Interval-Valued Intuitionistic Fuzzy Structures in Z-Algebras

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In 1989, K. T. Atanassov and G. Gargov [10] proposed interval-valued intuitionistic fuzzy set based on the comparative analysis of interval-valued fuzzy sets and intuitionistic fuzzy sets. This chapter is divided into two sections. In the first section, we introduce the notion of interval-valued intuitionistic fuzzy Z-Subalgebras in Z-algebras while in the second section, we introduce the notion of interval-valued intuitionistic fuzzy Z-ideals in Z-algebras and acquired some interesting results.

8.1 Interval-Valued Intuitionistic Fuzzy Z-Subalgebras in Z-algebras

In this section, we establish the notion of interval-valued intuitionistic fuzzification of the concept of Z-Subalgebras in Z-algebras and investigate some of its properties.

Definition 8.1.1: An interval-valued intuitionistic fuzzy set $A = \{\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\rangle | x \in X\}$ in a Z-algebra $(X, *, 0)$ is called an **interval-valued intuitionistic fuzzy Z-Subalgebra** of X if it satisfies the following conditions:

(i) $\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

(ii) $\tilde{\nu}_A(x * y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$, for all $x, y \in X$.

Example 8.1.2: Consider a Z-algebra X= $\{0,1,2,3\}$ be a set with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	2	1
2	0	2	2	1
3	0	1	1	3

Define an interval-valued intuitionistic fuzzy set $A_1 = (\tilde{\mu}_{A_1}, \tilde{\nu}_{A_1})$ in X by

$$\tilde{\mu}_{A_1}(x) = \begin{cases} [0.4, 0.8] & \text{if } x \in \{0,2\} \\ [0.3, 0.5] & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\nu}_{A_1}(x) = \begin{cases} [0.1, 0.2] & \text{if } x \in \{0,2\} \\ [0.2, 0.3] & \text{otherwise} \end{cases}.$$

Then, A_1 is an interval-valued intuitionistic fuzzy Z-Subalgebra of X.

Theorem 8.1.3: An interval-valued intuitionistic fuzzy set $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ in a Z-algebra X is an interval-valued intuitionistic fuzzy Z-Subalgebra of X if and only if $A^U = (\mu_A^U, \nu_A^U)$ and $A^L = (\mu_A^L, \nu_A^L)$ are intuitionistic fuzzy Z-Subalgebras of X.

Proof: Suppose A^U and A^L are intuitionistic fuzzy Z-Subalgebras in a Z-algebra X. Then,

$\mu_A^U(x*y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$, $\nu_A^U(x*y) \leq \max\{\nu_A^U(x), \nu_A^U(y)\}$,

$\mu_A^L(x*y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\nu_A^L(x*y) \leq \max\{\nu_A^L(x), \nu_A^L(y)\}$, for all $x, y \in X$.

Consider $\tilde{\mu}_A(x*y) = [\mu_A^L(x*y), \mu_A^U(x*y)] \geq [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}]$

$$= r\min\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

and $\tilde{\nu}_A(x*y) = [\nu_A^L(x*y), \nu_A^U(x*y)] \leq [\max\{\nu_A^L(x), \nu_A^L(y)\}, \max\{\nu_A^U(x), \nu_A^U(y)\}]$

$$= r\max\{[\nu_A^L(x), \nu_A^U(x)], [\nu_A^L(y), \nu_A^U(y)]\}$$

$$= r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$$

Thus A is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Conversely, for any $x, y \in X$,

$[\mu_A^L(x*y), \mu_A^U(x*y)] = \tilde{\mu}_A(x*y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

$$= r\min\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\}$$

$$= [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}]$$

and $[\nu_A^L(x*y), \nu_A^U(x*y)] = \tilde{\nu}_A(x*y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$

$$= r\max\{[\nu_A^L(x), \nu_A^U(x)], [\nu_A^L(y), \nu_A^U(y)]\}$$

$$= [\max\{\nu_A^L(x), \nu_A^L(y)\}, \max\{\nu_A^U(x), \nu_A^U(y)\}]$$

Hence, $\mu_A^U(x*y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$, $\nu_A^U(x*y) \leq \max\{\nu_A^U(x), \nu_A^U(y)\}$

$\mu_A^L(x*y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$, $\nu_A^L(x*y) \leq \max\{\nu_A^L(x), \nu_A^L(y)\}$, for all $x, y \in X$.

Therefore, $A^U = (\mu_A^U, \nu_A^U)$ and $A^L = (\mu_A^L, \nu_A^L)$ are intuitionistic fuzzy Z-Subalgebras of a Z-algebra X.

Theorem 8.1.4: Let $A_1 = (\tilde{\mu}_{A_1}, \tilde{\nu}_{A_1})$ and $A_2 = (\tilde{\mu}_{A_2}, \tilde{\nu}_{A_2})$ be interval-valued intuitionistic fuzzy Z-Subalgebras of a Z-algebra X. Then $A_1 \cap A_2 = (\tilde{\mu}_{A_1 \cap A_2}, \tilde{\nu}_{A_1 \cap A_2})$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of X.

Proof: Let $x, y \in A_1 \cap A_2$. Then, $x, y \in A_1$ and A_2.

Clearly, $\tilde{\mu}_{A_1 \cap A_2}(x * y) \geq r\min\{\tilde{\mu}_{A_1 \cap A_2}(x), \tilde{\mu}_{A_1 \cap A_2}(y)\}$

Also, $\tilde{\nu}_{A_1 \cup A_2}(x * y) = r\max\{\tilde{\nu}_{A_1}(x * y), \tilde{\nu}_{A_2}(x * y)\}$

$= r\max\{[\nu_{A_1}^L(x * y), \nu_{A_1}^U(x * y)], [\nu_{A_2}^L(x * y), \nu_{A_2}^U(x * y)]\}$

$= [\max\{\nu_{A_1}^L(x * y), \nu_{A_2}^L(x * y)\}, \max\{\nu_{A_1}^U(x * y), \nu_{A_2}^U(x * y)\}]$

$\leq [\max\{\max\{\nu_{A_1}^L(x), \nu_{A_1}^L(y)\}, \max\{\nu_{A_2}^L(x), \nu_{A_2}^L(y)\}\}, \max\{\max\{\nu_{A_1}^U(x), \nu_{A_1}^U(y)\}, \max\{\nu_{A_2}^U(x), \nu_{A_2}^U(y)\}\}]$

$= [\max\{\max\{\nu_{A_1}^L(x), \nu_{A_2}^L(x)\}, \max\{\nu_{A_1}^L(y), \nu_{A_2}^L(y)\}\}, \max\{\max\{\nu_{A_1}^U(x), \nu_{A_2}^U(x)\}, \max\{\nu_{A_1}^U(y), \nu_{A_2}^U(y)\}\}]$

$= [\max\{\nu_{A_1 \cup A_2}^L(x), \nu_{A_1 \cup A_2}^L(y)\}, \max\{\nu_{A_1 \cup A_2}^U(x), \nu_{A_1 \cup A_2}^U(y)\}]$

$= r\max\{[\nu_{A_1 \cup A_2}^L(x), \nu_{A_1 \cup A_2}^U(x)], [\nu_{A_1 \cup A_2}^L(y), \nu_{A_1 \cup A_2}^U(y)]\}$

$= r\max\{\tilde{\nu}_{A_1 \cup A_2}(x), \tilde{\nu}_{A_1 \cup A_2}(y)\}$

Hence $A_1 \cap A_2$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Corollary 8.1.5: Let $\{A_i \mid i \in \Omega\}$ be a family of interval-valued intuitionistic fuzzy Z-Subalgebras of a Z-algebra X. Then $\bigcap_{i \in \Omega} A_i$ is also an interval-valued intuitionistic fuzzy Z-Subalgebra of X.

Theorem 8.1.6: An interval-valued intuitionistic fuzzy set $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X if and only if the interval-valued fuzzy sets $\tilde{\mu}_A$ and $(\tilde{\nu}_A)^c$ are interval-valued fuzzy Z-Subalgebras of X.

Proof: Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X then

$\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and (1)

$\tilde{\nu}_A(x * y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\} \quad \forall\, x, y \in X$. (2)

By (1), the interval-valued fuzzy set $\tilde{\mu}_A$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X.

For every $x, y \in X$,

$$(\tilde{\nu}_A)^c(x*y) = [1,1] - \tilde{\nu}_A(x*y) \geq [1,1] - r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\} = r\min\{[1,1]-\tilde{\nu}_A(x), [1,1]-\tilde{\nu}_A(y)\}$$
$$= r\min\{(\tilde{\nu}_A)^c(x), (\tilde{\nu}_A)^c(y)\}$$

$$(\tilde{\nu}_A)^c(x*y) \geq r\min\{(\tilde{\nu}_A)^c(x), (\tilde{\nu}_A)^c(y)\}$$

Hence $(\tilde{\nu}_A)^c$ is an interval-valued fuzzy Z-Subalgebra of a Z-algebra X.

