Introduction

INTRODUCTION

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. This problem has attracted the attention of many mathematicians during the twentieth century.

The linear algebra over semirings is a subject of intensive research because of its purely algebraic interest and its numerous applications. There have been numerous investigations into the theory of matrices over algebraic structures. A fair amount of effort has been directed towards discovering various properties of matrices when the underlying algebraic structure is a semiring.

There are many research articles on linear operators that preserve the rank of matrices over several semirings and semifields.

Beasley, L.B., and Pullman, N.J., [3] defined the perimeter of a Boolean rank-1 matrix in order to characterize the linear operators preserving Boolean rank.

Song, S.Z., [23] obtained characterization of linear operators that preserve column rank over the fuzzy scalars.

Song, S.Z., and Kang, K.T., [25, 28] characterized the linear operators that preserve the rank and perimeter of the rank-1 matrices over chain semirings and over semifields.

Beasley, L.B., Song, S.Z., and Lee, S.G., [9] obtained characterizations of zero-term rank preservers of matrices over anti-negative semirings.

Beasley, L.B., et al. [13] obtain characterizations of the linear operators that preserve zero-term rank of real matrices.

Beasley, L.B., and Gutterman, A.E., [2] characterized linear preservers for set of matrix ordered tuples which satisfy extremal properties with respect to factor rank. They also characterized linear operators on matrices over semirings that preserve the extremal cases in the bounds on term ranks and zero-term ranks of sums and products of matrices.

Linear operators preserving pairs of Hermitian matrices on which the rank is additive is studied by Tang, X.M., and Cao, C.G., [30].

This thesis is devoted to the study of

(1) Linear operators preserving zero-term rank of real matrices.

(2) Linear operators preserving factor rank of matrices over semirings.

(3) Linear operators preserving term rank and zero-term rank of matrices over semirings.

(4) Linear operators preserving rank and perimeter of rank-1 matrices over semirings.

(5) Linear operators preserving rank and perimeter of rank-1 matrices over semifields.

(6) Linear operators preserving pairs of Hermitian matrices on which the rank is additive.

The first chapter deals with preliminary definitions and notations.

Chapter II deals with the linear operators preserving zero-term rank of real matrices.

" Zero-term rank of a matrix is the minimum number of lines (rows or columns) needed to cover all the zero entries of the given matrix ".

In this chapter, the linear operators preserving the zero-term rank of $m \times n$ real matrices is characterized (Theorem 2.10). Combinatorial equivalent condition for the zero-term rank of a real matrix is also obtained as follows :

" For a $m \times n$ real matrix A, the zero-term rank of A is equal to the maximal number of zeros in A with no two of the zeros on a line ".

Linear operators preserving factor rank of matrices over semirings are studied in chapter III.

" The matrix $A \in \mathbf{M}_{m,n}(\mathbf{S})$ (set of $m \times n$ matrices with entries in a semiring **S**) is said to be of factor rank k (rank (A)=k) if there exist matrices $B \in \mathbf{M}_{m,k}(\mathbf{S})$ and $C \in \mathbf{M}_{k,n}(\mathbf{S})$ such that A=BC and k is the smallest positive integer such that such a factorization exists ".

In this chapter, the linear preservers for sets of matrix ordered tuples which satisfy extremal properties with respect to factor rank are characterized.

The arithmetic properties of factor rank depend on the structure of the semiring **S**. It is restricted by the following list of inequalities :

If the semiring is arbitrarily antinegative, then

(1) $rank(A+B) \le rank(A) + rank(B);$

(2) $rank(AB) \le \min\{rank(A), rank(B)\}.$

If the semiring uses Boolean arithmetics, then

(3) $rank(A+B) \ge \begin{cases} rank(A) & if \quad B=O, \\ rank(B) & if \quad A=O, \\ 1 & if \quad A \ne O \quad and \quad B \ne O; \end{cases}$

(4) $rank(AB) \ge \begin{cases} 0 & if \quad rank(A)+rank(B) \le n, \\ 1 & if \quad rank(A)+rank(B) > n. \end{cases}$

If the semiring is a subsemiring of the set $\mathbb{R}^+$ of positive real numbers, we have

(5) $rank(A+B) \ge |\rho(A) - \rho(B)|;$

(6) $rank(AB) \ge \begin{cases} 0 & if \quad \rho(A)+\rho(B) \le n, \\ \rho(A)+\rho(B)-n & if \quad \rho(A)+\rho(B) > n; \end{cases}$

(7) $\rho(AB) + \rho(BC) \le rank(ABC) + rank(B),$

where ρ(A) is the usual rank function for any matrix A .

