# Chapter II

# CHAPTER II

# TOPOLOGICAL ORDERED SPACES

**Definition: 2.1**

A **topological ordered space** is a triple $(X, \tau, \leq)$, where $\tau$ is a topology on X and $\leq$ is a partial order on X.

**Notation: 2.2**

Let $(X, \tau, \leq)$ be a topological ordered space. For any $x \in X$ , $\{y \in X / x \leq y\}$ will be denoted by $[x, \rightarrow]$ and $\{y \in X / y \leq x\}$ will be denoted by $[\leftarrow, x]$.

**Definition: 2.3**

A subset A of a topological ordered space $(X, \tau, \leq)$ is said to be:

(1) **Increasing** if A = i(A),

(2) **Decreasing** if A = d(A),

where $i(A) = \bigcup_{a \in A} [a, \rightarrow]$ and $d(A) = \bigcup_{a \in A} [\leftarrow, a]$

**Definition: 2.4**

The **complement** of a decreasing (resp. an increasing) set is an increasing (resp. a decreasing) set. C(A) denotes the complement of A in X.

**Definition: 2.5**

For a subset A of a topological ordered space $(X, \tau, \leq)$,

icl(A) = $\cap$ {F / F is an increasing closed subset of X containing F},

dcl(A) = $\cap$ {F / F is a decreasing closed subset of X containing F},

bcl(A) = $\cap$ {F / F is a closed subset of X containing F with F = i(F) = d(F)},

$A^{io}$ = $\cup$ {G / G is an increasing open subset of X contained in A},

$A^{do}$ = $\cup$ {G / G is a decreasing open subset of X contained in A} and

$A^{bo}$ = $\cup$ {G / G is both increasing and decreasing open subset of X contained in A}.

**Remark: 2.6**

Clearly icl(A) (resp. dcl(A), bcl(A)) is the smallest increasing (resp. decreasing, both increasing and decreasing) closed set containing A. Moreover cl(A) $\subseteq$ icl(A) $\subseteq$ bcl(A) and dcl(A) $\subseteq$ bcl(A), where cl(A) stands for the closure of A in (X, $\tau$). Further A is increasing (resp. decreasing) closed if and only if A = icl(A) (resp. A = dcl(A)).

Clearly $A^{io}$ (resp. $A^{do}$, $A^{bo}$) is the largest increasing (resp. decreasing, both increasing and decreasing) open set contained in A. Moreover $A^{bo} \subseteq A^{io} \subseteq A^{o}$ and $A^{bo} \subseteq A^{do}$, where $A^{o}$ denotes the interior of A in (X, $\tau$). If A and B are any two subsets of a topological ordered space (X, $\tau$, $\leq$) such that $A \subseteq B$, then $A^{io} \subseteq B^{io} \subseteq B^{o}$. **IO(X)** (resp. **DO(X), BO(X)**) denotes the collection of all increasing (resp. decreasing, both increasing and decreasing) open subsets of a topological ordered space (X, $\tau$, $\leq$).

**Definition: 2.7**

A function f : (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) is called

(1) an **I-continuous** map if $f^{-1}(G) \in$ IO(X),

(2) a **D-continuous** map if $f^{-1}(G) \in$ DO(X),

(3) a **B-continuous** map if $f^{-1}(G) \in$ BO(X),

whenever G is an open subset of ($X^*$, $\tau^*$).

**Remark: 2.8**

It is evident that every x-continuous map is continuous for x = I, and that every B-continuous map is both I-continuous and D-continuous.

The following Example shows that a continuous map need nc x-continuous for x = I, D, B.

**Example: 2.9**

Let X = {a, b, c}, $\tau$ = {$\phi$, X, {a}, {b}, {a, b}} and $\leq$ = {(a, a), (b, b), (a, b), (b, c), (a, c)}. Clearly (X, $\tau$, $\leq$) is a topological ordered space. L the identity map from (X, $\tau$, $\leq$) onto itself. {b} is open but $f^{-1}$({b}) = {b} is an increasing nor a decreasing open set. Thus f is not x-continuc x = I, D, B. However f is continuous.