Conversely, for every $x, y \in X$,

$\tilde{\mu}_A(x*y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and (3)

$$[1,1] - \tilde{\nu}_A(x*y) = (\tilde{\nu}_A)^c(x*y) \geq r\min\{(\tilde{\nu}_A)^c(x), (\tilde{\nu}_A)^c(y)\} = r\min\{[1,1]-\tilde{\nu}_A(x), [1,1]-\tilde{\nu}_A(y)\}$$
$$= [1,1] - r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$$

i.e., $\tilde{\nu}_A(x*y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$ (4)

By (3) and (4) we get, $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

This completes the proof.

Theorem 8.1.7: $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X if and only if $\oplus A = (\tilde{\mu}_A, (\tilde{\mu}_A)^c)$ and $\otimes A = ((\tilde{\nu}_A)^c, \tilde{\nu}_A)$, both are interval-valued intuitionistic fuzzy Z-Subalgebras of X.

Proof: Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

To prove: $\oplus A$ and $\otimes A$, both are interval-valued intuitionistic fuzzy Z-Subalgebras of a Z-algebra X.

Let $x, y \in X$. Then,

(i) $\tilde{\mu}_A(x*y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

(ii) $\tilde{\nu}_A(x*y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$

(iii) $(\tilde{\mu}_A)^c(x*y) = [1,1] - \tilde{\mu}_A(x*y) \leq [1,1] - r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$
$$= r\max\{[1,1]-\tilde{\mu}_A(x), [1,1]-\tilde{\mu}_A(y)\}$$

$$= r\max\left\{(\widetilde{\mu}_A)^c(x),(\widetilde{\mu}_A)^c(y)\right\}$$

(iv) $(\widetilde{\nu}_A)^c(x*y)=[1,1]-\widetilde{\nu}_A(x*y)\geq[1,1]-r\max\{\widetilde{\nu}_A(x),\widetilde{\nu}_A(y)\}$

$$= r\min\left\{[1,1]-\widetilde{\nu}_A(x),[1,1]-\widetilde{\nu}_A(y)\right\}$$

$$= r\min\{(\widetilde{\nu}_A)^c(x),(\widetilde{\nu}_A)^c(y)\}$$

From (i) and (iii), we get $\oplus A$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

And, from (ii) and (iv), we get $\otimes A$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Conversely, assume that $\oplus A=\left(\widetilde{\mu}_A,(\widetilde{\mu}_A)^c\right)$ and $\otimes A=\left((\widetilde{\nu}_A)^c,\widetilde{\nu}_A\right)$ are interval-valued intuitionistic fuzzy Z-Subalgebras of a Z-algebra X.

For any $x,y\in A$,

$\widetilde{\mu}_A(x*y)\geq r\min\{\widetilde{\mu}_A(x),\widetilde{\mu}_A(y)\}$

and $\widetilde{\nu}_A(x*y)\leq r\max\{\widetilde{\nu}_A(x),\widetilde{\nu}_A(y)\}$

Hence $A=\left(\widetilde{\mu}_A,\widetilde{\nu}_A\right)$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 8.1.8: Let $A=\left(\widetilde{\mu}_A,\widetilde{\nu}_A\right)$ be an interval-valued intuitionistic fuzzy set in a Z-algebra X. Then A is an interval-valued intuitionistic fuzzy Z-Subalgebra of X if and only if the nonempty set

$$U\left(\widetilde{\mu}_A;[s_1,s_2]\right)=\{x\in X\mid \widetilde{\mu}_A(x)\geq[s_1,s_2]\}$$

and

$$L\left(\widetilde{\nu}_A;[t_1,t_2]\right)=\{x\in X\mid \widetilde{\nu}_A(x)\leq[t_1,t_2]\}$$

are interval-valued Z-Subalgebras of X for any $[s_1,s_2],[t_1,t_2]\in D[0,1]$.

Proof: Assume that A is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X and let $[s_1,s_2]\in D[0,1]$ be such that $x,y\in U\left(\widetilde{\mu}_A;[s_1,s_2]\right)$.

This implies $\widetilde{\mu}_A(x)\geq[s_1,s_2]$ and $\widetilde{\mu}_A(y)\geq[s_1,s_2]$.

$$\widetilde{\mu}_A(x*y)\geq r\min\left\{\widetilde{\mu}_A(x),\widetilde{\mu}_A(y)\right\}\geq r\min\{[s_1,s_2],[s_1,s_2]\}=\left[\min\{s_1,s_1\},\min\{s_2,s_2\}\right]$$

$$=[s_1,s_2]$$

and so $\widetilde{\mu}_A(x*y)\geq[s_1,s_2]$

Thus $U\left(\widetilde{\mu}_A;[s_1,s_2]\right)$ is an interval-valued Z-Subalgebra of a Z-algebra X.

Similarly, we can prove $L(\tilde{\nu}_A;[t_1,t_2])$ is an interval-valued Z-Subalgebra of a Z-algebra X for all $[t_1,t_2] \in D[0,1]$.

Conversely, assume that the nonempty sets $U(\tilde{\mu}_A;[s_1,s_2])$ and $L(\tilde{\nu}_A;[t_1,t_2])$ are an interval-valued Z-Subalgebra of a Z-algebra X for any $[s_1,s_2],[t_1,t_2] \in D[0,1]$.

Suppose there exists $x_0, y_0 \in X$ such that $\tilde{\mu}_A(x_0 * y_0) < r\min\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}$. (1)

Let $\tilde{\mu}_A(x_0) = [r_1,r_2]$, $\tilde{\mu}_A(y_0) = [r_3,r_4]$ and $\tilde{\mu}_A(x_0 * y_0) = [s_1,s_2]$.

Then (1) becomes, $[s_1,s_2] < r\min\{[r_1,r_2],[r_3,r_4]\} = [\min\{r_1,r_3\}, \min\{r_2,r_4\}]$

so $s_1 < \min\{r_1,r_3\}$ and $s_2 < \min\{r_2,r_4\}$.

If we take $[\lambda_1,\lambda_2] = \frac{1}{2}[\tilde{\mu}_A(x_0 * y_0) + r\min\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}]$

then $[\lambda_1,\lambda_2] = \frac{1}{2}[[s_1,s_2] + r\min\{[r_1,r_2],[r_3,r_4]\}] = \frac{1}{2}[[s_1,s_2] + [\min\{r_1,r_3\}, \min\{r_2,r_4\}]]$

$$= \left[\frac{1}{2}(s_1 + \min\{r_1,r_3\}), \frac{1}{2}(s_2 + \min\{r_2,r_4\})\right]$$

This implies $\lambda_1 = \frac{1}{2}(s_1 + \min\{r_1,r_3\})$ and $\lambda_2 = \frac{1}{2}(s_2 + \min\{r_2,r_4\})$

Therefore, $\min\{r_1,r_3\} > \lambda_1 = \frac{1}{2}(s_1 + \min\{r_1,r_3\}) > s_1$ and $\min\{r_2,r_4\} > \lambda_2 = \frac{1}{2}(s_2 + \min\{r_2,r_4\}) > s_2$

Hence, $[\min\{r_1,r_3\}, \min\{r_2,r_4\}] > [\lambda_1,\lambda_2] > [s_1,s_2] = \tilde{\mu}_A(x_0 * y_0)$

Therefore $x_0 * y_0 \notin U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$, which is a contradiction.

But, $\tilde{\mu}_A(x_0) = [r_1,r_2] \geq [\min\{r_1,r_3\}, \min\{r_2,r_4\}] > [\lambda_1,\lambda_2]$

and $\tilde{\mu}_A(y_0) = [r_3,r_4] \geq [\min\{r_1,r_3\}, \min\{r_2,r_4\}] > [\lambda_1,\lambda_2]$

implies $x_0, y_0 \in U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$, which is a contradiction.

Suppose there exists $x_0, y_0 \in X$ such that $\tilde{\nu}_A(x_0 * y_0) > r\max\{\tilde{\nu}_A(x_0), \tilde{\nu}_A(y_0)\}$ (2)

Let $\tilde{\nu}_A(x_0) = [\delta_1,\delta_2]$, $\tilde{\nu}_A(y_0) = [\delta_3,\delta_4]$ and $\tilde{\nu}_A(x_0 * y_0) = [t_1,t_2]$.

Then (2) becomes, $[t_1,t_2] > r\max\{[\delta_1,\delta_2],[\delta_3,\delta_4]\} = [\max\{\delta_1,\delta_3\}, \max\{\delta_2,\delta_4\}]$

so $t_1 > \max\{\delta_1,\delta_3\}$ and $t_2 > \max\{\delta_2,\delta_4\}$.

