Fixed Point Theorems

By

Kavitha A.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN - DEEMED UNIVERSITY, COIMBATORE - 641 043 IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

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Introduction

INTRODUCTION

This thesis is an attempt to discuss the fixed point theorems obtained by ***M. R. Taskovic*** [37]. ***Nikola Jotic*** [24], ***G. Jungck*** [17] and ***Zeqing Liu*** [40]. It is interesting to note that a number of fixed point theorems have been derived for metric spaces, compact Hausdorff spaces.

In chapter I the author has discussed fixed point theorems taken from the papers, "***Some results in the fixed point theory III*** ",by ***Milan, R. Taskovic*** [37] and "***Some fixed point theorems in metric spaces*** " by ***Nikola Jotic*** [24]. In the first section of this chapter ***localization monotone principle*** in the fixed point theory is defined as follows : "Let T be a mapping of a topological space x into itself, where x satisfies the condition of TCS - convergence. Suppose that there exists $y \in x$ and a mapping $\varphi : R_+^0 \to R_+^0$ such that $\phi(t) < t$ and $\lim_{z \to t+0} \sup \varphi(z) < t$ for every $t > 0$ and $B(Tx) \leq \phi(B(x))$, for every $x \in O(y)$ where $B : X \to R$ is T-orbitally lower semicontinous and $B(x) = 0$ implies $T x = x$. Then T has a unique fixed point $\xi \in X$". The applications of this principle are studied in detail. Many results published earlier can be obtained as corollaries of localization monotone principle. In the second section the author has discussed three fixed point theorems [1.2.1, 1.2.2, 1.2.3] in metric spaces for self mappings satisfying certain conditions with variable coefficients. These theorems generalize the corresponding results of ***Ciric*** [6,7], ***Hardy and Rogers*** [13]. ***Pal and Maiti*** [26]. ***Zamfirescu*** [39]. Many other generalizations of Banach contraction principle can be proved using these theorems.

Chapter II first section deals with characterization of self mappings with a fixed point interms of a communting map [Theorem : 2.1.1]. This characterization is proved in the main theorem and two corallaries are proved in the paper "***Commuting mappings and fixed points***" by ***Gerald Jungck***[17].

In the second section the fixed point theorems in compact Hausdorff spaces are studied. The results discussed here are contained in the paper. "***Some fixed point theorems in compact Hausdorff spaces***" by ***Zeqing Liu***[40]. The primary aim of this paper is to extend few results by ***Jungck*** [17], ***Singh and Rao*** [34] to a more general case. Four theorems [2.2.1, 2.2.2, 2.2.3, 2.2.4] are proved in this section. The last theorem [2.2.4] extends the theorem [2.1.1] proved by ***Gerald Jungck*** [17] studied in section 2.1 of this chapter.

Review of Literature

REVIEW OF LITERATURE

The first important result on fixed point theory is the ***Brouwer's*** fixed point theorem which states as follows :"Every continuous map $f : B^n \rightarrow B^n$ has a fixed point". Here B^n denotes the unit ball in R^n.

The second fundamental result on fixed point theory is the Banach contraction principle which states as follows: "A contraction is a mapping of a metric space (x,d) into itself such that $d(f(x), f(y)) \leq \alpha\ d(x,y)$ for some fixed $\alpha \in (0,1)$ and for all x, y $\in$ X. The ***Banach contraction principle*** states that any contraction of a complete metric space has a unique fixed point".

Fixed point theorems are of great interest in mathematics. It turns out that many problems such as problems concerning existence of solutions for systems of equations, for instance, can be formulated as fixed point problems. As an example we have the following ***classical theorem of Frobenius***, "Let A be a 3 X 3 matrics of positive real numbers then A has positive eigen value".

Recently a number of fixed point theorems on compact Hausdorff spaces and metric spaces are established by various authors. In this review of literature, we shall list below a few important articles published during '80 and '97.

1. M.R. Taskovic, "Some results in the fixed point theory, III", Indian J. Pure. appl. Math., 17(9) : 1094 - 1100, Sep. 1986.
2. Nikola Jotic, "Some fixed point theorems in - Metric spaces", Indian J. Pure. appl. Math., 26(10) : 947 - 952, October 1995.
3. Gerald Jungck, "Commuting mappings and fixed points", Mathematical notes, Department of Mathematics, University of Rhode Island, Kingston, RI 02881.

4. Zeqing Liu, "Some fixed point theorems in compact Hausdorff spaces", Indian Journal of Mathematics, Vol. 36, No. 3, December 1994, 235 - 239.

5. S. L. Singh and K. P. R. Rao. "Coincidence and fixed point for four mappings", Indian Journal of Mathematics , 31(1989) 215 - 223, MR 91b : 54088.

6. G. Jungck, "Common fixed points for commuting and compatible maps on compacta", Proc. Amer. Math. Soc. 103 (1988) 977 - 983. MR 89h : 54030.

7. G. Jungck, "Periodic and fixed points and commuting mappings", Proc. Amer. Math. Soc. 76(1979) 333 - 338. MR 80e : 54057.

8. V.W. Bryant, "A remark on a fixed point theorems for iterated mappings", Amer. Math. Monthly, 75(1968), 399 - 400.

9. B. Fisher, "Common fixed point of four mappings", Bull. Inst. Math. Acad. Sinica, 11(1983), 103 - 113.

10. G. Jungck, "Compatible mappings and common fixed points", Intern J. Math. Math. Sci., 9(1986), 771 - 779.

11. V. Popa, "Results on common fixed points", Math. Rev. Anal. number. Theor Approximation, Math., 26(49) (1984), No. 1, 75 - 79.

12. I.H.N. Rao and K.P.R. Rao, "Fixed point theorems on complete and compact spaces", Indian J. Pure appl. Math., 15 (1984), 1302 - 1307.

13. S. L. Singh & C. Kulshrestha, "Coincidence theorems", Indian J, Phy. Natur. Sci., 3B (1983), 5 - 10.

In this thesis we have discussed the four papers (1), (2), (3) and (4) in detail.

Chapter I

CHAPTER I

Section 1.1.

Localization Monotone Principle And Its Applications In Fixed Point Theory

Introduction

In this section localization monotone principle in the fixed point theory is defined and applications of this principle are studied in detail. Many authors like ***Chakrabarty*** [5], ***Popa*** [28], ***Ray*** [30], ***Jaggi*** [16], ***Bose and Mukherjee*** [3], ***Kasahara*** [20], ***Dhage*** [8], ***Furi and Vignoli*** [11] and ***Chatterjee*** [31] have studied fixed point theory. Many of their results are stated here as corollary of our localization monotone principle and as applications of the same principle. We collect in this section the applications of localization monotone principle studied by ***Milan R. Taskovic*** [37] in his paper "***some results in the fixed point theory II***".

Definition : Lower semi continuous

A real valued function g on a topological space X is lower semi continuous iff the set $\{ X : g(x) \leq a\}$ is closed for each real number a.

Definition: T- Orbitally Lower Semi Continuous Function At P.

Let X be a topological space, T: $X \to X$ and let B: $X \to R_+^0 = [0, \infty)$ be a T-orbitally lower semicontinuous function on X. The function B is T-orbitally lower semicontinuous at p if $\{x_n\}$ is a sequence in $O(x) := \{x, Tx, T^2x, \ldots\}$, $x \in X$ and $x_n \to p$ implies that $B(p) \leq \liminf B(x_n)$.

Definition: T- Orbitally Complete

A space X is said to be T-orbitally complete if every Cauchy sequence which is contained in O(x) for some $x \in X$ converges in X (*c.f. **Taskovic*** [36]).

Definition: TCS-Convergence in the space X

A topological space X satisfies the condition of TCS-convergence if there exists a point $x \in X$ such that

$B(T^n x) \to 0 (n \to \infty)$ implies $\{T^n x\}$ has a convergent subsequence.