The following Example supports that a D-continuous map need n a B-continuous map.

**Example: 2.10**

Let X = {a, b, c} = $X^*$, $\tau$ = {$\phi$, X, {a}, {b}, {a, b}} = $\tau^*$, $\leq$ = {(a, a), (| (c, c), (a, c)} and $\leq^*$ = {(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)}. Let g b identity map from (X, $\tau$, $\leq$) onto ($X^*$, $\tau^*$, $\leq^*$). g is not B-continuous. Howe is a D - continuous map.

The following Example supports that an I-continuous map need not B-continuous map.

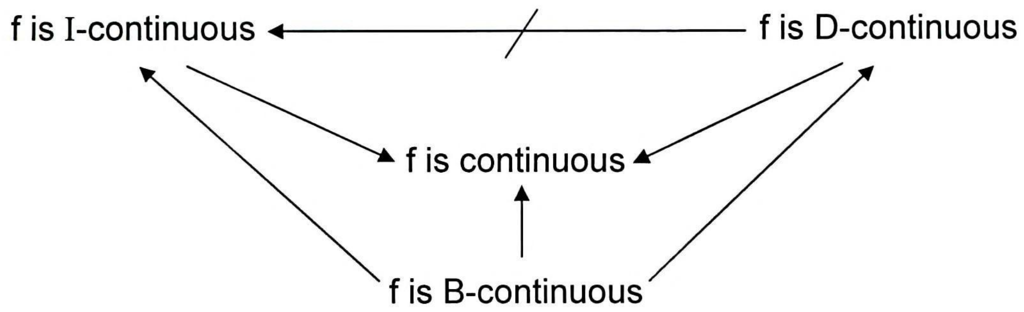
**Example: 2.11**

Let X = {a, b, c} = $X^*$, $\tau$ = {$\phi$, X, {a}, {b}, {a, b}} and $\tau^*$ = {$\phi$, $x^*$, {a}} $\leq$ = {(a, a), (b, b), (c, c), (a, b), (c, b)} = $\leq^*$. Define h : (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau$

by h(a) = b, h(b) = a and h(c) = c. h is I-continuous but not a B-continuous map.

Thus we have the following diagram.

For a function f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$.




**Characterization of I-continuous, D-continuous and B-continuous maps.**

**Theorem: 2.12**

For a function f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent:

(1) f is I-continuous.

(2) $f(dcl(A)) \subseteq cl(f(A))$ for any $A \subseteq X$.

(3) $dcl(f^{-1}(B)) \subseteq (cl(B))$ for any $B \subseteq X^*$.

(4) For every closed subset K of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is a decreasing closed subset of $(X, \tau, \leq)$.

**Proof**

(1) $\Rightarrow$ (2). Since C(cl(f(A))) is open in $X^*$ and f is I-continuous, then $f^{-1}(C(cl(f(cl(A)))))$ is an increasing open set in X. Then $C(f^{-1}(C(cl(f(A)))))$ is a decreasing closed subset of X. Since $C(f^{-1}(C(cl(f(A))))) = f^{-1}(cl(f(A)))$, then $f^{-1}(cl(f(A)))$ is a decreasing closed subset of X. Since $A \subseteq f^{-1}(cl(f(A)))$ and dcl(A) is the smallest decreasing closed set containing A, then $dcl(A) \subseteq f^{-1}(cl(f(A)))$. $f(f^{-1}(cl(f(A)))) \subseteq cl(f(A)$. Thus $f(dcl(A)) \subseteq cl(f(A))$.

(2) $\Rightarrow$ (3). Let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. This implies $cl(f(A)) \subseteq cl(B)$. Now $dcl(f^{-1}(B)) \subseteq dcl(A) \subseteq f^{-1}(f(dcl(A))) \subseteq f^{-1}(cl(f(A)))$. But $f^{-1}(cl(f(A))) \subseteq f^{-1}(cl(B))$. Thus $dcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(3) $\Rightarrow$ (4). $dcl(f^{-1}(K)) \subseteq f^{-1}(cl(K))$ for any closed set K of $(X^*, \tau^*, \leq^*)$. Thus $f^{-1}(K)$ is a decreasing closed in $(X, \tau, \leq)$, whenever K is a closed set in $(X^*, \tau^*, \leq^*)$.