If we take $[\alpha_1,\alpha_2]=\frac{1}{2}[\tilde{\nu}_A(x_0*y_0)+r\max\{\tilde{\nu}_A(x_0),\tilde{\nu}_A(y_0)\}]$

then $[\alpha_1,\alpha_2]=\frac{1}{2}[[t_1,t_2]+r\max\{[\delta_1,\delta_2],[\delta_3,\delta_4]\}]=\frac{1}{2}[[t_1,t_2]+[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}]]$

$$=\left[\frac{1}{2}(t_1+\max\{\delta_1,\delta_3\}),\frac{1}{2}(t_2+\max\{\delta_2,\delta_4\})\right]$$

This implies $\alpha_1=\frac{1}{2}(t_1+\max\{\delta_1,\delta_3\}),\alpha_2=\frac{1}{2}(t_2+\max\{\delta_2,\delta_4\})$

Therefore, $\max\{\delta_1,\delta_3\}<\alpha_1=\frac{1}{2}(t_1+\max\{\delta_1,\delta_3\})<t_1$

and $\max\{\delta_2,\delta_4\}<\alpha_2=\frac{1}{2}(t_2+\max\{\delta_2,\delta_4\})<t_2$

Hence, $[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}]<[\alpha_1,\alpha_2]<[t_1,t_2]=\tilde{\nu}_A(x_0*y_0)$

This implies that $[\alpha_1,\alpha_2]<\tilde{\nu}_A(x_0*y_0)$. Therefore $x_0*y_0\notin L(\tilde{\nu}_A;[\alpha_1,\alpha_2])$.

But $\tilde{\nu}_A(x_0)=[\delta_1,\delta_2]\le[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}]<[\alpha_1,\alpha_2]\Rightarrow\tilde{\nu}_A(x_0)<[\alpha_1,\alpha_2]$

and $\tilde{\nu}_A(y_0)=[\delta_3,\delta_4]\le[\max\{\delta_1,\delta_3\},\max\{\delta_2,\delta_4\}]<[\alpha_1,\alpha_2]\Rightarrow\tilde{\nu}_A(y_0)<[\alpha_1,\alpha_2]$

implies $x_0,y_0\in L(\tilde{\nu}_A;[\alpha_1,\alpha_2])$ which is a contradiction.

Thus $\tilde{\mu}_A(x*y)\ge r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$, $\tilde{\nu}_A(x*y)\le r\max\{\tilde{\nu}_A(x),\tilde{\nu}_A(y)\}$, for all $x,y\in X$.

Hence, $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 8.1.9: Any Z-Subalgebra of a Z-algebra X can be realized as both the interval-valued upper level and interval-valued lower level Z-Subalgebras of some interval-valued intuitionistic fuzzy Z-Subalgebras of X.

Proof: Let Y be a Z-Subalgebra of a Z-algebra X and A be an interval-valued intuitionistic fuzzy set on X defined by

$$\tilde{\mu}_A(x)=\begin{cases}[s_1,s_2] & \text{if } x\in Y\\ [0,0] & \text{otherwise}\end{cases}\quad\text{and}\quad \tilde{\nu}_A(x)=\begin{cases}[t_1,t_2] & \text{if } x\in Y\\ [1,1] & \text{otherwise}\end{cases}$$

where $[s_1,s_2],[t_1,t_2]\in D[0,1]$ with $[s_1,s_2]>[0,0]$ and $[t_1,t_2]<[1,1]$ and $s_1+t_1\le 1; s_2+t_2\le 1$.

It is clear that $U(\tilde{\mu}_A;[s_1,s_2])=Y=L(\tilde{\nu}_A;[t_1,t_2])$.

Let $x, y \in X$. We consider the following cases.

Case (1): If $x, y \in Y$ then $x * y \in Y$. Also $\tilde{\mu}_A(x) = \tilde{\mu}_A(y) = [s_1, s_2]$, $\tilde{\nu}_A(x) = \tilde{\nu}_A(y) = [t_1, t_2]$.

Now, $\tilde{\mu}_A(x * y) = [s_1, s_2] = r\min\{[s_1, s_2],[s_1, s_2]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

and $\tilde{\nu}_A(x * y) = [t_1, t_2] = r\max\{[t_1, t_2],[t_1, t_2]\} = r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$

Case (2): If $x, y \notin Y$ then $\tilde{\mu}_A(x) = [0,0] = \tilde{\mu}_A(y)$ and $\tilde{\nu}_A(x) = [1,1] = \tilde{\nu}_A(y)$.

Thus, $\tilde{\mu}_A(x * y) = [0,0] = r\min\{[0,0],[0,0]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

and $\tilde{\nu}_A(x * y) = [1,1] = r\max\{[1,1],[1,1]\} = r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$

Case (3): If $x \in Y$ and $y \notin Y$ then $\tilde{\mu}_A(x) = [s_1, s_2]$, $\tilde{\mu}_A(y) = [0,0]$, $\tilde{\nu}_A(x) = [t_1, t_2]$ and $\tilde{\nu}_A(y) = [1,1]$.

Thus, $\tilde{\mu}_A(x * y) \geq [0,0] = r\min\{[s_1, s_2],[0,0]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

and $\tilde{\nu}_A(x * y) \leq [1,1] = r\max\{[t_1, t_2],[1,1]\} = r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$

Case (4): If $y \in Y$ and $x \notin Y$ then $\tilde{\mu}_A(x) = [0,0]$, $\tilde{\mu}_A(y) = [s_1, s_2]$, $\tilde{\nu}_A(x) = [1,1]$ and $\tilde{\nu}_A(y) = [t_1, t_2]$.

Thus, $\tilde{\mu}_A(x * y) \geq [0,0] = r\min\{[0,0],[s_1, s_2]\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

and $\tilde{\nu}_A(x * y) \leq [1,1] = r\max\{[1,1],[t_1, t_2]\} = r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$

Therefore A is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 8.1.10: Let Y be a subset of a Z-algebra X and A be an interval-valued intuitionistic fuzzy set on X which is given in the proof of Theorem 8.1.9. If Y be realized as interval-valued upper $[s_1, s_2]$-level Z-Subalgebra and interval-valued lower $[t_1, t_2]$- level Z-Subalgebra of some interval-valued intuitionistic fuzzy Z-Subalgebra of X, then Y is a Z-Subalgebra of X.

Proof: Let Y be a subset of a Z-algebra X and Let A be an interval-valued intuitionistic fuzzy Z-Subalgebra of X such that $U(\tilde{\mu}_A;[s_1, s_2]) = Y = L(\tilde{\nu}_A;[t_1, t_2])$.

Let $x, y \in Y$. Then, $\tilde{\mu}_A(x) = [s_1, s_2] = \tilde{\mu}_A(y)$ and $\tilde{\nu}_A(x) = [t_1, t_2] = \tilde{\nu}_A(y)$.

Thus,

$\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = r\min\{[s_1, s_2],[s_1, s_2]\} = [\min\{s_1, s_1\}, \min\{s_2, s_2\}] = [s_1, s_2]$

and

$\tilde{\nu}_A(x * y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\} = r\max\{[t_1, t_2],[t_1, t_2]\} = [\max\{[t_1, t_1\}, \max\{t_2, t_2\}] = [t_1, t_2]$

Thus $x * y \in Y$.

Hence Y is a Z-Subalgebra of a Z-algebra X.

Theorem 8.1.11: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and A be an interval-valued intuitionistic fuzzy Z-Subalgebra of X with rsup-rinf property. Then the image of A, $h(A)=\{\langle y,\tilde{\mu}_{h(A)}(y),\tilde{\nu}_{h(A)}(y)\rangle | y \in Y\}$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of Y.

Proof: Let $a,b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that

$$\tilde{\mu}_A(x_0) = \underset{t\in h^{-1}(a)}{\mathrm{rsup}}\,\tilde{\mu}_A(t)\ ;\ \tilde{\mu}_A(y_0) = \underset{t\in h^{-1}(b)}{\mathrm{rsup}}\,\tilde{\mu}_A(t) \qquad \text{and}$$

$$\tilde{\nu}_A(x_0) = \underset{t\in h^{-1}(a)}{\mathrm{rinf}}\,\tilde{\nu}_A(t);\ \tilde{\nu}_A(y_0) = \underset{t\in h^{-1}(b)}{\mathrm{rinf}}\,\tilde{\nu}_A(t)$$

Now,

$$\tilde{\mu}_{h(A)}(a *' b) = \underset{t\in h^{-1}(a*'b)}{\mathrm{rsup}}\,\tilde{\mu}_A(t) \geq \tilde{\mu}_A(x_0 * y_0) \geq \mathrm{r}\min\{\tilde{\mu}_A(x_0),\tilde{\mu}_A(y_0)\}$$

$$= \mathrm{r}\min\left\{\underset{t\in h^{-1}(a)}{\mathrm{rsup}}\,\tilde{\mu}_A(t),\ \underset{t\in h^{-1}(b)}{\mathrm{rsup}}\,\tilde{\mu}_A(t)\right\}$$

$$= \mathrm{r}\min\{\tilde{\mu}_{h(A)}(a),\tilde{\mu}_{h(A)}(b)\}$$

Also $\tilde{\nu}_{h(A)}(a *' b) = \underset{t\in h^{-1}(a*'b)}{\mathrm{rinf}}\,\tilde{\nu}_A(t) \leq \tilde{\nu}_A(x_0 * y_0) \leq \mathrm{r}\max\{\tilde{\nu}_A(x_0),\tilde{\nu}_A(y_0)\}$

$$= \mathrm{r}\max\left\{\underset{t\in h^{-1}(a)}{\mathrm{rinf}}\,\tilde{\nu}_A(t),\ \underset{t\in h^{-1}(b)}{\mathrm{rinf}}\,\tilde{\nu}_A(t)\right\}$$

$$= \mathrm{r}\max\{\tilde{\nu}_{h(A)}(a),\tilde{\nu}_{h(A)}(b)\}$$

Hence h(A) is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra Y.

