Fuzzy Total Coloring and Chromatic Number of a Complete Fuzzy Graph

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Abstract:

For a graph $\hat{G}_k = (V, \sigma, \mu)$, a fuzzy coloring function assigns values to each of the vertices and edges in such a way that no two adjacent vertices and incident edges share the same color. In this paper we introduce the concept of k-fuzzy total coloring on a complete fuzzy graph. We determine the chromatic number for a complete fuzzy graph $\hat{G}_k$, with fuzzy set of vertices and fuzzy set of edges in terms of family of fuzzy sets.

Keywords: Complete Fuzzy Graph, Fuzzy Total Coloring, Chromatic Number, Executive Committee Problem.

1. Introduction:

Graph coloring finds its origin during late 1850 and the term "total coloring" was independently introduced by Behzad and Vizing in numerous occasions between1964 and 1968. As graph theory has numerous applications in system analysis, operations research, transportation, economics etc., it is sometime uncertain. In such cases the uncertainty can be dealt using fuzzy logic.

Fuzzy graph was first introduced by Rosenfeld [1975] and further discussed and developed by many others. Computing chromatic sum of an arbitrary graph introduced by Kubica [1989] is known as NP-complete problem. Graph coloring is the most studied problem of combinatorial optimization. As an advancement fuzzy coloring of a fuzzy graph was defined by authors Eslahchi and Onagh in 2004, and later developed by them as Fuzzy vertex coloring [1] in 2006.

This fuzzy vertex coloring was extended to fuzzy total coloring in terms of family of fuzzy sets by Lavanya. S and Sattanathan. R [2].

In this paper we focus on k-fuzzy total coloring of a complete fuzzy graph by taking fuzzy sets of vertices and fuzzy set of edges. In section 2.1 we review the classical definition of fuzzy sets and other basic definitions of fuzzy graphs. In section 2.2 we introduce definition of k-fuzzy total coloring of a complete fuzzy graph and an executive committee problem will be used to motivate and illustrate our definition in this section. In section 2.3 we generalize the "Fuzzy total chromatic number of a complete fuzzy graph" using the example in previous section.

2.1 Preliminary Definitions:

Definition 1:

A fuzzy set A defined on a non empty set X is the family $A = \{(x, \mu_A(x))/x \in X\}$ where $\mu_A : X \rightarrow I$ is the membership function. In classical fuzzy set theory the set I is usually defined as the interval [0,1] such that $\mu_A(x) = \begin{cases} 0\,; x \notin A \\ 1\,; x \in A. \end{cases}$

It takes any intermediate value between 0 and 1 represents the degree in which $x \in A$. The set I could be discrete set of the form $I = \{0, 1, \ldots k\}$ where $\mu_A(x) < \mu_A(x')$ indicates that the degree of membership of x to A is lower than the degree of membership of x'.

Definition 2:

Let V be a finite nonempty set. The triple $\hat{G} = (V, \sigma, \mu)$ is called a fuzzy graph on V where σ and μ are fuzzy sets on V and E, respectively, such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ and $uv \in E$. For fuzzy graph $\hat{G} = (V, \sigma, \mu)$, the elements V and E are called set of vertices and set of edges of G respectively.

Definition 3:

A fuzzy graph $\hat{G} = (V, \sigma, \mu)$ is called a complete fuzzy graph if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ and $uv \in E$. We denote this complete fuzzy graph by $\hat{G}_K$.

Definition 4:

Two vertices u and v in $\hat{G}$ are called adjacent if $(\frac{1}{2})[\sigma(u) \wedge \sigma(v)] \leq \mu(uv)$.

Definition 5:

The edge uv of $\hat{G}$ is called strong if u and v are adjacent. Otherwise it is called weak.

Definition 6:

The degree of vertex v in $\hat{G}$, denoted by $deg_{\hat{G}}(v)$ is the number of adjacent vertices to v and the maximum degree of $\hat{G}$ is defined by $\Delta(\hat{G}) = \max\{deg_{\hat{G}}(v) / v \in V\}$.

Definition 7:

Two edges v_iv_j and v_iv_k are said to be incident if $2\{\mu(v_iv_j) \wedge \mu(v_iv_k)\} \leq \sigma(v_i)$ for $i=1,2,\ldots|v|$ and $1 \leq j,k \leq |v|$.

Definition 8:

A family $\Gamma = \{\gamma_1,\gamma_2, \ldots,\gamma_k\}$ of fuzzy sets on V is called a k-fuzzy coloring of $\hat{G} = (V, \sigma, \mu)$ if

a) $\vee\Gamma = \sigma$
b) $\gamma_i \wedge \gamma_j = 0$
c) For every strong edge uv of $\hat{G}$, $\gamma_i(u) \wedge \gamma_i(v) = 0$ for $1 \leq i \leq k$.