(4) $\Rightarrow$ (1). Let G be an open set in $(X^*, \tau^*)$. Then $f^{-1}(C(G))$ is a decreasing closed set in $(X, \tau)$. Since C(G) is a closed set in $(X^*, \tau^*)$. But $C(f^{-1}(G)) = f^{-1}(C(G))$. Thus $C(f^{-1}(G))$ is a decreasing closed set in $(X, \tau, \leq)$. So $f^{-1}(G)$ is an increasing open set in $(X, \tau, \leq)$. Thus f is I-continuous.

In a similar manner the following theorems can be proved.

**Theorem: 2.13**

For a function $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent:

(1) f is D-continuous.

(2) $f(icl(A)) \subseteq f^{-1}(cl(A))$ for any $A \subseteq X$.

(3) $icl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for any $B \subseteq X^*$.

(4) For every closed subset K of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is an increasing closed subset of $(X, \tau, \leq)$.

**Theorem: 2.14**

For a function $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent:

(1) f is B-continuous.

(2) $f(bcl(A)) \subseteq cl(f(A))$ for any $A \subseteq X$.

(3) $bcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for any $B \subseteq X^*$.

(4) For every closed subset K of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is both increasing and decreasing closed subset of $(X, \tau, \leq)$.

Regarding compositions of these maps we have the following theorem.

**Theorem: 2.15**

Let f: $(X, \tau, \leq_1) \to (Y, \sigma, \leq_2)$ and g : $(Y, \sigma, \leq_2) \to (Z, \eta, \leq_3)$ be any two mappings. Then

(1) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-continuous if f and g are x-continuous for x = I, D, B.

(2) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-continuous if f is x-continuous and g is continuous for x = I, D, B.

(3) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-continuous if f is x-continuous and g is y-continuous for all x, y$\in$\{ I, D, B\}.

**Theorem: 2.16**

A function f: $(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ is called

(1) an **I-open** map if $f(G) \in IO(X^*)$

(2) a **D-open** map if $f(G) \in DO(X^*)$

(3) a **B-open** map if $f(G) \in BO(X^*)$

whenever G is an open subset of $(X, \tau)$.

**Remark: 2.17**

It is evident that every x-open map in an open map for x = I, D, B and that every B-open map is both I-open and D-open.

The following Example shows that an open map need not be x-open for x = I, D, B.

**Example: 2.18**

Let $(X, \tau, \leq)$ and f be as in the Example 2.9. f is an open map but f is not x-open for x = I, D, B.

The following Example shows that a D-open map need not be a B-open map.

**Example: 2.19**

Let $X$, $X^*$, $\tau$, $\tau^*$, $\leq$ and $\leq^*$ be as in the Example 2.10. Let $\theta$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. $\theta$ is D-open but not a B-open map.

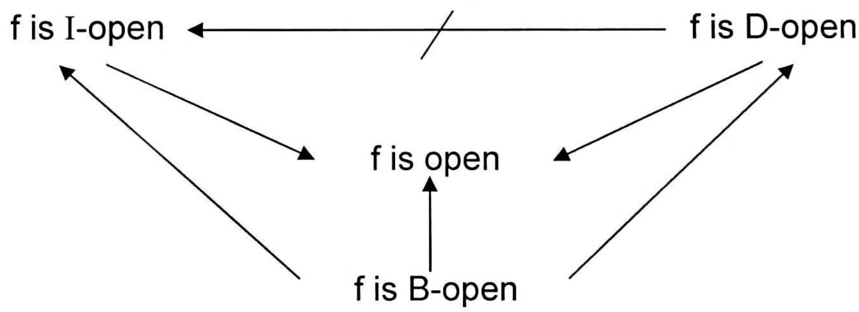
The following Example shows that an I-open map need not be a B-open map.