Theorem 8.1.12: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras and $B=\{\langle x,\tilde{\mu}_B(x),\tilde{\nu}_B(x)\rangle | x \in Y\}$ be an interval-valued intuitionistic fuzzy Z-Subalgebra of Y. Then the pre-image of B, $h^{-1}(B)=\left\{\langle x,\tilde{\mu}_{h^{-1}(B)}(x),\tilde{\nu}_{h^{-1}(B)}(x)\rangle | x\in X\right\}$ of B under h is an interval-valued intuitionistic fuzzy Z-Subalgebra of X. Converse is true if h is an Z-epimorphism.

Proof: Let $B=\{\langle x,\tilde{\mu}_B(x),\tilde{\nu}_B(x)\rangle | x \in Y\}$ be an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra Y.

Let $x, y \in X$,

Then $\tilde{\mu}_{h^{-1}(B)}(x * y) = \tilde{\mu}_B(h(x * y)) = \tilde{\mu}_B(h(x) *' h(y)) \geq r\min\{\tilde{\mu}_B(h(x)), \tilde{\mu}_B(h(y))\}$

$$= r\min\{\tilde{\mu}_{h^{-1}(B)}(x), \tilde{\mu}_{h^{-1}(B)}(y)\}$$

and $\tilde{\nu}_{h^{-1}(B)}(x * y) = \tilde{\nu}_B(h(x * y)) = \tilde{\nu}_B(h(x) *' h(y)) \leq r\max\{\tilde{\nu}_B(h(x)), \tilde{\nu}_B(h(y))\}$

$$= r\max\{\tilde{\nu}_{h^{-1}(B)}(x), \tilde{\nu}_{h^{-1}(B)}(y)\}$$

Therefore $h^{-1}(B) = \left\{\left\langle x, \tilde{\mu}_{h^{-1}(B)}(x), \tilde{\nu}_{h^{-1}(B)}(x)\right\rangle \middle| x \in X\right\}$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

On the other hand, let $y_1, y_2 \in Y$. Since h is an Z-epimorphism there exists $x_1, x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

This implies $x_1 = h^{-1}(y_1)$ and $x_2 = h^{-1}(y_2)$.

Now, $\tilde{\mu}_B(y_1 *' y_2) = \tilde{\mu}_B(h(x_1) *' h(x_2)) = \tilde{\mu}_B(h(x_1 * x_2)) = \tilde{\mu}_{h^{-1}(B)}(x_1 * x_2)$

$$\geq r\min\{\tilde{\mu}_{h^{-1}(B)}(x_1), \tilde{\mu}_{h^{-1}(B)}(x_2)\}$$

$$= r\min\{\tilde{\mu}_B(h(x_1)), \tilde{\mu}_B(h(x_2))\}$$

$$= r\min\{\tilde{\mu}_B(y_1), \tilde{\mu}_B(y_2)\}$$

and $\tilde{\nu}_B(y_1 *' y_2) = \tilde{\nu}_B(h(x_1) *' h(x_2)) = \tilde{\nu}_B(h(x_1 * x_2)) = \tilde{\nu}_{h^{-1}(B)}(x_1 * x_2)$

$$\leq r\max\{\tilde{\nu}_{h^{-1}(B)}(x_1), \tilde{\nu}_{h^{-1}(B)}(x_2)\}$$

$$= r\max\{\tilde{\nu}_B(h(x_1)), \tilde{\nu}_B(h(x_2))\}$$

$$= r\max\{\tilde{\nu}_B(y_1), \tilde{\nu}_B(y_2)\}$$

Hence B is an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra Y.

Theorems 8.1.13: Let A and B be any two interval-valued intuitionistic fuzzy Z-Subalgebras of a Z-algebra X. Then $A \times B$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of $X \times X$.

Proof: Take (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then clearly,

$$\tilde{\mu}_{A\times B}[(x_1, y_1) * (x_2, y_2)] \geq r\min\{\tilde{\mu}_{A\times B}(x_1, y_1), \tilde{\mu}_{A\times B}(x_2, y_2)\}$$

$$\tilde{\nu}_{A\times B}[(x_1, y_1) * (x_2, y_2)] = \tilde{\nu}_{A\times B}[(x_1 * x_2), (y_1 * y_2)]$$

$$= r\max\{\tilde{\nu}_A(x_1 * x_2), \tilde{\nu}_B(y_1 * y_2)\}$$

$$\leq \text{r}\max\{\text{r}\max\{\tilde{\nu}_A(x_1), \tilde{\nu}_A(x_2)\}, \text{r}\max\{\tilde{\nu}_B(y_1), \tilde{\nu}_B(y_2)\}\}$$
$$= \text{r}\max\{\text{r}\max\{\tilde{\nu}_A(x_1), \tilde{\nu}_B(y_1)\}, \text{r}\max\{\tilde{\nu}_A(x_2), \tilde{\nu}_B(y_2)\}\}$$
$$= \text{r}\max\{\tilde{\nu}_{A\times B}(x_1, y_1), \tilde{\nu}_{A\times B}(x_2, y_2)\}$$

This proves that the Cartesian product of any two interval-valued intuitionistic fuzzy Z-Subalgebras is again an interval-valued intuitionistic fuzzy Z-Subalgebra of a Z-algebra X.

8.2 Interval-Valued Intuitionistic Fuzzy Z-Ideals in Z-algebras

In this section, we introduce the notion of Interval-Valued Intuitionistic Fuzzy Z-ideals of Z-algebras. Also we have proved some simple interesting results.

Definition 8.2.1: An interval-valued intuitionistic fuzzy set $A = \{\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\rangle | x \in X\}$ in a Z-algebra $(X,*,0)$ is called an **interval-valued intuitionistic fuzzy Z-ideal** of X if it satisfies the following conditions:

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\nu}_A(0) \leq \tilde{\nu}_A(x)$

(ii) $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$

(iii) $\tilde{\nu}_A(x) \leq r\max\{\tilde{\nu}_A(x*y), \tilde{\nu}_A(y)\}$, for all $x, y \in X$.

Example 8.2.2: Consider a Z-algebra X= {0,1,2,3} be a set with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	3	2
2	0	3	2	2
3	0	2	2	3

Define an interval-valued intuitionistic fuzzy set $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ in X as follows: $\tilde{\mu}_A(x)$=[0.4,0.5] for all $x = 0,1,2,3$ and $\tilde{\nu}_A(x)$=[0.3,0.4] for all $x = 0,1,2,3$. Then, $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of X.

Theorem 8.2.3 : Let $A = (\tilde{\mu}_A, \tilde{\nu}_A) = ([\mu_A^L, \mu_A^U],[\nu_A^L, \nu_A^U])$ be an interval-valued intuitionistic fuzzy set of a Z-algebra X then A is an interval-valued intuitionistic fuzzy Z-ideal of X $\Leftrightarrow A^L = (\mu_A^L, \nu_A^L)$ and $A^U = (\mu_A^U, \nu_A^U)$ are intuitionistic fuzzy Z-ideals of X.

Proof: Assume $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-Ideal of a Z-algebra X.

$\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\nu}_A(0) \leq \tilde{\nu}_A(x)$, for all $x \in X$.

$\Leftrightarrow$ $[\mu_A^L(0), \mu_A^U(0)] \geq [\mu_A^L(x), \mu_A^U(x)]$ and $[\nu_A^L(0), \nu_A^U(0)] \leq [\nu_A^L(x), \nu_A^U(x)]$, for all $x \in X$.