The above definition of k-fuzzy coloring was defined by the authors Eslahchi and Onagh [1] on fuzzy set of vertices. This has been extended to both fuzzy set of vertices and fuzzy set of edges by Lavanya. S and Sattanathan. R [2] as k-fuzzy total coloring as follows:

Definition 9:

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots \gamma_k\}$ of fuzzy sets on $V \cup E$ is called a *k*-fuzzy total coloring of $\hat{G}= (V, \sigma, \mu)$ if

a) $\max_i\{\gamma_i(v)\} = \sigma(v)$ for all $v \in V$ and $\max_i\{\gamma_i(uv)\} = \mu(uv)$ for all edge $uv \in E$.
b) $\gamma_i \wedge \gamma_j = 0$
c) For every adjacent vertices u, v of $\min\{\gamma_i(u) , \gamma_i(v)\} = 0$ and for every incident edges $\min\{\gamma_i(v_jv_k) / v_jv_k$ are set of incident edges from the vertex $v_j\} = 0$, $j = 1, 2,\ldots |v|$.

2.2. k-fuzzy total coloring on complete fuzzy graph:

We extend the definition of k-fuzzy total coloring on complete fuzzy graph in the definition given below. Since we deal with complete fuzzy graph for which $\mu(uv)$ strictly equals to $\sigma(u) \wedge \sigma(v)$, the definition can be stated as follows.

Definition 1:

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots \gamma_k\}$ of fuzzy sets on $V \cup E$ is called a *k*-fuzzy total coloring of $\hat{G}_k = (V, \sigma, \mu)$ if

a) $\vee \gamma_i (v) = \sigma (v)$ and $\vee \gamma_i (uv) = \mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u,v \in V$, $uv \in E$ and $1 \leq i \leq k$.

b) $\gamma_i \wedge \gamma_j = 0$ for $1 \leq i, j \leq k$.

c) For every strong edge uv of $\hat{G}_k$, $(\gamma_i(u) \wedge \gamma_i(v)) = 0$, $(1 \leq i \leq k)$ and for any set of incident edges uv on vertex $u \in V$ of $\hat{G}_k$, $\wedge\{ \gamma_i(uv)\} = 0$, $(1 \leq i \leq k)$.

The Executive Committee Problem:

In order to analyze the concept of total coloring of a complete fuzzy graph and its associated chromatic number a scheduling problem is presented in example 2.2.1. This problem has been modeled as an assignment problem and has been studied as an intersection graph in [4].

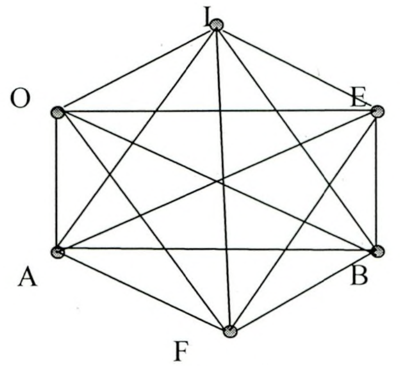
Example – 2.2.1:

The executive committee of the board of trustees of a small college has seven members, Kim, Smith, Jones, Gupta, Ramirez, Wang, and Chernov. It has six subcommittees with the following membership

a) Investments (I): W, R, G
b) Operations (O): G, J, S, K
c) Academic affairs (A): S, W, C
d) Fund Raising (F): W, C, K
e) Budget (B): G, R, C
f) Enrollment (E): R, S, J, K

Each time the executive committee has a meeting, first each of the subcommittees meets with appropriate college officers, and then the executive committee gets together as a whole to go over subcommittee recommendations and make decisions. Two committees cannot meet at the same time if they have a member in common, but committees that don't have a member in common can meet at the same time.

We draw a graph (fig-1) in which the vertices are the committee names and two vertices are adjacent if they have a member in common. Here we figure the minimum number of time slots needed to schedule the entire subcommittee meetings



[fig - 1]

The graph in fig-1 is an "intersection graph", which means its vertices correspond to sets and it has an edge between two vertices if and only if the corresponding sets intersect.fig-1 is a complete graph with six vertices, with each vertex of degree 5.Because each pair of vertices is adjacent and they require two different colors, the minimum number of colors needed to properly color G is 6, and we see that the chromatic number of graph in fig-1 is 6.Hence six time slots are needed to schedule all the subcommittee meetings. Obviously the subcommittee meetings depend on the intersection of sets. This concept could be fuzzy, and it could be associated with some numerical values and the problem in Example 2.2.1 could be modeled by means of a complete fuzzy graph $\hat{G}_k=(V,\sigma,\mu)$ with vertex set $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ and edge set $E=\{v_iv_j / ij=12,13,14,15,16,23,24,25,26,34,35,36,45,46,56\}$ and the corresponding membership functions which has been dealt in detail in section 2.3.