**Example: 2.20**

Let $X$, $X^*$, $\tau$, $\tau^*$, $\leq$ and $\leq^*$ be as in the Example 2.11. Define $\phi : (X^*, \tau^*, \leq^*) \rightarrow (X, \tau, \leq)$ by $\phi(a) = b$, $\phi(b) = a$ and $\phi(c) = c$. $\phi$ is an I-open map but not a B-open map.

Thus we have the following diagram:

For a function f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$,




**Characterization of I-open, D-open and B-open maps.**

**Lemma: 2.21**

Let A be any subset of a topological ordered space $(X, \tau, \leq)$. Then

(1) $C(dcl(A)) = (C(A))^{io}$

(2) $C(icl(A)) = (C(A))^{do}$ and

(3) $C(bcl(A)) = (C(A))^{bo}$

**Proof**

(1) $C(dcl(A))) = C(\cap \{F / F$ is a decreasing closed subset of X containing A}),

$= \cup\{C(F) / F$ is a decreasing closed subset of X containing A},

$= \cup \{G / G$ is an increasing open subset of X contained in C(A)},

$= (C(A))^{io}$

The proofs for (2) and (3) are analogous to that of (1).

The following Theorem characterizes I-open functions.

**Theorem: 2.22**

For any function f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statement are equivalent:

(1) f is an I-open map.

(2) $f(A^0) \subseteq [f(A)]^{i0}$ for any $A \subseteq X$.

(3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{i0})$ for any $B \subseteq X^*$.

(4) $f^{-1}(dcl(B)) \subseteq dcl(f^{-1}(B))$ for any $B \subseteq X^*$.

**Proof**

(1) $\Rightarrow$ (3). Since $[f^{-1}(B)]^0$ is open in X and f is I-open, then $f([f^{-1}(B)]^0)$ is an increasing open set in $X^*$. Also $f([f^{-1}(B)]^0) \subseteq f(f^{-1}(B)) \subseteq B$.

Then $f([f^{-1}(B)]^0) \subseteq B^{i0}$ since $B^{i0}$ is the largest increasing open set contained in B. Therefore $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{i0})$.

(3) $\Rightarrow$ (4). Replacing B by C(B) in (3), we get $(f^{-1}[C(B)]^0) \subseteq f^{-1}([C(B)]^{i0})$. Since $f^{-1}(C(B)) = C(f^{-1}(B))$, then $[C(f^{-1}(B))]^0 \subseteq f^{-1}([C(B)]^{i0})$. Now $C(dcl(f^{-1}(B))) = [C(f^{-1}(B))]^{i0} \subseteq [C(f^{-1}(B))]^0 \subseteq f^{-1}([C(B)]^{i0}) = f^{-1}(C(dcl(B))) = C(f^{-1}(dcl(B)))$ using the above Lemma 2.21. Therefore $f^{-1}(dcl(B)) \subseteq dcl(B))$.

(4) $\Rightarrow$ (3). All th steps in (3) $\Rightarrow$ (4) are reversable.

(3) $\Rightarrow$ (2). Replacing B by f(A) in (3), we get $[f^{-1}(f(A))]^0 \subseteq f^{-1}([f(A)]^{i0})$. Since $A^0 \subseteq [f^{-1}(f(A))]^0$, then we have $A^0 \subseteq f^{-1}([f(A)]^{i0})$. This implies that $f(A^0) \subseteq f(f^{-1}([f(A)]^{i0})) \subseteq [f(A)]^{i0}$. Hence $f(A^0) \subseteq [f(A)]^{i0}$.

(2) $\Rightarrow$ (1). Let G be any open subset of X. Then $f(G) = f(G^0) \subseteq [f(G)]^{0i}$. So f(G) is an increasing open set in $X^*$. Therefore f is an I-open map.

The following two Theorems give characterizations for D-open maps and B-open maps, whose proofs are similar to as that of the Theorem 2.21.

**Theorem: 2.23**

For any function f: $(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$, the following statements are equivalent:

(1) f is a D-open map.

(2) $f(A^0) \subseteq [f(A)]^{d0}$ for any $A \subseteq X$.

(3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{d0})$ for any $B \subseteq X^*$.

(4) $f^{-1}(icl(B)) \subseteq icl(f^{-1}(B))$ for any $B \subseteq X^*$.