In order to denote the sets of matrices that arise as extremal cases in the inequalities listed above, the following notations are used :

$\mathbf{F}_1(\mathbf{S}) = \{(X,Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid rank(X+Y) = rank(X) + rank(Y)\};$

$\mathbf{F}_{2B}(\mathbf{S}) = \{(X,Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid rank(X+Y) = 1\};$

$\mathbf{F}_{2R}(S)=\{(X,Y)\in \mathbf{M}_{m,n}(S)^2 \mid rank(X+Y)= \mid \rho(X)-\rho(Y) \mid \}$;

$\mathbf{F}_3(S)=\{(X,Y)\in \mathbf{M}_n(S)^2 \mid rank(XY)=min\{rank(X)+rank(Y)\}\}$;

$\mathbf{F}_{4N}(S)=\{(X,Y)\in \mathbf{M}_n(S)^2 \mid rank(XY)=0\}$;

$\mathbf{F}_{4B}(S)=\{(X,Y)\in \mathbf{M}_n(S)^2 \mid rank(XY)=1\}$;

$\mathbf{F}_{4R}(S)=\{(X,Y)\in \mathbf{M}_n(S)^2 \mid rank(XY)= \rho(X)+\rho(Y)-n\}$;

$\mathbf{F}_5(S)=\{(X,Y,Z)\in \mathbf{M}_n(S)^3 \mid rank(XYZ)+rank(Y)= \rho(XY)+\rho(YZ)\}$.

Linear preservers of $\mathbf{F}_1$, $\mathbf{F}_{2B}$, $\mathbf{F}_{2R}$, $\mathbf{F}_3$, $\mathbf{F}_{4N}$, $\mathbf{F}_{4B}$, $\mathbf{F}_{4R}$ and $\mathbf{F}_5$ are studied in this chapter.

The following theorem characterizes linear preserver of $\mathbf{F}_1$:

" Let S be a commutative antinegative semiring without zero divisors and $T : \mathbf{M}_{m,n}(S) \rightarrow \mathbf{M}_{m,n}(S)$ be a surjective linear operator. The operator T preserves $\mathbf{F}_1$ if and only if T is a (U,V)-operator, where U and V are invertible monomial matrices ".

Characterization theorems for linear preservers $\mathbf{F}_{2B}$, $\mathbf{F}_{2R}$, $\mathbf{F}_3$, $\mathbf{F}_{4N}$, $\mathbf{F}_{4B}$, $\mathbf{F}_{4R}$ and $\mathbf{F}_5$ are given in theorems 3.21, 3.24, 3.27, 3.29, 3.33, 3.36 and 3.39 respectively.

In chapter IV, the linear operators preserving term rank and zero-term rank of matrices over semirings are studied.

" A matrix $A\in \mathbf{M}_{m,n}(S)$ is said to be of term rank k (t(A)=k) if the least number of lines needed to include all nonzero elements of A is equal to k ".

In this chapter, linear operators on matrices over semirings that preserve the extremal cases in the bounds on term ranks and zero-term ranks of sums and products of matrices are characterized.

The arithmetic properties of term rank and zero-term rank are restricted by the following list of inequalities :

(1) $t(A+B) \leq t(A)+t(B)$;

(2) $t(A+B) \geq \max\{t(A),t(B)\}$;

(3) $t(AB) \leq \min\{c(A),r(B)\}$;

(4) $t(AB) \geq t(A) + t(B) - n$;

(5) if **S** is a subsemiring of the semiring of nonnegative reals, R_+, then $\rho(AB)+\rho(BC) \leq t(ABC)+t(B)$,

(6) $z(A+B) \geq 0$;

(7) $z(A+B) \leq \min\{z(A), z(B)\}$;

(8) $z(AB) \geq 0$;

(9) $z(AB) \leq z(A) + z(B)$.