$\Leftrightarrow$ $\mu_A^L(0) \geq \mu_A^L(x)$, $\nu_A^L(0) \leq \nu_A^L(x)$ and $\mu_A^U(0) \geq \mu_A^U(x)$, $\nu_A^U(0) \leq \nu_A^U(x)$, for all $x \in X$. (1)

$\widetilde{\mu}_A(x) \geq r\min\{\widetilde{\mu}_A(x*y), \widetilde{\mu}_A(y)\} = r\min\{[\mu_A^L(x*y), \mu_A^U(x*y)], [\mu_A^L(y), \mu_A^U(y)]\}$

$\Leftrightarrow [\mu_A^L(x), \mu_A^U(x)] \geq [\min\{\mu_A^L(x*y), \mu_A^L(y)\}, \min\{\mu_A^U(x*y), \mu_A^U(y)\}]$

$\Leftrightarrow \mu_A^L(x) \geq \min\{\mu_A^L(x*y), \mu_A^L(y)\}$ and $\mu_A^U(x) \geq \min\{\mu_A^U(x*y), \mu_A^U(y)\}$,for all $x, y \in X$. (2)

Similarly, $\widetilde{\nu}_A(x) \leq r\max\{\widetilde{\nu}_A(x*y), \widetilde{\nu}_A(y)\} = r\max\{[\nu_A^L(x*y), \nu_A^U(x*y)], [\nu_A^L(y), \nu_A^U(y)]\}$

$\Leftrightarrow [\nu_A^L(x), \nu_A^U(x)] \leq [\max\{\nu_A^L(x*y), \nu_A^L(y)\}, \max\{\nu_A^U(x*y), \nu_A^U(y)\}]$

$\Leftrightarrow \nu_A^L(x) \leq \max\{\nu_A^L(x*y), \nu_A^L(y)\}$ and $\nu_A^U(x) \leq \max\{\nu_A^U(x*y), \nu_A^U(y)\}$, for all $x, y \in X$. (3)

By (1), (2) and (3) $\Leftrightarrow A^L = (\mu_A^L, \nu_A^L)$ and $A^U = (\mu_A^U, \nu_A^U)$ are intuitionistic fuzzy Z-Ideals of a Z-algebra X.

Lemma 8.2.4: An interval-valued intuitionistic fuzzy set $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X if and only if $\widetilde{\mu}_A$ and $(\widetilde{\nu}_A)^c$ are interval-valued fuzzy Z-ideals of X.

Proof : Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

Clearly, $\widetilde{\mu}_A$ is an interval-valued fuzzy Z-ideal of a Z-algebra X.

For any $x, y \in X$,

$(\widetilde{\nu}_A)^c(0) = [1,1] - \widetilde{\nu}_A(0) \geq [1,1] - \widetilde{\nu}_A(x) = (\widetilde{\nu}_A)^c(x)$

$(\widetilde{\nu}_A)^c(x) = [1,1] - \widetilde{\nu}_A(x) \geq [1,1] - r\max\{\widetilde{\nu}_A(x*y), \widetilde{\nu}_A(y)\}$

$= r\min\{[1,1] - \widetilde{\nu}_A(x*y), [1,1] - \widetilde{\nu}_A(y)\} = r\min\{(\widetilde{\nu}_A)^c(x*y), (\widetilde{\nu}_A)^c(y)\}$

Hence $(\widetilde{\nu}_A)^c$ is an interval-valued fuzzy Z- ideal of a Z-algebra X.

Conversely, for every $x, y \in X$, we have,

(i) $\widetilde{\mu}_A(0) \geq \widetilde{\mu}_A(x)$

(ii) $[1,1] - \widetilde{\nu}_A(0) = (\widetilde{\nu}_A)^c(0) \geq (\widetilde{\nu}_A)^c(x) = [1,1] - \widetilde{\nu}_A(x)$. That is, $\widetilde{\nu}_A(0) \leq \widetilde{\nu}_A(x)$

(iii) $\widetilde{\mu}_A(x) \geq r\min\{\widetilde{\mu}_A(x*y), \widetilde{\mu}_A(y)\}$

(iv) $[1,1] - \widetilde{\nu}_A(x) = (\widetilde{\nu}_A)^c(x) \geq r\min\{(\widetilde{\nu}_A)^c(x*y), (\widetilde{\nu}_A)^c(y)\}$

$= r\min\{[1,1] - \widetilde{\nu}_A(x*y), [1,1] - \widetilde{\nu}_A(y)\} = [1,1] - r\max\{\widetilde{\nu}_A(x*y), \widetilde{\nu}_A(y)\}$

That is, $\widetilde{\nu}_A(x) \leq r\max\{\widetilde{\nu}_A(x*y), \widetilde{\nu}_A(y)\}$

From (i), (ii), (iii) and (iv) we get $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

Theorem 8.2.5: Let $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy set in a Z-algebra X. Then $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of X if and only if $\oplus A=(\tilde{\mu}_A,(\tilde{\mu}_A)^c)$ and $\otimes A=((\tilde{\nu}_A)^c,\tilde{\nu}_A)$ are interval-valued intuitionistic fuzzy Z-ideals of X.

Proof : Let $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

$\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x)\Rightarrow\ [1,1]-(\tilde{\mu}_A)^c(0)\ge[1,1]-(\tilde{\mu}_A)^c(x)$

$\Rightarrow\ (\tilde{\mu}_A)^c(0)\le(\tilde{\mu}_A)^c(x)$, for any $x\in X$. (1)

$(\tilde{\mu}_A)^c(x)=[1,1]-\tilde{\mu}_A(x)\le[1,1]-r\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}$

$=r\max\{[1,1]-\tilde{\mu}_A(x*y),[1,1]-\tilde{\mu}_A(y)\}$

$=r\max\{(\tilde{\mu}_A)^c(x*y),(\tilde{\mu}_A)^c(y)\}$ (2)

$\tilde{\nu}_A(0)\le\tilde{\nu}_A(x)\Rightarrow\ [1,1]-(\tilde{\nu}_A)^c(0)\le[1,1]-(\tilde{\nu}_A)^c(x)$

$\Rightarrow\ (\tilde{\nu}_A)^c(0)\ge(\tilde{\nu}_A)^c(x)$, for any $x\in X$ and (3)

$(\tilde{\nu}_A)^c(x)=[1,1]-\tilde{\nu}_A(x)\ge[1,1]-r\max\{\tilde{\nu}_A(x*y),\tilde{\nu}_A(y)\}$

$=r\min\{[1,1]-\tilde{\nu}_A(x*y),[1,1]-\tilde{\nu}_A(y)\}$

$=r\min\{(\tilde{\nu}_A)^c(x*y),(\tilde{\nu}_A)^c(y)\}$ (4)

Hence $\oplus A=(X,\tilde{\mu}_A,(\tilde{\mu}_A)^c)$ and $\otimes A=(X,(\tilde{\nu}_A)^c,\tilde{\nu}_A)$ are interval-valued intuitionistic fuzzy Z-ideals of a Z-algebra X. Clearly, converse is true.

Theorem 8.2.6: If $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X then $\otimes A=((\tilde{\nu}_A)^c,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of X.

$\Leftrightarrow\otimes A^L=((\nu_A^L)^c,\nu_A^L)$ and $\otimes A^U=((\nu_A^U)^c,\nu_A^U)$ are intuitionistic fuzzy Z-ideals of X.

Proof: We have $(\tilde{\nu}_A)^c(0)\ge(\tilde{\nu}_A)^c(x)$

$\Leftrightarrow\ [1,1]-\tilde{\nu}_A(0)\ge[1,1]-\tilde{\nu}_A(x)$

$\Leftrightarrow\ [1,1]-[\nu_A^L(0),\nu_A^U(0)]\ge[1,1]-[\nu_A^L(x),\nu_A^U(x)]$

$\Leftrightarrow\ [1-\nu_A^L(0),1-\nu_A^U(0)]\ge[1-\nu_A^L(x),1-\nu_A^U(x)]$

$\Leftrightarrow \quad [(\nu_A^L)^c(0),(\nu_A^U)^c(0)] \geq [(\nu_A^L)^c(x),(\nu_A^U)^c(x)]$

$\Leftrightarrow \quad (\nu_A^L)^c(0) \geq (\nu_A^L)^c(x)$ and $(\nu_A^U)^c(0) \geq (\nu_A^U)^c(x)$

Also

$(\tilde{\nu}_A)^c(x) \geq r\min\{(\tilde{\nu}_A)^c(x*y),(\tilde{\nu}_A)^c(y)\}$

$\Leftrightarrow \quad [1,1]-\tilde{\nu}_A(x) \geq r\min\{[1,1]-\tilde{\nu}_A(x*y),[1,1]-\tilde{\nu}_A(y)\}$

$\Leftrightarrow \quad [1,1]-[\nu_A^L(x),\nu_A^U(x)] \geq r\min\{[1,1]-[\nu_A^L(x*y),\nu_A^U(x*y)],[1,1]-[\nu_A^L(y),\nu_A^U(y)]\}$

$\Leftrightarrow \quad [1-\nu_A^L(x),1-\nu_A^U(x)] \geq r\min\{[1-\nu_A^L(x*y),1-\nu_A^U(x*y)],[1-\nu_A^L(y),1-\nu_A^U(y)]\}$

$\Leftrightarrow \quad [(\nu_A^L)^c(x),(\nu_A^U)^c(x)] \geq r\min\{[(\nu_A^L)^c(x*y),(\nu_A^U)^c(x*y)],[(\nu_A^L)^c(y),(\nu_A^U)^c(y)]\}$

$\Leftrightarrow \quad [(\nu_A^L)^c(x),(\nu_A^U)^c(x)] \geq [\min\{(\nu_A^L)^c(x*y),(\nu_A^L)^c(y)\},\min\{(\nu_A^U)^c(x*y),(\nu_A^U)^c(y)\}]$

Therefore, $(\nu_A^L)^c(x) \geq \min\{(\nu_A^L)^c(x*y),(\nu_A^L)^c(y)\}$ and

$(\nu_A^U)^c(x) \geq \min\{(\nu_A^U)^c(x*y),(\nu_A^U)^c(y)\}$, for all $x, y \in X$.