Localization Monotone Principle

Let T be a mapping of a topological space X into itself, where X satisfies the condition of TCS- convergence. Suppose that there exists $y \in X$ and a mapping $\varphi: R_+^0 \to R_+^0$ such that

$$\varphi(t) < t \text{ and } \quad \lim_{z \to t+0} \sup \varphi(z) < t \tag{1}$$

for every $t > 0$ and

$$B(Tx) \leq \varphi(B(x)), \text{ for every } x \in O(y) \tag{2}$$

where $B : X \to R$ is T-orbitally lower semicontinuous and

$$B(x) = 0 \text{ implies } Tx = x \tag{3}$$

Then T has a unique fixed point $\xi \in X$.

We get the following from the preceeding principle:

Corollary : 1

Let T be a mapping of a topological space X into itself. Suppose that there exist $\alpha \in [0, 1)$ and

$B(Tx) \le \alpha B(x)$, for all $x \in X$,

where B: $X \to R_+^0$ is T-orbitally lower semi continuous and $B(x) = 0$ implies $Tx = x$. If for some $x \in X$ the sequence $\{T^n x\}$ has a convergent subsequence, then T has a fixed point $\xi \in X$.

Proof

By taking α in corollary 1 as φ in localization monotone principle, α is easily verified to satisfy conditions of φ. The last part of corollary 1 proves T is TCS-convergent.

Hence by localization monotone principle T has a fixed point.

Hence the corollary 1

The following result of ***chakrabarty*** [5] can be easily proved by using localization monotone priciple.

Corollary : 2

Let T be a mapping of metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exist $\alpha \in [0, 1)$ and

$A(Tx, Ty) \le \alpha A(x, y)$, for all $x, y \in X$. where A: $X \times X \to R_+^0$, $x \to A(x, Tx)$ is lower semi continuous and $\rho[x, y] \le A(x, y)$ for all $x, y \in X$. Then T has a unique fixed point $\xi \in X$ and $T^n x \to \xi$ for each $x \in X$.

We assume that (x, ρ) is a complete metric space and that Φ is a mapping of R_+^0 into itself satisfying the conditions: Φ is nondecreasing and continuous on the right,

$\Phi(t) < t$ for all $t > 0$. Further we assume that ψ is a continuous mapping of X x X into R_+^0 satisfying the conditions $\psi(x, x) = 0$ and $\rho(x, y) \leq \psi(x, y)$, for any x, y in X.

The following result of ***chakrabarty*** [5] is proved here using localization monotone principle.

Corollary : 3 (Chakrabarty [5])

Let M be a subset of X and T be a mapping of M into itself. Suppose that

$$\psi(Tx, Ty) \leq \Phi(\psi(x, y)), \text{ for all } x, y \in M \qquad (4)$$

where $\psi(x, T^n x) \leq A(x)$, $n \in N$ for every $x \in M$ and $A(x)$ a positive number. Then there is a point ξ in X such that for any $x \in X$, $T^n x \to \xi$ $(n \to \infty)$. If M is a closed, then ξ is the unique fixed point of T.

Proof

Let $B(x) = \psi(x, Tx)$, for every $x \in X$, and $\varphi(t) = \Phi(t)$.

Therefore, $\varphi(t) = \Phi(t)$

$$\Rightarrow \varphi(t) < t.$$

Hence (1) of localization monotone principle is satisfied.

consider

$$B(Tx) = \psi(Tx, T^2x)$$

$$\leq \Phi(\psi(x, Tx))$$

$$\leq \varphi(B(x)) \text{ for every } x \in X.$$

Hence (2) of localization monotone principle is satisfied.

And if $B(x) = 0 = \psi(x, Tx)$

We know $\rho(x, Tx) \leq \psi(x, Tx) = 0$

since ρ is metric.

Therefore, $\rho(x, Tx) = 0$

$$\Rightarrow x = Tx.$$

Hence (3) of localization monotone principle is satisfied.

From (4),

If $\{x_n\} \subseteq O(x)$, $x \in X$ and $x_n \to p$ implies that

$$B(p) = \psi(p, Tp) \le \liminf \psi(x_n, Tx_n)$$

$$= \liminf B(x_n)$$

Hence B is T-orbitally lower semi continuous.

Hence B and φ satisfies all the required hypotheses in localization monotone principle. Also from (4),

$$B(T^n x) = \psi(T^n x, T^{n+1} x)$$

$$\le \varphi(\psi(T^{n-1} x, T^n x))$$

$$\le \varphi^2 (\psi(T^{n-2} x, T^{n-1} x))$$

$$\vdots$$

$$\le \varphi^n (\psi(x, Tx))$$

$$\le \varphi^n B(x).$$

As $n \to \infty$, $B(T^n x) \to 0$. (Since $\varphi \in [0, 1)$).

Hence X satisfies the condition of TCS-convergence.

This completes the proof.

Next to ***chakrabarty*** [5], now we shall see the results of ***Popa*** [28] as special case of localization monotone principle.

Theorem 1.1.1 (Popa [28])

Let T be a continuous mapping of a Hausdorff space X into itself and let f be a continuous mapping of X x X into the non negative reals such that

$$f(Tx, Ty) \le a\, f(x, Tx)\, f(y, Ty)\, (f(x, y))^{-1} + bf(x, y) \qquad (5)$$

and

$$f(x, x)\, f(y, y) \le f^2 (x, y) \text{ and } f(x, y) \ne 0. \qquad (6)$$

If for some $x \in X$ the sequence of iterates $\{T^n x\}$ has a convergent subsequence, then T has a unique fixed point.

Proof

Let x be an arbitrary point in X. Then for $y = Tx \ne x$, from (5)

we have,

$$f(Tx, Ty) = f(Tx, T^2x) \le a\, f(x, Tx) + f(Tx, T^2x)\, (f(x, Tx))^{-1} + bf(x, Tx)$$

$$\Rightarrow f(Tx, T^2x) \le a\, f(Tx, T^2x) + b\, f(x, Tx)$$

$$\Rightarrow f(Tx, T^2x) \le \frac{b}{1-a}\, f(x, Tx)$$

$$\le \frac{a+b}{a+(1-a)}\, f(x, Tx)$$

$$f(Tx, T^2x) \le (a+b)\, f(x, Tx), \text{ for all } x \in X.$$

Let $B(x) = f(x, Tx)$ which is lower semi continuous on X, and let $\varphi(t) = (a+b)t$, for $t \in R_+$; Consider

$$B(Tx) = f(Tx, T^2x) \le (a+b)\, f(x, Tx)$$

$\leq \varphi(f(x, Tx))$

$\leq \varphi(B(x))$.

And if, $B(x) = 0 = f(x, Tx)$.

Since if $x \neq Tx$, then $f(x, Tx) \neq 0$

A contradiction.

Therefore $x = Tx$.

Hence B and φ satisfies all the required bypotheses in our localization monotone principle since X satisfies the condition of TCS - convergence, applying our localization monotone principle we obtain $T\xi = \xi$ for some $\xi \in X$.

Uniqueness follows immediately from condition (5) and (6). Hence the theorem 1.1.1.

Bose and Mukherjee [3] has proved a theorem in the fixed point theory analogous to the result of theorem 1.1.1. (***Popa*** [28]). It is given in the next corollary.

Corollary : 4 (Jaggi [16], Bose and Mukherjee [3])

Let T be a mapping of a metric space X into itself satisfying the following condition.

$\rho\,[Tx, Ty] \leq \alpha\, \rho\,[x, Tx]\, \rho\,[y, Ty]\, (\rho\,(x, y))^{-1} + b\, \rho\,[x, y]\;,\; x \neq y$

Where a and b are non negative real numbers such that $a + b < 1$. If there exists an orbit $O(x_0)$ which contains a convergent subsequence of which T is orbitally continuous, then T has a unique fixed point.

Definition : L- Space

Let N denote the set of all nonnegative integers. A pair (x, $\rightarrow$) of a set X and a subset $\rightarrow$ of the set X^N x X is called an L-space if the following two conditions are satisfied:

(i) If $x_n = x \in X$ for all $n \in N$,

then$(\{x_n\}, x) \in \rightarrow$.