**Theorem: 2.24**

For any function f: $(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$, the following statements are equivalent:

(1) f is an B-open map.

(2) $f(A^0) \subseteq [f(A)]^{b0}$ for any $A \subseteq X$.

(3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{b0})$ for any $B \subseteq X^*$.

(4) $f^{-1}(bcl(B)) \subseteq bcl(f^{-1}(B))$ for any $B \subseteq X^*$.

Regarding compositions of these maps we have the following theorem.

**Theorem: 2.25**

Let f: $(X, \tau, \leq_1) \to (Y, \sigma, \leq_2)$ and g: $(Y, \sigma, \leq_2) \to (Z, \eta, \leq_3)$ be any two mappings. Then

(1) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x - open if f is open and g is x-open for x = I, D, B.

(2) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-open if both f and g are x-open for x = I, D, B.

(3) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-open if f is y-open and g is x-open for all x, y$\in$\{I, D, B\}.

**Definition: 2.26**

A function f: $(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ is called

(1) an **I-closed** map if $f(G) \in IC(x^*)$

(2) a **D-closed** map if $f(G) \in DC(x^*)$

(3) a **B-closed** map if $f(G) \in BC(x^*)$

whenever G is a closed subset of X, where $IC(X^*)$ (resp. $DC(X^*)$, $BC(X^*)$) is the collection of all increasing (resp. decreasing, both increasing and decreasing) closed subsets of $(X^*, \tau^*, \leq^*)$.

**Remark: 2.27**

Clearly every x-closed map for x = I, D, B and every B-closed map is both I-closed and D-closed.

The following Example shows that a closed map need not be x-closed for x = I, D, B.

**Example: 2.28**

Let $(X, \tau, \leq)$ and f be as in the Example 2.9. f is a closed map but f is not x-closed for x = I, D, B.

The following Example shows that an I-closed map need not be a B-closed map.

**Example: 2.29**

Let X, $X^*$, $\tau$, $\tau^*$, $\leq$, $\leq^*$, and $\theta$ be as in the Example 2.19. $\theta$ is I-closed but not a B-closed map.

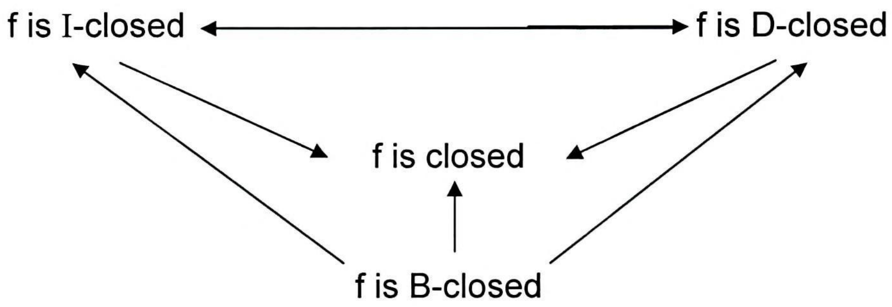
The following Example shows that a D-closed map need not be a B-closed map.

**Example: 2.30**

Let X, $X^*$, $\tau$, $\tau^*$, $\leq$, $\leq^*$ and $\phi$ be as in the Example 2.20. $\phi$ is a D-closed map but not a B-closed map.

Thus we have the following diagram:

For a function f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$,




**Characterization of I-closed, D-closed and B-closed maps.**

**Theorem: 2.31**

Let f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be any map. Then f is I-closed if and only if $icl(f(A)) \subseteq f(cl(A))$ for every $A \subseteq X$.

**Proof**

**Necessity:** Since f is I-closed, then f(cl(A)) is an increasing closed subset of X and $f(A) \subseteq f(cl(A))$. Therefore $icl(f(A)) \subseteq f(cl(A))$ since icl(f(A)) is the smallest increasing closed set in $X^*$ containing f(A).

**Sufficiency:** Let F be any closed subset of X. Then $f(F) \subseteq icl(f(F)) \subseteq f(cl(F)) = f(F)$. Thus $f(F) = icl(f(F))$. So f(F) is an increasing closed subset of $X^*$. Therefore f is an I-closed map.