Theorem 8.2.7: If $A=(\tilde{\mu}_A,\tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X then $\oplus A=(\tilde{\mu}_A,(\tilde{\mu}_A)^c)$ is an interval-valued intuitionistic fuzzy Z-ideal of X.

$\Leftrightarrow \quad \oplus A^L=(\mu_A^L,(\mu_A^L)^c)$ and $\oplus A^U=(\mu_A^U,(\mu_A^U)^c)$ are intuitionistic fuzzy Z-ideals of X.

Proof: We have $(\tilde{\mu}_A)^c(0) \leq (\tilde{\mu}_A)^c(x)$

$\Leftrightarrow \quad [1,1]-\tilde{\mu}_A(0) \leq [1,1]-\tilde{\mu}_A(x)$

$\Leftrightarrow \quad [1,1]-[\mu_A^L(0),\mu_A^U(0)] \leq [1,1]-[\mu_A^L(x),\mu_A^U(x)]$

$\Leftrightarrow \quad [1-\mu_A^L(0),1-\mu_A^U(0)] \leq [1-\mu_A^L(x),1-\mu_A^U(x)]$

$\Leftrightarrow \quad [(\mu_A^L)^c(0),(\mu_A^U)^c(0)] \leq [(\mu_A^L)^c(x),(\mu_A^U)^c(x)]$

$\Leftrightarrow \quad (\mu_A^L)^c(0) \leq (\mu_A^L)^c(x)$ and $(\mu_A^U)^c(0) \leq (\mu_A^U)^c(x)$

And

$(\tilde{\mu}_A)^c(x) \leq r\max\{(\tilde{\mu}_A)^c(x*y),(\tilde{\mu}_A)^c(y)\}$

$$\Leftrightarrow \quad [1,1]-\tilde{\mu}_A(x) \le r\max\{[1,1]-\tilde{\mu}_A(x*y),[1,1]-\tilde{\mu}_A(y)\}$$

$$\Leftrightarrow \quad [1,1]-[\mu_A^L(x),\mu_A^U(x)] \le r\max\{[1,1]-[\mu_A^L(x*y),\mu_A^U(x*y)],[1,1]-[\mu_A^L(y),\mu_A^U(y)]\}$$

$$\Leftrightarrow \quad [1-\mu_A^L(x),1-\mu_A^U(x)] \le r\max\{[1-\mu_A^L(x*y),1-\mu_A^U(x*y)],[1-\mu_A^L(y),1-\mu_A^U(y)]\}$$

$$\Leftrightarrow \quad [(\mu_A^L)^c(x),(\mu_A^U)^c(x)] \le r\max\{[(\mu_A^L)^c(x*y),(\mu_A^U)^c(x*y)],[(\mu_A^L)^c(y),(\mu_A^U)^c(y)]\}$$

$$\Leftrightarrow \quad [(\mu_A^L)^c(x),(\mu_A^U)^c(x)] \le [\max\{(\mu_A^L)^c(x*y),(\mu_A^L)^c(y)\},\max\{(\mu_A^U)^c(x*y),(\mu_A^U)^c(y)\}]$$

Therefore, $(\mu_A^L)^c(x) \le \max\{(\mu_A^L)^c(x*y),(\mu_A^L)^c(y)\}$ and

$(\mu_A^U)^c(x) \le \max\{(\mu_A^U)^c(x*y),(\mu_A^U)^c(y)\}$, for all $x, y \in X$.

Theorem 8.2.8 : Intersection of any two interval-valued intuitionistic fuzzy Z-ideals of a Z-algebra X is again an interval-valued intuitionistic fuzzy Z-ideal of X.

Proof : Let $A = \{\langle x,\tilde{\mu}_A(x),\tilde{\nu}_A(x)\rangle | x \in X\}$ and $B = \{\langle x,\tilde{\mu}_B(x),\tilde{\nu}_B(x)\rangle | x \in X\}$ be an interval-valued intuitionistic fuzzy Z-ideals of a Z-algebra X .

Now, for every $x, y \in X$

$$\tilde{\mu}_{A\cap B}(0) = r\min\{\tilde{\mu}_A(0),\tilde{\mu}_B(0)\} \ge r\min\{\tilde{\mu}_A(x),\tilde{\mu}_B(x)\} = \tilde{\mu}_{A\cap B}(x)$$

$$\tilde{\nu}_{A\cup B}(0) = r\max\{\tilde{\nu}_A(0),\tilde{\nu}_B(0)\} \le r\max\{\tilde{\nu}_A(x),\tilde{\nu}_B(x)\} = \tilde{\nu}_{A\cup B}(x)$$

$$\begin{aligned}\tilde{\mu}_{A\cap B}(x) &= r\min\{\tilde{\mu}_A(x),\tilde{\mu}_B(x)\}\\ &\ge r\min\{r\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\},r\min\{\tilde{\mu}_B(x*y),\tilde{\mu}_B(y)\}\}\\ &= r\min\{r\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_B(x*y)\},r\min\{\tilde{\mu}_A(y),\tilde{\mu}_B(y)\}\}\\ &= r\min\{\tilde{\mu}_{A\cap B}(x*y),\tilde{\mu}_{A\cap B}(y)\}\end{aligned}$$

$$\begin{aligned}\tilde{\nu}_{A\cup B}(x) &= r\max\{\tilde{\nu}_A(x),\tilde{\nu}_B(x)\}\\ &\le r\max\{r\max\{\tilde{\nu}_A(x*y),\tilde{\nu}_A(y)\},r\max\{\tilde{\nu}_B(x*y),\tilde{\nu}_B(y)\}\}\\ &= r\max\{r\max\{\tilde{\nu}_A(x*y),\tilde{\nu}_B(x*y)\},r\max\{\tilde{\nu}_A(y),\tilde{\nu}_B(y)\}\}\\ &= r\max\{\tilde{\nu}_{A\cup B}(x*y),\tilde{\nu}_{A\cup B}(y)\}\end{aligned}$$

Hence $A \cap B$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

We generalize the above theorem as follows.

Theorem 8.2.9: Let $\{A_i \mid i \in \Omega\}$ be a family of interval-valued intuitionistic fuzzy Z-ideals of a Z-algebra X. Then $\bigcap_{i\in\Omega} A_i$ is an interval-valued intuitionistic fuzzy Z-ideal of X.

Theorem 8.2.10: An interval-valued intuitionistic fuzzy set $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X if and only if for all $[s_1, s_2], [t_1, t_2] \in D[0,1]$, the sets $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\nu}_A; [t_1, t_2])$ are either empty or Z-ideals of X.

Proof : Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

Assume that $U(\tilde{\mu}_A; [s_1, s_2]) \neq \phi \neq L(\tilde{\nu}_A; [t_1, t_2])$ for any $[s_1, s_2], [t_1, t_2] \in D[0,1]$.

Since $\tilde{\mu}_A(0) \geq [s_1, s_2]$ and $\tilde{\nu}_A(0) \leq [t_1, t_2]$,

It is clear that $0 \in U(\tilde{\mu}_A; [s_1, s_2]) \cap L(\tilde{\nu}_A; [t_1, t_2])$ and $U(\tilde{\mu}_A; [s_1, s_2])$ is an Z-ideal of a Z-algebra X.

Let $x, y \in X$ such that $x * y \in L(\tilde{\nu}_A; [t_1, t_2])$ and $y \in L(\tilde{\nu}_A; [t_1, t_2])$. Then $\tilde{\nu}_A(x * y) \leq [t_1, t_2]$ and $\tilde{\nu}_A(y) \leq [t_1, t_2]$.

Now, $\tilde{\nu}_A(x) \leq r\max\{\tilde{\nu}_A(x * y), \tilde{\nu}_A(y)\} \leq r\max\{[t_1, t_2], [t_1, t_2]\} = [t_1, t_2]$

$\Rightarrow \quad x \in L(\tilde{\nu}_A; [t_1, t_2])$.