(ii) If $(s, x) \in \rightarrow$, then $(t, x) \in \rightarrow$ for every subsequence t of s.

Definition : Separated L-Space

Let (x, $\rightarrow$) be an L-space. It is said to be seperated if each sequence in X converges to at most one point of X.

Definition : d - Complete L-space

Let d be a nonnegative extended real valued function on X x X. The L-space (X, $\rightarrow$) is said to be d-complete if each sequence $\{x_n\}_{n\in N}$ in X with

$$\sum_{n=0}^{\infty} d(x_n - 1, x_n) < \infty$$ converges to at least one point of X.

A result of ***Kasahara*** [20] in L-space is proved here applying localization monotone principle.

Theorem 1.1.2. (Kasahara [20])

Let (X, $\rightarrow$) be a separated L-Space which is d-complete for a nonnegative extended real valued function d on X x X, and f be a continuous mapping of X into itself satisfying the following conditions for some α ,β with $0 \leq \alpha < 1$ and $0 < \beta \leq \infty$:

$$d(f(x), f^2(x)) \leq \alpha\, d(x, f(x)) < \alpha\, \beta, \text{ for all } x \in X \qquad (7)$$

Then f has a fixed point, and the sequence $\{T^{n}x\}$ converges to a fixed point of f.

Proof

As can readily be seen by induction, $d(f^{n}(a), f^{n+1}(a)) \leq \alpha^{n} d(a, f(a))$, for every $n \in N$, where N denote the set of all nonnegative integers.

Hence the d-completeness of the space implies that the sequence $\{f^{n} x\}$ converges to some $\xi \in X$. This implies that X satisfies the condition of TCS- convergence.

Let $B(x) = d(x, f(x))$ and

$\varphi(t) = \alpha t$

since $\varphi(t) = \alpha t$

$\Rightarrow \varphi(t) < t$ (since $\alpha < 1$)

Hence (1) of localization monotone principle is satisfied.

Consider
$$\begin{aligned} B(f(x)) &= d(f(x), f(f(x))) \\ &= d(f(x), f^{2}(x))) \\ &\leq \alpha\, (d(x, f(x)) \\ &\leq \alpha\, B(x) \end{aligned}$$

$$B(f(x)) \leq \varphi(B(x))$$

Hence (2) of localization monotone principle is satisfied

and if $B(x) = 0 = d(x, f(x))$

$\Rightarrow x = f(x)$.

Hence (3) of localization monotone principle is satisfied.

Consider $B(p) = d(p, f(p))$

$\leq \liminf d(x^{n}, f(x^{n}))$

$$= \lim_{n \to \infty} \inf B(x^n)$$

Hence B is T-orbitally lower semicontinuous.

Applying localization monotone principle we get $\xi = f\,\xi$ for some $\xi \in X$.

This completes the proof.

Corollary 5 given below is an immediate corollary of our localization monotone principle. The below given corollary is used in the next theorem.

Corollary : 5

Let T be a mapping of metric space (x, ρ) into itself, suppose that there exists $\alpha \in [0, 1)$ such that

$\rho\,[\,Tx, T^2x] \leq \alpha\,\rho\,[x, Tx]$, for all $x \in X$. Where $x \to \rho\,[x, Tx]$ is T-orbitally lower semicontinuous and $\{T^nx\}$ has a convergent subsequence. Then T has a fixed point in X.

Applying Localization monotone principle, we obtained the proof of the result by ***Dhage*** [8] in the following theorem.

Theorem 1.1.3 (Dhage [8])

Let $T : X \to X$ be an orbitally continuous self-map of a metric space X and let X be T- orbitally complete. If T satisfies the condition

$$\min\{\,\rho\,[Tx, Ty], \rho\,[x, Tx]\,\rho\,[y, Ty]\,\} + a \min\{\rho\,[x, Ty],\ \rho\,[v, Tx]\,, \leq p\rho\,[x, y] + q\rho\,[x, Tx] \qquad (8)$$

for all $x, y \in X$ and a, p and q are real numbers such that $0 < p + q < 1$, then for each $x \in X$, then sequence $\{T^nx\}$ converges to a fixed point of T.

Proof

Let $x \in X$ be an arbitrary point in X. Then, for $y = Tx$, from (8) we have,

$\min \{ \rho [Tx, T^2 x], \rho [x, Tx], \rho [Tx, T^2 x] \} + a \min \{\rho [x, T^2 x], \rho [Tx, Tx]\} \leq p\rho$ $[x, Tx] + q\rho [x, Tx]$.

$\Rightarrow \min \{\rho [Tx, T^2 x], \rho [x, Tx]\} + a(0) \leq (p + q) \rho [x, Tx]$ (since ρ is metric)

$\Rightarrow \min \{\rho [Tx, T^2 x], \rho [x, Tx]\} \leq (p + q) \rho [x, Tx]$

Hence, for $B(x) = \rho [x, Tx]$

$\varphi(t) = (p + q) t$.

Consider $\varphi(t) = (p + q) t$

$< t$. (Since $(p + q) < 1$).

Hence (1) of localization monotone principle is satisfied and

consider $B(Tx) = \rho [Tx, T^2 x]$

$\leq (p + q) \rho [x, Tx]$

$\leq (p + q) B(x)$

$B(Tx) \leq \varphi(B(x))$

Hence (2) of localization monotone principle is satisfied.

And if $Bx = 0 = \rho [x, Tx]$

since ρ is metric

$\Rightarrow x = Tx$

Hence (3) of localization monotone principle is satisfied

Consider $B(p) = \rho(p, T(p))$

$\leq \lim_{n \to \infty} \inf \rho(x_n , T(x_n))$

$= \lim_{n \to \infty} \inf B(x_n)$

Hence T is T-orbitally lower semicontinuous.

And since X satisfies the condition of TCS-convergence (X is T-orbitally complete metric space and

$$\rho[T^n x, T^{n+s} x] \leq (p+q)^n (1-p-q)^{-1} \rho[x, Tx]),$$

applying our localization monotone principle we obtain T $\xi = \xi$ for some $\xi \in X$.

Hence the theorem.

Some other results are proved as theorems 2,3 and 4 in his paper by ***Dhage*** [8] as special cases of localization monotone principle. The author ***Milan.R. Taskovic*** [37] also has proved the following statement which generalizes theorem 1.1.3. of ***Dhage*** [8].

Theorem 1.1.4. (Taskovic [35])

Let $T : X \to X$ be a mapping on X and let X be a T-orbitally complete metric space. If there exists real numbers α_i, β for every $x, y \in X$ such that

$\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $\beta - \alpha_2 \geq 0$ or $\beta - \alpha_3 \geq 0$, and

$$\alpha_1 \rho[Tx, Ty] + \alpha_2 \rho[x, Tx] + \alpha_3 \rho[y, Ty] + \alpha_4 \min\{\rho[x, Ty], \rho[y, Tx]\} \leq \beta\, \rho[x, y] \quad (9)$$

then for each $x \in X$, then sequence $\{T^n x\}$ converges to a fixed point ξ of T.

Proof is analogous to the proof of the theorem 1.1.2.

Let X denote a metric space and K a bounded subset of X.

Following ***Kuratowski*** [22] we denote by α (K) the infimum of all $\in > 0$ such that K admits a finite convering with subset of diameters less that $\in$. We use the following properties of the number α.

1) $\alpha(K) = 0$ if any only if K is precompact. For this reason $\alpha(K)$ is called the measure of non compactness of K.

2) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$, $(A, B \subset X)$.

3) If Kis compact then $\alpha(K) = 0$. Also, $\alpha(C\ 1\ K) = 0 \Leftrightarrow \alpha(K) = 0$, and $0 \leq \alpha(K) \leq \delta(K)$.

4) $\alpha(K) = 0$ and X complete imply K is compact.