In order to denote the sets of matrices that arise as extremal cases in the inequalities listed above, the following notations are used :

$\mathbf{T}_1(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid t(X+Y)=t(X)+t(Y)\}$;

$\mathbf{T}_2(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid t(X+Y)=\max(t(X),t(Y))\}\}$;

$\mathbf{T}_3(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid t(XY)=\min\{r(X),c(Y)\}\}$;

$\mathbf{T}_4(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{n}(\mathbf{S})^2 \mid t(XY)=t(X)+t(Y)-n\}$;

$\mathbf{T}_5(\mathbf{S})=\{(X,Y,Z)\in \mathbf{M}_{m,n}(\mathbf{S})^3 \mid t(XYZ)+t(Y)=\rho(XY)+ \rho(YZ)\}$;

$\mathbf{Z}_1(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(X+Y)=\min\{z(X),z(Y)\}\}$;

$\mathbf{Z}_2(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(X+Y)=0\}$;

$\mathbf{Z}_3(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(XY)=0\}$;

$\mathbf{Z}_4(\mathbf{S})=\{(X,Y)\in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(XY)=z(X)+z(Y)\}$.

Characterization theorems for linear preservers of $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$, $\mathbf{T}_5$, $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$ are given in theorems 4.2, 4.4, 4.5, 4.8, 4.11, 4.12, 4.13, 4.14 and 4.15 respectively.

Linear operators preserving rank and perimeter of rank-1 matrices over semirings are studied in chapter V.

" The rank of $A\in \mathbf{M}_{m,n}(\mathbf{S})$ is 1 if and only if there exist nonzero vectors $\mathbf{a}\in \mathbf{M}_{m,1}(\mathbf{S})$ and $\mathbf{b}\in \mathbf{M}_{n,1}(\mathbf{S})$ such that $A = \mathbf{ab}^t$ ".

" Let A be any rank-1 matrix in $\mathbf{M}_{m,n}(\mathbf{S})$. We define the perimeter of A, P(A), as $|\mathbf{a}|+|\mathbf{b}|$ for arbitrary factorization $A = \mathbf{ab}^t$ ".

" A linear operator on $T \in \mathbf{M}_{m,n}(\mathbf{S})$ preserve rank-1 if r(T(A))=1 whenever r(A)=1 for all $A \in \mathbf{M}_{m,n}(\mathbf{S})$ and preserve perimeter k of rank-1 matrices if $\rho(T(A))=k$ whenever $\rho(A)=k$ for all $A \in \mathbf{M}_{m,n}(\mathbf{S})$ with r(A)=1 ".

The set of linear operators preserving the rank and perimeter of every rank-1 matrix over any chain semiring is characterized as follows :

" For a linear operator T on $\mathbf{M}_n(\mathbf{S})$, where **S** is a chain semiring, the following statements are equivalent :

(i) T is a (P,Q,B)-operator

(ii) T preserves both rank and perimeter of rank-1 matrices ".

Chapter VI deals with linear operators preserving rank and perimeter of rank-1 matrices over semifields.

It is proved here that a linear operator T preserves the rank and perimeter of rank-1 matrices over semifields if and only if it has the form T(A)=UAV, or $T(A)=UA^tV$ with some invertible matrices U and V (A^t denotes transpose of A).

Linear operators preserving pairs of Hermitian matrices on which the rank is additive is studied in chapter VII.

" An $n \times n$ matrix is Hermitian if $A=A^*$, where A^* is the conjugate transpose of A ".

" A pair of $n \times n$ Hermitian matrices (A,B) is said to be rank-additive if rank(A+B)=rank(A)+rank(B) ".

The linear maps from $\mathbf{H}_n$ (the set of all $n \times n$ Hermitian matrices) into itself which preserve the set of rank-additive pairs is characterized as follows :

" Let $n \geq 2$ be an integer. Let $\mathbf{H}_n$ be the set of all complex $n \times n$ Hermitian matrices, and $f : \mathbf{H}_n \rightarrow \mathbf{H}_n$ be a linear map preserving the set $\Theta_+^{H_n}$, where $\Theta_+^{H_n}$ denote the subset of $\mathbf{H}_n \times \mathbf{H}_n$ consisting of all rank additive pairs.

Then either

$$f \equiv 0$$

or f is of the form

$$f(A) = cPAP^{*}, \qquad \forall A \in \mathbf{H}_n,$$

or of the form

$$f(A) = cPA^{t}P^{*}, \qquad \forall A \in \mathbf{H}_n,$$

where $c \in R^{*}$, the multiplicative group of real numbers where, the term multiplicative group refers to any group whose binary operation is written in multiplication notation and $P \in GL_n$, where GL_n is the general linear group consists of all $n \times n$ invertible matrices over the field of complex numbers ".