Hence $L(\tilde{\nu}_A; [t_1, t_2])$ is an Z-ideal of a Z-algebra X.

Conversely, for any $x \in X$, let $\tilde{\mu}_A(x) = [s_1, s_2]$ and $\tilde{\nu}_A(x) = [t_1, t_2]$.

Then $x \in U(\tilde{\mu}_A; [s_1, s_2]) \cap L(\tilde{\nu}_A; [t_1, t_2])$

$\Rightarrow \quad U(\tilde{\mu}_A; [s_1, s_2]) \neq \phi \neq L(\tilde{\nu}_A; [t_1, t_2])$

Since $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\nu}_A; [t_1, t_2])$ are Z-ideals of a Z-algebra X,

$0 \in U(\tilde{\mu}_A; [s_1, s_2]) \cap L(\tilde{\nu}_A; [t_1, t_2])$.

Hence, $\tilde{\mu}_A(0) \geq [s_1, s_2] = \tilde{\mu}_A(x)$ and $\tilde{\nu}_A(0) \leq [t_1, t_2] = \tilde{\nu}_A(x) \quad \forall\, x \in X$. (1)

Suppose there exist $x', y' \in X$ such that $\tilde{\mu}_A(x') < r\min\{\tilde{\mu}_A(x' * y'), \tilde{\mu}_A(y')\}$.

By taking $[\lambda_1, \lambda_2] = \frac{1}{2}\left[\tilde{\mu}_A(x') + r\min\{\tilde{\mu}_A(x' * y'), \tilde{\mu}_A(y')\}\right]$

We have, $\tilde{\mu}_A(x') < [\lambda_1, \lambda_2] < r\min\{\tilde{\mu}_A(x' * y'), \tilde{\mu}_A(y')\}$

Hence $x' \notin U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$ but $x' * y' \in U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$ and $y' \in U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$.

That is, $U(\tilde{\mu}_A;[\lambda_1,\lambda_2])$ is not an Z-ideal of a Z-algebra X, which is a contradiction.

Therefore, $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}$, for all $x, y \in X$. (2)

If there exist a, $b \in X$ such that, $\tilde{\nu}_A(a) > r\max\{\tilde{\nu}_A(a*b),\tilde{\nu}_A(b)\}$.

Then by taking $[\alpha_1,\alpha_2] = \frac{1}{2}[\tilde{\nu}_A(a) + r\max\{\tilde{\nu}_A(a*b),\tilde{\nu}_A(b)\}]$

We have, $r\max\{\tilde{\nu}_A(a*b),\tilde{\nu}_A(b)\} < [\alpha_1,\alpha_2] < \tilde{\nu}_A(a)$

Therefore, $a*b \in L(\tilde{\nu}_A;[\alpha_1,\alpha_2])$ and $b \in L(\tilde{\nu}_A;[\alpha_1,\alpha_2])$

But $a \notin L(\tilde{\nu}_A;[\alpha_1,\alpha_2])$, that is, $L(\tilde{\nu}_A;[\alpha_1,\alpha_2])$ is not an Z-ideal of a Z-algebra X ,which is a contradiction.

Hence, $\tilde{\nu}_A(x) \leq r\max\{\tilde{\nu}_A(x*y),\tilde{\nu}_A(y)\}$, for all $x, y \in X$. (3)

By (1), (2) and (3), $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

Theorem 8.2.11: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and A be an interval-valued intuitionistic fuzzy Z-ideal of X with rsup-rinf property. Then image of A, $h(A) = \{\langle y, \tilde{\mu}_{h(A)}(y), \tilde{\nu}_{h(A)}(y)\rangle \mid y \in Y\}$ is an interval-valued intuitionistic fuzzy Z-ideal of Y.

Proof: Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that

$$\tilde{\mu}_A(x_0) = \underset{t\in h^{-1}(a)}{r\sup}\,\tilde{\mu}_A(t)\ ;\ \tilde{\mu}_A(y_0) = \underset{t\in h^{-1}(b)}{r\sup}\,\tilde{\mu}_A(t)$$

$$\tilde{\nu}_A(x_0) = \underset{t\in h^{-1}(a)}{r\inf}\,\tilde{\nu}_A(t)\ ;\ \tilde{\nu}_A(y_0) = \underset{t\in h^{-1}(b)}{r\inf}\,\tilde{\nu}_A(t)$$

Now,

$$\tilde{\mu}_{h(A)}(0') = \underset{t\in h^{-1}(0')}{r\sup}\,\tilde{\mu}_A(t) \geq \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x_0) = \underset{t\in h^{-1}(a)}{r\sup}\,\tilde{\mu}_A(t) = \tilde{\mu}_{h(A)}(a)$$

$$\tilde{\nu}_{h(A)}(0') = \underset{t\in h^{-1}(0')}{r\inf}\,\tilde{\nu}_A(t) \leq \tilde{\nu}_A(0) \leq \tilde{\nu}_A(x_0) = \underset{t\in h^{-1}(a)}{r\inf}\,\tilde{\nu}_A(t) = \tilde{\nu}_{h(A)}(a)$$

$$r\min\{\tilde{\mu}_{h(A)}(a*'b),\tilde{\mu}_{h(A)}(b)\} = r\min\left\{\underset{t\in h^{-1}(a*'b)}{r\sup}\,\tilde{\mu}_A(t), \underset{t\in h^{-1}(b)}{r\sup}\,\tilde{\mu}_A(t)\right\} \leq r\min\{\tilde{\mu}_A(x_0*y_0),\tilde{\mu}_A(y_0)\}$$

$$\leq \tilde{\mu}_A(x_0)$$

$$= r\sup_{t\in h^{-1}(a)} \tilde{\mu}_A(t)$$

$$= \tilde{\mu}_{h(A)}(a)$$

$$r\max\{\tilde{\nu}_{h(A)}(a *' b), \tilde{\nu}_{h(A)}(b)\} = r\max\left\{ r\inf_{t\in h^{-1}(a*'b)} \tilde{\nu}_A(t), r\inf_{t\in h^{-1}(b)} \tilde{\nu}_A(t)\right\} \geq r\max\{\tilde{\nu}_A(x_0 * y_0), \tilde{\nu}_A(y_0)\}$$

$$\geq \tilde{\nu}_A(x_0)$$

$$= r\inf_{t\in h^{-1}(a)} \tilde{\nu}_A(t)$$

$$= \tilde{\nu}_{h(A)}(a)$$

Hence h(A) is an interval-valued intuitionistic fuzzy Z- ideal of a Z-algebra Y.

Theorem 8.2.12: Let $h:(X,*,0)\to(Y,*',0')$ is a Z-homomorphism of Z-algebras and B be an interval-valued intuitionistic fuzzy Z-ideal of Y. Then the inverse image of B, $h^{-1}(B) = \left\{\left\langle x, \tilde{\mu}_{h^{-1}(B)}(x), \tilde{\nu}_{h^{-1}(B)}(x)\right\rangle \middle| x\in X\right\}$ is an interval-valued intuitionistic fuzzy Z-ideal of X.

Proof: Let $x, y \in X$. Now it is clear that

$$\tilde{\mu}_{h^{-1}(B)}(0) = \tilde{\mu}_B(h(0)) \geq \tilde{\mu}_B(h(x)) = \tilde{\mu}_{h^{-1}(B)}(x)$$

$$\tilde{\nu}_{h^{-1}(B)}(0) = \tilde{\nu}_B(h(0)) \leq \tilde{\nu}_B(h(x)) = \tilde{\nu}_{h^{-1}(B)}(x)$$

Then, $\tilde{\mu}_{h^{-1}(B)}(x) = \tilde{\mu}_B(h(x)) \geq r\min\{\tilde{\mu}_B(h(x) *' h(y)), \tilde{\mu}_B(h(y))\} = r\min\{\tilde{\mu}_B(h(x * y)), \tilde{\mu}_B(h(y))\}$

$$= r\min\{\tilde{\mu}_{h^{-1}(B)}(x * y), \tilde{\mu}_{h^{-1}(B)}(y)\}$$

Also, $\tilde{\nu}_{h^{-1}(B)}(x) = \tilde{\nu}_B(h(x)) \leq r\max\{\tilde{\nu}_B(h(x) *' h(y)), \tilde{\nu}_B(h(y))\} = r\max\{\tilde{\nu}_B(h(x * y)), \tilde{\nu}_B(h(y))\}$

$$= r\max\{\tilde{\nu}_{h^{-1}(B)}(x * y), \tilde{\nu}_{h^{-1}(B)}(y)\}$$

Hence $h^{-1}(B)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

Theorem 8.2.13: Let $h:(X,*,0)\to(Y,*',0')$ be an Z-epimorphism of Z-algebras. Then B is an interval-valued intuitionistic fuzzy Z-ideal of Y, if $h^{-1}(B)$ is an interval-valued intuitionistic fuzzy Z-ideal of X .