Different types of mappings considered by ***Furi and Vignoli*** [11], ***Ray and Chatterjee*** [31], ***Singh*** [33], ***Khan*** [21], ***Bose and Mukherjee*** [3] are special cases of mappings mentioned in localization monotone principle. In mappings by these authors are stated below:

(i) (***Furi and Vignoli*** : [11]).

The continuous mapping T is called densifying, if for every bounded subset A of X, such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$. Let F be a real lower semicontinuous function defined in X x X. The densifying mapping T is said to be weakly F-contractive if the condition $F(Tx, Ty) < F(x, y)$ holds for all $x, y \in X$, $(x \neq y)$.

(ii) (***Ray and Chatterjee*** [31], ***singh*** [33], ***Khan*** [21]).

Let $F : X \times X \rightarrow R_+^0$ be continuous and $T : X \rightarrow X$ be densifying mapping such that $F(Tx, Ty) < \alpha F(x, y) + \beta F(x, Tx) + \gamma F(y, Ty)$ for each pair, of distinct points $x, y \in X$ and for non negative real numbers α, β, γ with $\alpha + \beta + \gamma < 1$, then T is called generalized densifying.

(iii) (***Bose and Mukherjee*** [3])

Let F be a continuous symmetric mapping of X x X into the set of nonnegative reals, such that $F(x, y) = 0$ iff $x = y$ and

$$F(Tx, Ty) \leq a\, F(x, Tx)\, F(y, Ty)(F(x, y))^{-1} + bF(x, y),\ x \neq y$$

for each pair of distinct points $x, y \in X$ where $a + b < 1$ $(a, b \geq 0)$.

Section 1.2

Fixed Point Theorems in - Metric Spaces

Introduction

The results in this section are taken from the paper ***"Some fixed point theorems in-metric spaces" by Nikola Jotic*** [24].

The author has proved three fixed point theorems in metric spaces for self mappings satisfying certain conditions with variable coefficients. These theorems generalize corresponding results of ***Ciric*** [6, 7], ***Hardy and Rogers*** [13], ***Hicks and Rhoades*** [14], ***Pal and Maiti*** [26], ***Zamfirescu*** [39] and many other generalizations of Banach contraction principle.

Theorem : 1.2.1

Let (X, d) be a complete metric space, $f : X \to X$ a self-mapping and $\varphi : [0, \infty) \to [0, \infty)$ a real function such that

$$\varphi(r) < r \text{ for } r > 0, \tag{1}$$

$$\lim_{t \to r^+} \sup \varphi(t) < r \text{ for } r > 0, \tag{2}$$

If $f : X \to X$ satisfies the condition

$$d[f(x), f^2(x)] \leq \varphi\, [d(x, f(x))] \tag{3}$$

for every $x \in X$, then $\{f^n(x)\}$ is a Cauchy sequence. Furthermore, if X is complete and if a mapping $G(x) = d(x. f(x))$ is lower semi-continuous at a limit point of $\{f^n(x)\}$, say x^* ; then x^* is a fixed point of f.

Proof

Let $x_0 \in X$ be an arbitrary element of X and define

$$a_n = d\,[f^n(x), f^{n+1}(x)] \qquad (4)$$

We may assume that $a_n \neq 0$ for all $n \in N$, since otherwise the assertion of the theorem trivially holds.

Then by $d\,[f(x), f^2(x)] \leq \varphi\,[d(x, f(x))]$,

we get,

$$a_{n+1} = d\,[f^{n+1}(x), f^{n+2}(x)] \leq \varphi\,[d(f^n(x), f^{n+1}(x))]$$

$$= \varphi\,(a_n)$$

And by using $\varphi(r) < r$, for $r > 0$, we get,

$$\varphi[\,d\,(f^n(x), f^{n+1}(x))] \quad = \varphi\,(a_n) < a_n.$$

Therefore, $a_{n+1} < a_n$.

Therefore $\{a_n\}$ decreases and hence has a limit say a, such that $a \geq 0$.

Claim : 1

$a = 0$.

Proof

Now let $a_n \to a$.

And we have $a_{n+1} \leq \varphi\,(a_n) < a_n$.

Take lim sup on both sides, we get

$$\lim_{n \to \infty} \sup a_{n+1} \leq \lim_{a_n \to a} \sup \varphi(a_n) < \lim_{n \to \infty} \sup a_n$$

$$a \leq \lim_{t \to a^+} \varphi(t) < a.$$

$$\Rightarrow a < a.$$

A contradiction.

Therefore, a= 0

Hence the claim.

Therefore,

$$\lim_{n \to \infty} d(f^n(x), f^{n+1}(x)) = \lim_{n \to \infty} a_n = 0 \qquad (7)$$

Now we shall show that $\{f^n(x)\}$ is a Cauchy sequence.

Claim : 2

$\{f^n(x)\}$ is a Cauchy sequence.

Proof

If $\{f^n(x)\}$ is not a Cauchy sequence.

Then there exists an $\varepsilon > 0$ and a sequence of integers $\{m_k\}$, $\{n_k\}$ with $m_k > n_k \geq k$, and such that

$$d\left(f^{m_k}(x), f^{n_k}(x)\right) \geq \varepsilon \qquad (8)$$

And call $d\left(f^{m_k}(x), f^{n_k}(x)\right)$ as h_k, for k = 1, 2, ...

Hence $h_k \geq \varepsilon$, for k = 1, 2, ...

We may assume that,

$$d\left(f^{m_k - 1}(x), f^{n_k}(x)\right) < \varepsilon \qquad (9)$$

by choosing m_k is the smallest number exceeding n_k for which (8) holds.

From the definition of a_n and triangle inequality and applying result (9), we have

$$h_k = d\left(f^{m_k}(x), f^{n_k}(x)\right)$$

$$\leq d\left(f^{m_k}(x), f^{m_k-1}(x)\right) + d\left(f^{m_k-1}(x), f^{n_k}(x)\right)$$

$$\leq a_{m_k-1} + \varepsilon$$

$$\leq a_{n_k} + \varepsilon \qquad (10)$$

(since a_n is a decreasing function)

Hence by (7),

$$\lim_{k\to\infty} h_k = \lim_{k\to\infty} d\left(f^{m_k}(x), f^{n_k}(x)\right)$$

$$\leq \lim_{k\to\infty} a_{n_k} + \varepsilon$$

$$= 0 + \varepsilon \qquad \text{(since by (7))}$$

Therefore as $k \to \infty$, $h_k \to \varepsilon$.

Now using the triangle inequality, (3), (4), (5) and (8), we have,

$$h_k = d\left(f^{m_k}(x), f^{n_k}(x)\right)$$

$$\leq d\left(f^{m_k}(x), f^{m_k+1}(x)\right) + d\left(f^{m_k+1}(x), f^{n_k+1}(x)\right) + d\left(f^{n_k+1}(x), f^{n_k}(x)\right)$$

$$\leq a_{m_k} + d\left[f\left(f^{m_k}(x)\right), f\left(f^{n_k}(x)\right)\right] + a_{n_k}$$

$$\leq a_{m_k} + \varphi\left[d\left(f^{m_k}(x)\right), \left(f^{n_k}(x)\right)\right] + a_{n_k}$$

$$\leq a_{n_k} + \varphi(h_k) + a_{n_k}$$

$$\leq 2a_{n_k} + \varphi(h_k) \qquad (11)$$

Hence using (2), (7) and (10), we get

$$\lim_{k \to \infty} \sup h_k \leq \lim_{a_{n_k} \to \infty} 2a_{n_k} + \lim_{k \to \infty} \sup \varphi(h_k) \quad (\text{since } a_{m_k} < a_{n_k})$$

$$\varepsilon \leq 0 + \lim_{t \to \varepsilon^+} \sup \varphi(t) < \varepsilon$$

$$\Rightarrow \varepsilon < \varepsilon$$

A contradiction.

Therefore, $\{f^n(x)\}$ is a Cauchy sequence.