The following two Theorems 2.32 and 2.33 characterize D-closed maps and B-closed maps.

**Theorem: 2.32**

Let $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be any map. Then f is D-closed if and only if $dcl(f(A)) \subseteq f(cl(A))$ for every $A \subseteq X$.

**Theorem: 2.33**

Let $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be any map. Then f is B-closed if and only if $bcl(f(A)) \subseteq f(cl(A))$ for every $A \subseteq X$.

**Theorem: 2.34**

Let $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a bijection. Then

(1) f is I-open if and only if f is D-closed.

(2) f is I-closed if and only if f is D-open.

(3) f is B-open if and only if f is B-closed.

**Proof**

**Necessity:** Let F be any closed subset of X. Then f(C(F) is an increasing open subset of $X^*$ since f is an I-open map and C(F) is an open subset of X. Since f is a bijection, then we have $f(C(F)) = C(f(F))$. So f(F) is a decreasing closed subset of $X^*$. Therefore f is D-closed.

**Sufficiency:** Let G be any open subset of X. Then f(C(G)) is a decreasing closed subset of $X^*$ since f is a D-closed map and C(G) is a closed subset of X. Since f is a bijection, then we have that f(C(G)) = C(f(G)). So f(G) is an increasing open subset of $X^*$. Therefore f is an I-open map.

The proofs for (2) and (3) are similar to that of (1).

Regarding compositions of these maps we have the following theorem.

**Theorem: 2.35**

Let f: $(X, \tau, \leq_1) \to (Y, \sigma, \leq_2)$ and g: $(Y, \sigma, \leq_2) \to (Z, \eta, \leq_3)$ be any two mappings. Then

(1) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-closed if f is closed and g is x-closed for x = I, D, B.

(2) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-closed if both f and g are x-closed for x = I, D, B.

(3) $g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is x-closed if f is y-closed and g is x-closed for all x, y$\in$\{I, D, B\}.

In a similar manner the following Theorems can be proved.

**Theorem: 2.36**

Let f: $(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be a bijection. Then the following statements are equivalent:

(1) f is an **I-open** map.

(2) f is an **D-closed** map.

(3) $f^{-1}$ is an **I-continuous**.

**Theorem: 2.37**

Let f: (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) be a bijection. Then the following statements are equivalent:

(1) f is a **D-open** map.

(2) f is an **I-closed** map.

(3) $f^{-1}$ is **D-continuous**.

**Theorem: 2.38**

Let f: (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) be a bijection. Then the following statements are equivalent:

(1) f is a **B-open** map.

(2) f is a **B-closed** map.

(3) $f^{-1}$ is a **B-continuous**.

**Theorem: 2.39**

Let f: (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) be an I-closed map and B, C $\subseteq X^*$. Then

(1) If U is an open neighbourhood of $f^{-1}(B)$, then there exists a decreasing open neighbourhood V and B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint neighbourhoods, then $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint decreasing open neighbourhoods.

**Proof**

(1) Let U be an open neighbourhood of $f^{-1}(B)$. Take C(V) = f(C(U)). Since f is an I-closed map and C(U) is a closed set, then C(V) = f(C(U)) is an increasing closed subset of X.

Thus V is a decreasing open subset of X. Since $f^{-1}(B) \subseteq U$, then $C(V) = f(C(U)) \subseteq f(f^{-1}(C(B))) \subseteq C(B)$. So $B \subseteq V$. Thus V is a decreasing open neighbourhood of B. Further $C(U) \subseteq f^{-1}(f(C(U))) = f^{-1}(C(V))$. Thus $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) follows from (1).

Similarly, we have the following two Theorems regarding D-closed maps and B-closed maps.

**Theorem: 2.40**

Let $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a D-closed map and $B, C \subseteq X^*$. Then:

(1) If U is an open neighbourhood of $f^{-1}(B)$, then there exists an increasing open neighbourhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint neighbourhoods, then $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint increasing open neighbourhoods.