2.3 Fuzzy total chromatic number of a complete fuzzy graph:

Considering the fig-1, a complete fuzzy graph $\hat{G}_k=(V,\sigma,\mu)$ with vertex set $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ and edge set $E=\{v_iv_j/ij=12,13,14,15,16,23,24,25,26,34,35,36,45,46,56\}$ the membership functions are defined as follows:

$$\sigma(v_i)=\begin{cases}1, & for\ i=2,5\\ 0.4, & for\ i=1,4\\ 0.3, & for\ i=3,6\end{cases}\qquad \mu(v_iv_j)=\begin{cases}1, & for\ ij=25.\\ 0.4, & for\ ij=12,14,15,24,45.\\ 0.3, & for\ ij=13,16,23,26,34,35,36,46,56.\end{cases}$$

We see that the membership functions satisfy the definition of complete fuzzy graph. Let $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6\}$ be a family of fuzzy sets defined on $V\cup E$ as follows:

$$\gamma_1(v_i)=\begin{cases}0.4, & i=1\\ 0, & otherwise\end{cases}\qquad \gamma_1(v_iv_j)=\begin{cases}1, & ij=25\\ 0.3, & ij=34\\ 0, & otherwise.\end{cases}$$

$$\gamma_2(v_i)=\begin{cases}1, & i=2\\ 0, & otherwise\end{cases}\qquad \gamma_2(v_iv_j)=\begin{cases}0.3, & ij=35,46\\ 0.4, & ij=12\\ 0, & otherwise.\end{cases}$$

$$\gamma_3(v_i)=\begin{cases}0.3, & i=3\\ 0, & otherwise\end{cases}\qquad \gamma_3(v_iv_j)=\begin{cases}0.3, & ij=13,56\\ 0.4, & ij=24\\ 0, & otherwise.\end{cases}$$

$$\gamma_4(v_i)=\begin{cases}0.4, & i=4\\ 0, & otherwise\end{cases}\qquad \gamma_4(v_iv_j)=\begin{cases}0.3, & ij=36\\ 0.4, & ij=14\\ 0, & otherwise.\end{cases}$$

$$\gamma_5(v_i)=\begin{cases}1, & i=5\\ 0, & otherwise\end{cases}\qquad \gamma_5(v_iv_j)=\begin{cases}0.3, & ij=26\\ 0.4, & ij=15\\ 0, & otherwise.\end{cases}$$

$$\gamma_6(v_i)=\begin{cases}0.3, & i=6\\ 0, & otherwise\end{cases}\qquad \gamma_6(v_iv_j)=\begin{cases}0.3, & ij=16,23\\ 0.4, & ij=45\\ 0, & otherwise.\end{cases}$$

Hence the family $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6\}$ satisfies our definition of total coloring of a complete fuzzy graph. We find that any family of fuzzy sets having less than six members could not satisfy the definition. From the tables given below, we can see the values of $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6$ clearly. Hence in this case the total chromatic number $\chi_T(\hat{G}_k)$ is 6.

Table:1- For set of vertices (v_i):

Vertices	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	Max
1	0.4	0	0	0	0	0	0.4
2	0	1	0	0	0	0	1
3	0	0	0.3	0	0	0	0.3
4	0	0	0	0.4	0	0	0.4
5	0	0	0	0	1	0	1
6	0	0	0	0	0	0.3	0.3

Table:2- For set of edges (v_iv_j):

Edges	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	Max
12	0	0.4	0	0	0	0	0.4
13	0	0	0.3	0	0	0	0.3
14	0	0	0	0.4	0	0	0.4
15	0	0	0	0	0.4	0	0.4
16	0	0	0	0	0	0.3	0.3
23	0	0	0	0	0	0.3	0.3
24	0	0	0.4	0	0	0	0.4
25	1	0	0	0	0	0	1
26	0	0	0	0	0.3	0	0.3
34	0.3	0	0	0	0	0	0.3
35	0	0.3	0	0	0	0	0.3
36	0	0	0	0.3	0	0	0.3
45	0	0	0	0	0	0.4	0.4
46	0	0.3	0	0	0	0	0.3
56	0	0	0.3	0	0	0	0.3

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