Hence the Claim.

To prove the result of the thorem, assume that X is complete.

$\{f^n(x)\}$ being a Cauchy sequence from the above claim, converges in the complete metric space X.

Let $\lim_{n \to \infty} f^n(x) = x^*$, for some $x^* \in X$.

Let G be a mapping such that

$G : X \to [0, \infty)$ such that $G(x) = d(x, f(x))$, and is lower semi-continuous at x^*.

$$G(x^*) = d(x^*, f(x^*))$$

$$= d(\lim_{n \to \infty} f^n(x), f(\lim_{n \to \infty} f^n(x)))$$

$$\leq \lim \inf d(f^n(x), f^{n+1}(x)) \quad (\text{since by the definition of lower semicontinuous})$$

$$= 0 \quad (\text{since by (7)})$$

Therefore, $d(x^*, f(x^*)) = 0$. Hence $f(x^*) = x^*$.

Which completes the proof of the theorem.

Remark : 1

If in theorem 1.2.1 φ is defined by $\varphi(r) = hr$ for each $r \geq 0$, where $0 < h < 1$, then theorem 1.2.1 reduces to the theorem ***Hicks and Rhoades*** [14].

Now we shall prove the following main result.

Theorem 1.2.2.

Let (x, d) be a complete metric space and f a self-mapping of X. If there exists real functions $\alpha, \beta, \gamma, \delta : (0, \infty) \to (-\infty, +\infty)$ which are continuous from the right and such that for each $x, y \in X$ and $r = d(x, y)$ the following inequalities hold:

$$\alpha(r)\, d(x, y) + \beta(r) d(f(x),f(y)) + \gamma\, [d(x, f(x))+d(y, f(y))]+$$

$$\delta(r)\, [d(x, f(y)) + d(y, f(x))] \geq 0, \qquad (12)$$

$$\alpha(r) + \beta(r) + 2\,\gamma(r) + \delta(r) + |\delta(r)| < 0 \qquad (13)$$

$$\beta(r) + \gamma(r) + \delta(r) < 0 \qquad (14)$$

$$\alpha(r) + \beta(r) + 2\delta(r) < 0 \qquad (15)$$

then f has a fixed point in X.

Proof

Let $x \in X$ be arbitrary and let $y = f(x)$.

Put $d_x = d(x, y) = d[x, f(x)]$.

Then by (12) we have,

$$\alpha\,(d_x)\, d_x + \beta(d_x)\, d(f(x), f^2(x)) + \gamma(d_x)\, [d_x + d(f(x), f^2(x))]$$

$$+\delta(d_x)(d(x, f^2(x)))+ d(f(x), f(x)) \geq 0.$$

As d(f(x), f(x)) = 0, we have,

$\alpha(d_x)\, d_x + \beta(d_x)\, d(f(x), f^2(x)) + \gamma(d_x)\, [d_x + d(f(x), f^2(x))] + \delta(d_x)(d(x, f^2(x))) \geq 0.$

We shall consider two cases.

Case : 1

$\delta(d_x) \geq 0$,

By the triangle inequality and $\delta(d_x) \geq 0$, we get,

$$\delta(d_x)\, d(x, f^2(x)) \leq \delta(d_x) d(x, f(x)) + \delta(d_x)\, d(f(x), f^2(x))$$

$$= \delta(d_x) d_x + \delta(d_x)\, d(f(x), f^2(x)) \qquad (*)$$

Substituting (*) in (16)

$$\alpha(d_x)d_x + \beta(d_x)\, d(f(x), f^2(x)) + \gamma(d_x)\,(d_x + d(f(x), f^2(x))) + \delta(d_x)d_x + \delta(d_x)\, d[f(x), f^2(x)] \geq 0.$$

$$\Rightarrow [\alpha(d_x) + \gamma(d_x) + \delta(d_x)]\, d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)]\, d[f(x), f^2(x)] \geq 0.$$

And hence;

$$[\alpha(d_x) + \gamma(d_x) + \delta(d_x)]\, d_x \geq -[\beta(d_x) + \gamma(d_x) + \delta(d_x)]\, d[f(x), f^2(x)] \qquad (17)$$

From (14), $\beta(d_x) + \gamma(d_x) + \delta(d_x)$ is negative and so substituting in (17) we get

$[\alpha d_x + \gamma d_x + \delta d_x]\, d_x \geq 0.$

As $\delta(d_x) \geq 0$ it can be replaced by $|\delta(d_x)|$ and the above inequality becomes

$$\alpha(d_x) + \gamma(d_x) + |\delta(d_x)| \geq 0 \qquad (18)$$

Again from (17), we have,

$$d(f(x), f^2(x)) \leq -\frac{\alpha(d_x) + \gamma(d_x) + |\delta(d_x)|}{\beta(d_x) + \gamma(d_x) + \delta(d_x)}\, d_x \qquad (19)$$

From (13), for $\delta(d_x) \geq 0$, we have,

$[\alpha(d_x) + \gamma(d_x) + \delta(d_x)] < -[\beta(d_x) + \gamma(d_x) + \delta(d_x)]$

Now (14) and (18) implies

$$0 < -\frac{\alpha(d_x)+\gamma(d_x)+|\delta(d_x)|}{\beta(d_x)+\gamma(d_x)+\delta(d_x)} < 1 \tag{20}$$

Define a function $\varphi : [0, \infty) \to [0, \infty)$ as follows that

$$\varphi(t) = -\frac{\alpha(t)+\gamma(t)+|\delta(t)|}{\beta(t)+\gamma(t)+\delta(t)}t \tag{21}$$

Then from (19) and (20) we have, for all $x \in X$, $d(f(x), f^2(x)) \leq \varphi(d(x), f(x)))$ (22)

Case : 2

Suppose now that $\delta(d_x) < 0$. By the triangle inequality,

$d(x, f^2(x)) \geq d(f(x), f^2(x)) - d(x, f(x))$ and $\delta(d_x) < 0$, we get,

$\delta(d_x)\, d(x, f^2(x)) \leq \delta(d_x) d(f(x), f^2(x) - \delta(d_x)\, d_x.$

Now from (16) we obtain,

$$\alpha(d_x)d_x + \beta(d_x)\, d(f(x), f^2(x)) + \gamma(d_x)\,(d_x + d(f(x), f^2(x))) + \delta(d_x)(d(f(x), f^2(x)) - d(x, f(x))) \geq 0$$

and hence,

$$[\alpha(d_x) + \gamma(d_x) - \delta(d_x)]\, d_x + [\beta(d_x) + \gamma(d_x) + \delta(d_x)]\,(d(f(x), f^2(x))) \geq 0.$$

Hence, using (14) and as $-\delta(d_x) = |\delta(d_x)|$, we get

$$[\alpha(d_x) + \gamma(d_x) - \delta(d_x)]\, d_x \geq -[\beta(d_x) + \gamma(d_x) + \delta(d_x)]\, d[f(x), f^2(x)]$$

$$\Rightarrow d(f(x), f^2(x)) \leq \frac{\alpha(d_x)+\gamma(d_x)-\delta(d_x)}{\beta(d_x)+\gamma(d_x)+\delta(d_x)} d_x$$

$$\Rightarrow d(f(x), f^2(x)) \leq \frac{\alpha(d_x)+\gamma(d_x)+|\delta(d_x)|}{\beta(d_x)+\gamma(d_x)+\delta(d_x)} d_x \qquad (\text{since } -\delta(d_x) = |\delta(d_x)|)$$

From $\delta(d_x) < 0$ and (13) it follows that

$$\alpha(d_x)+\gamma(d_x)+|\delta(d_x)| \; + \; \beta(d_x)+\gamma(d_x)+\delta(d_x) < 0$$

Hence we obtain the relation (20) again :

$$0 \leq -\frac{\alpha(d_x)+\gamma(d_x)+|\delta(d_x)|}{\beta(d_x)+\gamma(d_x)+\delta(d_x)} < 1$$

If we now, define a function

$\varphi : [0, \infty) \to [0, \infty)$, as before by formula (21), we obtain the relation (22).

