-2$, assign the color c_{i+1}
- For $b^i{}_j : 1 \leq i \leq n,\ j = n-2$, assign the color c_{i+n}

An easy check shows that the above said coloring satisfy the

b-coloring condition with maximum n+3 colors.

Case 2: For $n \geq 4$

Assign the colors to V ($C_n \circ F_{2,n}$) as follows.

- For $a_i : 1 \leq i \leq n$, assign the color c_i
- For , if i is odd assign the color c_{n+1}, if i is even assign the color c_{n+2}
- For $b^i{}_j : 1 \leq i \leq n, 2 \leq j \leq n+1$, assign the color $c_k, 1 \leq k \leq n+2$.

By this coloring procedure, we have that $\varphi[C_n \circ F_{2,n}] \geq n+2$.

To prove the lower bound, let us assume that b-chromatic number of corona product of cycle graph C_n with double fan graph $F_{2,n}$ is greater than n+2.

That is the b-chromatic number of $(C_n \text{ o } F_{2,n}) = n+3$

Assigning n+3 colors to the graph $C_n \text{ o } F_{2,n}$ is only possible, when the graph consists of n+3 vertices with degree n+2. And also all the n+3 colors should adjacent to remaining n+2 colors.

Even though the graph having n+3 vertices with degree n+2, the color class C_{n+3} is not adjacent to the color class C_{n+1} which is the contradiction.

$\therefore \quad \varphi(C_n \text{ o } F_{2,n}) \le n+2$

Hence $\varphi(C_n \text{ o } F_{2,n}) = n+2$

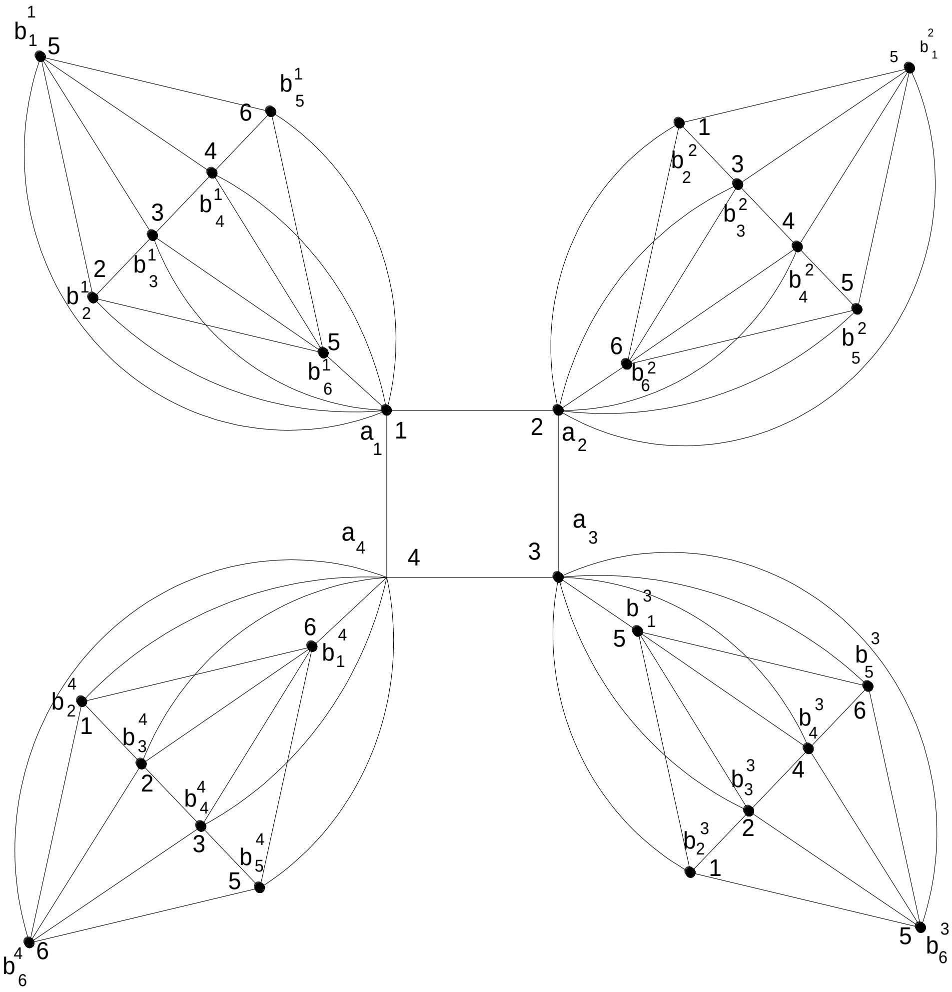

Fig 4.10: $\varphi(C_4 \text{ o} F_{2,4}) = 6$