**Theorem: 2.41**

Let $f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a B-closed map and $B, C \subseteq X^*$. Then:

(1) If U is an open neighbourhood of $f^{-1}(B)$, then there exists an open neighbourhood V of B, which is both increasing and decreasing such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint neighbourhoods, then $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint open neighbourhoods which are both increasing and decreasing.

**Definition: 2.42**

A bijection f: $(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ is called

(1) an **I-homeomorphism** if both f and $f^{-1}$ are I-continuous.

(2) a **D-homeomorphism** if both f and $f^{-1}$ are D-continuous.

(3) a **B-homeomorphism** if both f and $f^{-1}$ are B-continuous.

**Remark: 2.43**

Clearly every x-homeomorphism is a homeomorphism for x = I, D, B and every B-homeomorphism is both I-homeomorphism and D-homeomorphism.

The following Example shows that a homeomorphism need not be x-homeomorphism for x = I, D, B.

**Example: 2.44**

Let $(X, \tau, \leq)$ and f be as in the Example 2.9. f is a homeomorphism but not x-homeomorphism for x = I, D, B.

The following Example shows that a D-homeomorphism need not be a B-homeomorphism.

**Example: 2.45**

Let X, $X^*$, $\tau$, $\tau^*$ and $\leq^*$ be as in the Example 2.10. Let $\psi : (X, \tau, \leq^*) \rightarrow (X^*, \tau^*, \leq^*)$ be the identity map. $\psi$ is a D-homeomorphism but not a B-homeomorphism.

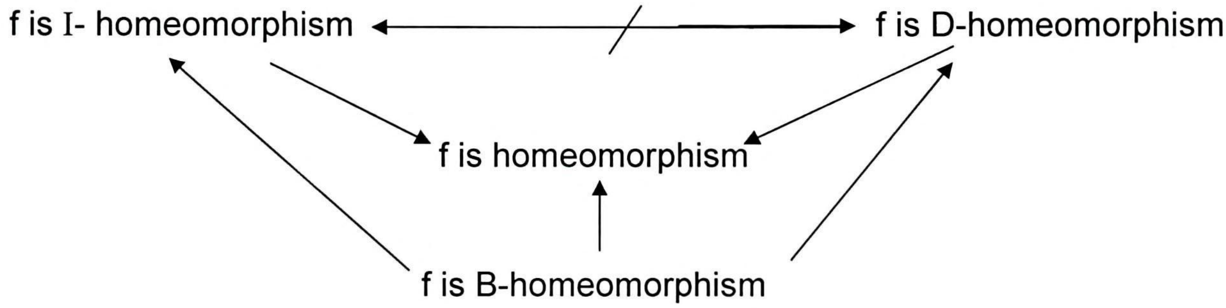
The following Example shows that an I-homeomorphism need not be a B-homeomorphism.

**Example: 2.46**

Let X = {a, b, c}, $\tau$ = {$\phi$, X, {b}, {c}, {b, c}} and $\leq$= {(a, a), (b, b), (c, c), (a, b), (a, c)}. Let j : (X, $\tau$, $\leq$) $\rightarrow$ (X, $\tau$, $\leq$) be the identity map. j is an I-homeomorphism but not a B-homeomorphism.

Thus we have the following diagram:

For the function f : (X, $\tau$, $\leq^*$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$),




**Characterization of I-homeomorphism, D-homeomorphism and B- homeomorphism.**

**Theorem: 2.47**

Let f : (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) be a bijective I-continuous map. Then the following statements are equivalent:

(1) f is an I-homeomorphism.

(2) f is an I-open map.

(3) f is an I-closed map.

**Theorem: 2.48**

Let f : (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) be a bijective D-continuous map. Then the following statements are equivalent:

(1) f is a D-homeomorphism.

(2) f is a D-open map.

(3) f is a D-closed map.

**Theorem: 2.49**

Let f : (X, $\tau$, $\leq$) $\rightarrow$ ($X^*$, $\tau^*$, $\leq^*$) be a bijective B-continuous map. Then the following statements are equivalent:

(1) f is a B-homeomorphism.

92) f is a B-open map.

(3) f is a B-closed map.