Therefore, in both cases $\delta(d_x) \geq 0$ and $\delta(d_x) < 0$ we have the relation (22), where φ is defined by (21).

Put $x_0 = x$ and let sequence $\{x_n\}_{n \in N}$ be defined by $x_n = f(x_{n-1})$.

From (20) we conclude that the function $\varphi(t)$ defined by (21) satisfies (1).

$$\text{i.e., } \varphi(t) = -\frac{\alpha(t)+\gamma(t)+|\delta(t)|}{\beta(t)+\gamma(t)+\delta(t)} t$$

$$< t.$$

Since α, β, γ and δ are continuous from the right and from (14),

$$\beta(r) + \gamma(r) + \delta(r) < 0,$$

it follows that,

$$\lim_{t \to r^+} \varphi(t) = -\frac{\alpha(r)+\gamma(r)+|\delta(r)|}{\beta(r)+\gamma(r)+\delta(r)} r$$

$$= \varphi(r).$$

So $\varphi(t)$ is continuous from the right. Therefore, for $r > 0$, we have

$$\lim_{t \to r^+} \sup \varphi(t) = \lim_{t \to r^+} \varphi(t)$$

$$= \varphi(r)$$

$$< r.$$

And we conclude that the function $\varphi(t)$ satisfies (2).

From (22) we see that $\varphi(t)$ also satisfies (3). Therefore, from the first part of theorem: 1.2.1., we conclude that $\{ x_n\}$ is a convergent sequence in X.

Claim : 1

$\lim_{n \to \infty} x_n = x^*$ ***implies*** $x^* = f(x^*)$.

Proof

Let $d_n = d(x_n, x^*)$

Put $r = d_n$

By (12) we obtain,

$$\alpha(d_n)d(x_n, x^*) + \beta(d_n)\, d(f(x_n), f(x^*)) + \gamma\,(d_n)\, [(d(x_n + (f(x_n)) + d(x^*, f(x^*))]$$

$$+\delta(d_n)[d(x_n\,, f(x^*)) + d(x^*, f(x_n))] \geq 0.$$

From this inequality and from the definition of f, $f(x_n) = x_{n+1}$, we get,

$$\alpha(d_n)d(x_n, x_{n+1}) + \beta(d_n)\, d(x_{n+1}, f(x^*)) + \gamma\,(d_n)\, [d(x_n\,, (x_{n+1})) + d(x^*, f(x^*))]$$

$$+\delta(d_n)[d(x_n\,, f(x^*)) + d(x^*, f(x_{n+1}))] \geq 0.$$

If now let n tend to infinity, we get,

(when $n \to \infty$, $x_n \to x^*$, $d_n \to 0$)

$$\alpha(0)\ d(x^*,\ x^*)\ +\ \delta(0)d[x^*,\ f(x^*)] +\ \gamma(0)[d(x^*,\ x^*)\ +\ d(x^*,\ f(x^*))] +\ \delta(0)$$
$$[d(x^*,\ f(x^*)) + d(x^*, x^*)] \geq 0.$$

$\Rightarrow [\ \beta(0) + \gamma(0) + \delta(0)]\ d(x^*, f(x^*)) \geq 0.$

Because α, β, γ, δ are continuous from the right. By this inequality and (14) it follows that

$d(x^*, f(x^*)) = 0$

Hence $x^* = f(x^*)$.

Hence the claim.

Claim : 2

The fixed point x^* is unique.

Proof

Let y_1, y_2 be two fixed points.

I.e., $y_1 = f(y_1)$, $y_2 = f(y_2)$

and $r = d(y_1, y_2)$.

From (12) it follows that

$$\alpha(r)d(y_1,y_2)+\beta(r)d(f(y_1),f(y_2))+\gamma(r)[d(y_1,f(y_1))+d(y_2,f(y_2))]$$

$$+\delta(r)[d(y_1,f(y_2))+d(y_2),f(y_1))] \geq 0.$$

$\Rightarrow\alpha(r)d+\beta(r)d+\gamma(r)[d(y_1, y_1)+d(y_2, y_2)]+\delta(r)[d(y_1, y_2)+d(y_2,y_1)] \geq 0$ (since $d = d(y_1, y_2)$.

$\Rightarrow(\alpha(r)d+\beta(r)d+\delta(r))(2d) \geq 0.$

Which implies

$(\alpha(r)+\beta(r)+ 2\delta(r)d \geq 0$

But (15) states that $\alpha(r) + \beta(r) + 2\delta(r)< 0$.

Hence the only possibility is that $d = 0$

i,e., $y_1 = y_2$.

Hence the claim. The proof is complete.

Corollary : 1

Let (x, d) be a complete metric space and f : X → X a self-mapping. If α, γ, δ : (0, +∞) → (-∞, +∞) are functions as in theorem 1.2.2 and such that the inequalities (12) - (!5) are satisfied with β = -1, then f has a unique fixed point in X.

Remark : 2

If in Corollary : 1 functions α, γ, δ are nonnegative, then corollary : 1 reduces to the theorem which contains Theorem 2 of ***Hardy and Rogers*** [13].

Remark : 3

If in Corollary 1 α, β and γ are constants, then corollary 1 reduces to theorems of ***Ciric*** [6], ***Zeemfirescu*** [39] and Theorem 1 of ***Hardy and Rogers*** [13].

Theorem 1.2.3.

Let (X, d), f and α, β, γ ,δ be as in Theorem: 1.2.2. such that the conditions (12) - (14) are satisfied. Then f has atleast one fixed point in X.

Proof

Condition (15) was used in the proof of the Theorem : 1.2.2. only in the part of the uniqueness of the fixed point.

This theorem is the same as Theorem 1.2.2. except for the uniqueness part. Proof follows as in Theorem 1.2.2.

Remark: 4

If in Theorem 1.2.3.

A) β = δ = 0 and γ = -1, or

B) $\beta = 0, \gamma = -1$ and $\alpha = \gamma$, or

C) $\alpha = 0, \beta = \gamma = -1$,

then Theorem : 1.2.3. reduces to corresponding Theorem of ***Pal and Maiti*** [26].

Similar results can be found in the papers of ***Achari*** [1], *Basu* [2], ***Ciric*** [6, 7], ***Fisher*** [10], ***Pachpatte*** [25], ***Ray*** [32] and ***Taskovic*** [37].

Chapter II

CHAPTER II

Section 2.1

Commuting mappings and Fixed Points

Introduction

This section deals with the inter dependence between the commuting mapping and fixed point concepts. ***Pfeffer*** [27] proves a proposition that any involution r of a circle S has a fixed point if and only if there exists a free involution (≠r) of S which commutes with r. This result depicts the interdependence between the commuting mapping and fixed point concepts. In his paper "***Commuting mappings and fixed points*** " - ***Gerald Jungck*** [17] has highlighted this interdependence in more general context. In this section we prove the main theorem of the above paper.

Proposition

Let f be a mapping of a set X into itself. Then f has a fixed point iff there is a constant map $h : X \to X$ which commutes with f (i.e., $h(f(x)) = f(h(x))$ for all x in X).

Proof

Sufficiency : ($\Leftarrow$)

By hypothesis there exists $a \in X$ and $h : X \to X$ such that $h(x) = a$, for all $x \in X$.

h commutes with f.

i.e., $h(f(x)) = f(h(x))$, for all $x \in X$. (1)

$h(x) = a$, for every $x \in X$

as a , $f(a) \in X$

$\Rightarrow h(a) = a$ (2)

$h(f(a)) = a$ (3)

Consider, $f(a) = f(h(a))$ (from (1))

$= h(f(a))$ (from (2))

$= a.$ (from (3))

Therefore, $f(a) = a$.

So that a is a fixed point of f. (The necessity portion of the proof is included in the proof of our main result).