Proof: Let y $\in$ Y, there exists x $\in$ X such that h (x) = y. Then

$$\tilde{\mu}_B(y) = \tilde{\mu}_B(h(x)) = \tilde{\mu}_{h^{-1}(B)}(x) \leq \tilde{\mu}_{h^{-1}(B)}(0) = \tilde{\mu}_B(h(0)) = \tilde{\mu}_B(0')$$

$\widetilde{\nu}_B(y) = \widetilde{\nu}_B(h(x)) = \widetilde{\nu}_{h^{-1}(B)}(x) \geq \widetilde{\nu}_{h^{-1}(B)}(0) = \widetilde{\nu}_B(h(0)) = \widetilde{\nu}_B(0')$

Let x, y $\in$ Y. Then there exists a, b $\in$ X such that h(a) = x and h (b) = y. It follows that

$$\begin{aligned}\widetilde{\mu}_B(x) = \widetilde{\mu}_B(h(a)) = \widetilde{\mu}_{h^{-1}(B)}(a) \geq r\min\{\widetilde{\mu}_{h^{-1}(B)}(a * b), \widetilde{\mu}_{h^{-1}(B)}(b)\} &= r\min\{\widetilde{\mu}_B(h(a * b)), \widetilde{\mu}_B(h(b))\} \\ &= r\min\{\widetilde{\mu}_B(h(a) *' h(b)), \widetilde{\mu}_B(h(b))\} \\ &= r\min\{\widetilde{\mu}_B(x *' y), \widetilde{\mu}_B(y')\} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\widetilde{\nu}_B(x) = \widetilde{\nu}_B(h(a)) = \widetilde{\nu}_{h^{-1}(B)}(a) \leq r\max\{\widetilde{\nu}_{h^{-1}(B)}(a * b), \widetilde{\nu}_{h^{-1}(B)}(b)\} &= r\max\{\widetilde{\nu}_B(h(a * b)), \widetilde{\nu}_B(h(b))\} \\ &= r\max\{\widetilde{\nu}_B(h(a) *' h(b)), \widetilde{\nu}_B(h(b))\} \\ &= r\max\{\widetilde{\nu}_B(x *' y), \widetilde{\nu}_B(y')\}\end{aligned}$$

Hence B is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra Y.

Theorem 8.2.14: Let A and B be two interval-valued intuitionistic fuzzy Z-ideals of a Z-algebra X. Then $A \times B$ is an interval-valued intuitionistic fuzzy Z-ideal of $X \times X$.

Proof: Take $(x_1, x_2), (y_1, y_2) \in X \times X$. Then, by Theorem 7.2.8,

$\widetilde{\mu}_{A\times B}(0,0) \geq \widetilde{\mu}_{A\times B}(x_1, x_2)$ and $\widetilde{\mu}_{A\times B}(x_1, x_2) \geq r\min\{\widetilde{\mu}_{A\times B}((x_1, x_2) * (y_1, y_2)), \widetilde{\mu}_{A\times B}(y_1, y_2)\}$

Then $\widetilde{\nu}_{A\times B}(0,0) = r\max\{\widetilde{\nu}_A(0), \widetilde{\nu}_B(0)\} \leq r\max\{\widetilde{\nu}_A(x_1), \widetilde{\nu}_B(x_2)\} = \widetilde{\nu}_{A\times B}(x_1, x_2)$

and

$$\begin{aligned}\widetilde{\nu}_{A\times B}(x_1, x_2) &= r\max\{\widetilde{\nu}_A(x_1), \widetilde{\nu}_B(x_2)\} \\ &\leq r\max\{r\max\{\widetilde{\nu}_A(x_1 * y_1), \widetilde{\nu}_A(y_1)\}, r\max\{\widetilde{\nu}_B(x_2 * y_2), \widetilde{\nu}_B(y_2)\}\} \\ &= r\max\{r\max\{\widetilde{\nu}_A(x_1 * y_1), \widetilde{\nu}_B(x_2 * y_2)\}, r\max\{\widetilde{\nu}_A(y_1), \widetilde{\nu}_B(y_2)\}\} \\ &= r\max\{\widetilde{\nu}_{A\times B}((x_1 * y_1), (x_2 * y_2)), \widetilde{\nu}_{A\times B}(y_1, y_2)\} \\ &= r\max\{\widetilde{\nu}_{A\times B}((x_1, x_2) * (y_1, y_2)), \widetilde{\nu}_{A\times B}(y_1, y_2)\}\end{aligned}$$

Hence $A \times B$ is an interval-valued intuitionistic fuzzy Z-ideal of $X \times X$.

Theorem 8.2.15: Let A and B be two interval-valued intuitionistic fuzzy sets of a Z-algebra X. If $A \times B$ is an interval-valued intuitionistic fuzzy Z-ideal of $X \times X$, the following are true.

(i) $\widetilde{\mu}_A(0) \geq \widetilde{\mu}_B(y)$ and $\widetilde{\mu}_B(0) \geq \widetilde{\mu}_A(x)$ for all $x, y \in X$.

(ii) $\widetilde{\nu}_A(0) \leq \widetilde{\nu}_B(y)$ and $\widetilde{\nu}_B(0) \leq \widetilde{\nu}_A(x)$ for all $x, y \in X$.

Proof : Assume that $\widetilde{\mu}_B(y) > \widetilde{\mu}_A(0)$ and $\widetilde{\mu}_A(x) > \widetilde{\mu}_B(0)$ for some $x, y \in X$.

Then $\widetilde{\mu}_{A\times B}(x, y) = r\min\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(y)\} > r\min\{\widetilde{\mu}_B(0), \widetilde{\mu}_A(0)\}$

$= \widetilde{\mu}_{A\times B}(0,0)$ which is a contradiction.

Similarly, assume that $\widetilde{\nu}_A(x) < \widetilde{\nu}_B(0)$ and $\widetilde{\nu}_B(y) < \widetilde{\nu}_A(0)$ for some $x, y \in X$.

Then $\widetilde{\nu}_{A\times B}(x, y) = r\max\{\widetilde{\nu}_A(x), \widetilde{\nu}_B(y)\} < r\max\{\widetilde{\nu}_B(0), \widetilde{\nu}_A(0)\}$

$= \widetilde{\nu}_{A\times B}(0,0)$ which is also a contradiction.Thus proving the result.

Theorem 8.2.16: Let A and B be two interval-valued intuitionistic fuzzy sets of a Z-algebra X such that $A\times B$ is an interval-valued intuitionistic fuzzy Z-ideal of $X\times X$ .Then either A or B is an interval-valued intuitionistic fuzzy Z-ideal of X.

Proof : Now by above Theorem 8.2.15 if we take $\widetilde{\mu}_A(0) \geq \widetilde{\mu}_B(y)$ and $\widetilde{\nu}_A(0) \leq \widetilde{\nu}_B(y)$ for all $y \in X$,

$\widetilde{\mu}_{A\times B}(0, y) = r\min\{\widetilde{\mu}_A(0), \widetilde{\mu}_B(y)\} = \widetilde{\mu}_B(y)$ and $\widetilde{\nu}_{A\times B}(0, y) = r\max\{\widetilde{\nu}_A(0), \widetilde{\nu}_B(y)\} = \widetilde{\nu}_B(y)$ (1)

Take $(x_1, x_2), (y_1, y_2) \in X\times X$.

Since $A\times B$ is an interval-valued intuitionistic fuzzy Z-ideal of $X\times X$.

$\widetilde{\mu}_{A\times B}(x_1, y_1) \geq r\min\{\widetilde{\mu}_{A\times B}((x_1, y_1) * (x_2, y_2)), \widetilde{\mu}_{A\times B}(x_2, y_2)\}$

$= r\min\{\widetilde{\mu}_{A\times B}(x_1 * x_2, y_1 * y_2), \widetilde{\mu}_{A\times B}(x_2, y_2)\}$ (2)

Putting $x_1 = x_2 = 0$ in (2) we get,

$\widetilde{\mu}_{A\times B}(0, y_1) \geq r\min\{\widetilde{\mu}_{A\times B}(0, y_1 * y_2), \widetilde{\mu}_{A\times B}(0, y_2)\}$ and by (1),

$\widetilde{\mu}_B(y_1) \geq r\min\{\widetilde{\mu}_B(y_1 * y_2), \widetilde{\mu}_B(y_2)\}$

Analogously, we can prove $\widetilde{\nu}_B(y_1) \leq r\max\{\widetilde{\nu}_B(y_1 * y_2), \widetilde{\nu}_B(y_2)\}$.

Hence B is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

By Theorem 8.2.15, assume that $\widetilde{\mu}_B(0) \geq \widetilde{\mu}_A(x)$ and $\widetilde{\nu}_B(0) \leq \widetilde{\nu}_A(x)$ then A is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

Therefore, either A or B is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X.

This completes the proof.