Main Theorem & Corollaries

The proof of our theorem appeals to the following lemma, which is easily verified upon noting that condition

(i) ensures that the sequence $\{y_n\}$ is cauchy.

Lemma

Let $\{y_n\}$ be a sequence in a complete metric space (x, d). If there exists $\alpha \in (0, 1)$ such that

(i) $d(y_{n+1}, y_n) \leq \alpha d(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Theorem 2.1.1.

Let f be a continuous mapping of a complete metric space (X, d) into itself. Then f has a fixed point in X iff there exists $\alpha \in (0, 1)$ and a mapping $g: X \rightarrow X$ which commutes with f and satisfies.

(*) $g(X) \subset f(X)$ and $d(g(x), g(y)) \leq \alpha\, d(f(x), f(y))$ for all $x, y \in X$.

Indeed, f and g have a unique common fixed point if (*) holds.

Proof

Necessary Part : ($\Rightarrow$)

Suppose that f has a fixed point, say a.

Then $f(a) = a$, for some $a \in X$.

Define $g: X \to X$ by $g(x) = a$, for all $x \in X$.

Then $g(f(x)) = a$ (Since $f(x) \in X$)

and $f(g(x)) = f(a)$ (Since $g(x) = a$, for every $x \in X$)

$= a$ (Since a is the fixed point of f)

So $g(f(x)) = f(g(x))$, for all $x \in X$ and g commutes with f.

More over, $g(x) = a = f(a)$, for all $x \in X$. So that $g(X) \subset f(x)$.

Finally, for any $\alpha \in (0, 1)$, we have for all x, y in X :

$d(g(x), g(y)) = d(a, a)$ (Since $g(x) = a$, for every $x \in X$).

$= 0$

$\leq \alpha\, (d(f(x), f(y))$.

Thus (*) holds.

Sufficiency Part : ($\Leftarrow$)

On the other hand, suppose there is a mapping g of X into itself which commutes with f and for which (*) holds. We show that this condition is sufficient to ensure that f and g have a unique common fixed point.

Let $x_0 \in X$ and let x_1 be such that $f(x_1) = g(x_0)$.

In general, choose x_n so that

$$f(x_n) = g(x_{n-1}) \tag{4}$$

We can do this since $g(X) \subset f(x)$.

The relation (*) and (4) imply that

$$d(f(x_{n+1}), f(x_n)) = d(g(x_n), g(x_{n-1})) \qquad \text{(since by (4))}$$

$$\leq \alpha\, d(f(x_n), f(x_{n-1})), \text{ for all } n. \qquad \text{(since by (*))}$$

The lemma yields $t \in X$ such that

$$f(x_n) \to t \tag{5}$$

$$g(x_n) \to t \tag{6}$$

Now since f is continuous, (*) implies that both f and g are continuous.

(Since $g(X) \subset f(X)$).

Hence (5) and (6) demand that $g(f(x_n)) \to g(t)$ and $f(g(x_n)) \to f(t)$.

But f and g commute so that,

$$g(f(x_n)) = f(g(x_n)) \text{ for all } n.$$

Thus $f(t) = g(t)$ and consequently,

$$f(f(t)) = f(g(t))$$

$$= g(f(t)) \qquad \text{(since by commutativity)}$$

$$= g(g(t)) \qquad \text{(since } f(t) = g(t))$$

We can therefore infer,

$$d(g(t), g(g(t))) \leq \alpha\, d(f(t), f(g(t)))$$

$$= \alpha\, d(g(t), g(g(t))) \qquad \text{(since } f(t) = g(t))$$

Hence,

$$d(g(t), g(g(t))) - \alpha\, d(g(t), g(g(t))) \leq 0.$$

$\Rightarrow d(g(t), g(g(t)))\,(1-\alpha) \leq 0$ Since $\alpha \in (0, 1)$

$\Rightarrow (1-\alpha) > 0$

$\Rightarrow d(g(t), g(g(t))) \leq 0$

$\Rightarrow d(g(t), g(g(t))) = 0$ (since $d(g(t), g(g(t)))$ cannot be less than zero).

$\Rightarrow g(t) = g(g(t))$

We now have,

$g(t) = g(g(t)) = f(g(t))$ (since $f(t) = g(t)$)

i.e, $g(t)$ is a common fixed point of f and g.

Claim

f and g have only one common fixed point

Proof

Suppose that x and y are two common fixed points of f and g.

Therefore, we have

$x = f(x) = g(x)$ and $y = g(y) = f(y)$.

Therefore, we get,

$d(g(x), g(y)) = d(x, y)$.

Then (*) implies,

$$d(x, y) = d(g(x), g(y))$$

$$\leq \alpha\, d(f(x), f(y))$$

$$= \alpha\, d(x, y)$$

$\Rightarrow d(x, y) \leq \alpha\, d(x, y)$

$\Rightarrow d(x, y)\,(1-\alpha) < 0$

$\Rightarrow d(x, y) = 0$ (since $\alpha \in (0, 1)$)

$\Rightarrow x = y.$

Hence the claim.

This completes the proof.

Corollary : 1

Let f and g be commuting mappings of a complete metric space (X, d) into itself. Suppose that f is continuous and $g(X) \subset f(X)$. If there exists $\alpha \in (0, 1)$ and a positive integer k such that

(i) $d(g^k(x), g^k(y)) \leq \alpha\, d(f(x), f(y))$

for all x and y in X, then f and g have a unique common fixed point.

Proof

Given that f and g are commuting mappings i.e., $f \cdot g = g \cdot f$

Consider $f \cdot g^2 = f \cdot g \cdot g$

$= g \cdot f \cdot g$

$= g \cdot g \cdot f$

$= g^2 \cdot f$

Proceeding like this we can generalize;

$f \cdot g^k = g^k \cdot f.$

Hence g^k commutes with f and

$g^k(X) \subset g(X) \subset f(X).$

Thus the theorem pertains to g^k and f, so there is a unique common fixed point for f and g^k, say a

$$\Rightarrow a = f(a) = g^k(a) \tag{7}$$

operating g on (7)

$$g(a) = g(f(a)) = g(g^k(a))$$

$$\Rightarrow g(a) = f(g(a)) = g^k\,(g(a)) \qquad \text{(since } f\,.\,g = g\,.\,f)$$

$$\Rightarrow f(g(a)) = g^k\,(g(a))$$

which says that g(a) is a common fixed point of f and g^k.

The uniqueness of 'a' implies

$$a = g(a) = f(a)$$

Therefore, f and g have a unique common fixed point.

Hence the corollary 1.

We obtain the Banach contraction principle as a consequence of corollary 1.

The ***Banach contraction principle*** states that any contraction of a complete metric space has a unique fixed point.

Proof

If we set k = 1 in corollary 1

$$\Rightarrow d(g(x), g(y)) \leq \alpha\, d(f(x), f(y)) \tag{8}$$

and let f be the identity map, i.

$$\Rightarrow g \text{ commutes with i, i.e., } g\,.\,i = i\,.\,g \tag{9}$$

and $g(X) \subset i(X) = X$.

i.e, $g(X) \subset X$ (10)

Therefore from (8), (9) and (10), i and g have a unique common fixed point from the corollary 1. This completes the proof.

In fact, if we let f be the identity map and keep the "general k", we obtain the generalization of Banach's theorem given in [4] and in [12].

Note that corollary 1 did not require that g be continuous.

On the other hand,

let $\alpha = \frac{1}{K}$ and $f = g^{n+1}$ in the statement of our main theorem 2.1.1.

(i) $g.\, g^{n+1} = g^{n+1} \cdot g = g^{n+2}$

Hence g and $f = g^{n+1}$ commute,

(ii) As we know,

$$g(X) \subseteq g^2(X) \subseteq \ldots\ldots\ldots g^n(X) \subseteq g^{n+1}(X).$$

We have $g(X) \subseteq g^{n+1}(X)$.

$$g(X) \subset f(X) = g^{n+1}(X).$$

and (iii) Consider $d(g(x), g(y)) \leq \frac{1}{K} d(g^{n+1}(x), g^{n+1}(y))$, for every $x, y \in X$.

$\Rightarrow K\, d(g(x), g(y)) \leq d(g^{n+1}(x), g^{n+1}(y))$, for every $x, y \in X$.

Put $n + 1 = n$ and $g = f = i$ (Since $ix = x$ for every $x \in X$)

We get,

$K\, d(f(x), f(y)) \leq d(g^n(x), g^n(y))$, for every $x, y \in X$.

$\Rightarrow d(g^n(x), g^n(y)) \geq K\, d(x, y)$ for every $x, y \in X$.

This substitution yields:

Corollary : 2

Let n be a positive integer and let K be a real number > 1. If g is a continuous mapping of a complete metric space (x, d) into itself such that

$d(g^n(x), g^n(y)) \geq K\,(d(x, y))$ for $x, y \in X$,

then g has a unique fixed point.

With n = 1 in corollary 2 we obtain the conclusion corollary of [29].

We conclude with an example of two functions neither of which is contractive (So that Banach's theorem doesn't pertain), or "expansive" in the sense of corollary 2 (with n = 1), which nevertheless satisfy the hypothesis of our theorem and hence have a unique common fixed point.

Example:

Let $X = R^2$ - Euclidean two space with the usual metric which we denote by d.

Define $f, g : X \to X$ for $p = (x, y)$ by $g(p) = (7x, (y/3) + 4)$ and $f(p) = (11x, (y/2) + 3)$.

$$\text{Then } f(g(p)) = \left(7(11x),\ \frac{(y/3)+4}{2} + 3\right)$$

$$= \left(77x,\ \frac{y + 12 + 18}{6}\right)$$

$$= \left(77x,\ (y/6) + 5\right)$$

$$\text{and } g(f(p)) = \left(11(7x),\ \frac{(y/2)+3}{3} + 4\right)$$

$$= \left(77x,\ \frac{y + 18 + 12}{6}\right)$$

$$= \left(77x,\ (y/6)+5\right)$$

Therefore f(g(p)) = g(f(p))

So that f and g commute.

Let q = (x', y')

and consider , d(f(p), f(q))

$$= \|(11x, (y/2)+3) - (11x', (y'/2)) + 3\|$$

$$= \|11(x-x') + (1/2)(y-y')\| \qquad (11)$$

and consider, d(g(p), g(q))

$$= \|(7x, (y/3)+4) - (7x', (y/3)) + 4\|$$

$$= \|7(x-x') + (1/3)(y-y')\|$$

$$3/2 d(g(p), g(q)) = (3/2)\ \|7(x-x') + (1/3)(y-y')\|$$

$$= \left\|\frac{21}{2}(x-x') + (1/2)(y-y')\right\| \qquad (12)$$

From (11) and (12) we get

$(x-x') \geq (21/2)\ (x-x')$

Therefore we get $d(f(p), f(q)) \geq (3/2)\ d(g(p), g(q))$.

We can therefore let $\alpha = 2/3$ inthe statement of the theorem 2.1.1 to conclude that f and g have a unique common fixed point.

Now q = (x', y')

If we set y = y' (see def of p),

we have, d(f(p), f(q)) = 11 d(p,q)

$\Rightarrow d(f(p), f(q)) > (3/2)\, d(p, q)$ (since $\alpha = 2/3$)

Thus f is not a contraction.

Similarly, if we let x = x',

$$d(f(p), f(q)) = (1/2)\, d(p,q)$$

$\Rightarrow d(f(p), f(q)) \leq (3/2)\, d(p, q)$ (since $\alpha = 2/3$)

So that f is not "expansion". Of course, the same substitution yield comparable conclusions for g.

In closing, we do observe that the requirement (*) of our theorem 2.1.1 can be appereciably weakened if we demand that (X, d) be compact. Consider the following new result which we state here without proof.

If we let C_f denote the set of all mappings $g : X \rightarrow X$ which commute with f, we can say the following.

Theorem 2.1.2

Let f be a continuous mapping of a compact metric space (X, d) into itself. Then f has a fixed point iff whenever $f(x) \neq f(y)$

$$d(g(x), g(y)) < d(f(x), f(y)) \text{ for atleast one } g \in C_f.$$

(Note that the choice of g need not be the same for different pair (x, y)).

Section 2.2.

Fixed Point Theorems in Compact Hausdorff Spaces

Introduction

This section deals with some fixed point theorems in compact Hausdorff spaces from the paper "***Some fixed point theorems in compact Hausdorff spaces***" by ***zeqing Liu***[40]. The intent of this paper is to extend few results by ***Jungck*** [17], ***singh and Rao*** [34], to a more general case. Four theorems are proved in this section. The theorems extend some main theorems of ***Jungck*** [17], ***Singh and Rao*** [34]. The last theorem of symmetric function extends the main theorem proved in the previous section 2.1. of this thesis. In this section we always assume that (x, τ) is a compact Hausdorff space, f and g are self mappings of X. F is a continuous function from X x X into $[0, \infty)$ such that $F(x, y) = 0$ iff $x = y$; N denotes the set of positive integers. A point $x \in X$ is a cluster point of a sequence $\{x_n\} \in X$ iff for each neighbourhood $\cup$ of x and $m \in N$, there exists $k > m$ such that $x_k \in \cup$.

Definition

$C_f : \{h/h: X \to X \text{ and } fh = hf\}$

and $H_f = \{h/h: X \to X \text{ and } h\left(\bigcap_{n=1}^{\infty} f^n X\right) \subset \bigcap_{n=1}^{\infty} f^n X$

Claim

$H_f \supset C_f.$

Proof

Let $h \in C_f$

$\Rightarrow hf = fh$ (by the definition of C_f)

To prove $h \in H_f$, i.e., $h\left(\bigcap_{n=1}^{\infty} f^n X\right) \subset \bigcap_{n=1}^{\infty} f^n X$

Consider $h(\cap f^n(x))$

$= \cap(h f^n(x))$

$= \cap f^n h(x)$

$= \cap f^n(x)$ (Since $h : X \to X$ and $h(X) \subseteq X$)

$\Rightarrow h \in H_f$

Theorem $H_f \supset C_f$

Hence the claim.

Clearly $C_f \supset \{f^n / n \in \{0\} \cup N\}$

Therefore $H_f \supset C_f \supset \{f^n / n \in \{0\} \cup N\}$

In this section we obtain the fixed points of H_{fg}, H_f and H_g and give two necessary and sufficient conditions for the existence of fixed points.

Lemma

Let f, g be self mappings of (X, τ), such that fg is continuous and $fg = gf$.

Let $A = \bigcap_{n=1}^{\infty} (fg)^n X$. Then

(i) $hA \subset A$ for $h \in C_{fg}$;

(ii) $A = fgA = fA = gA \neq \phi$; and

A is compact.

Proof

(i) Let $h \in C_{fg}$, where $C_{fg} = \{h / h(fg) = (fg)h\}$

$\Rightarrow h \in H_{fg}$ (since $H_{fg} \supset C_{fg}$)

$$\Rightarrow h\left(\bigcap_{n=1}^{\infty}(fg)^n X\right) \subset \bigcap_{n=1}^{\infty}(fg)^n X. \text{ (From the definition of } H_{fg})$$

$$\Rightarrow h\,A \subset A. \text{ (Since } A = \bigcap_{n=1}^{\infty}(fg)^n X)$$

(ii) Let $x \in X$

Since $fg : X \to X$ and $(fg)^2 : [fg(X)] \to X$.

Therefore $(fg)\,x \in X$.

Clearly $(fg)^2 X \subseteq fg(X)$.

Note that $(fg)^n X \supset (fg)^{n+1} X$, for $n \in \{0\} \cup N$.

From the compactness of X and the continuity of fg it follows that $\bigcap_{n=1}^{\infty}(fg)^n X = A$ is a non empty compact subset of X. Clearly $fgA \subset A$.

Claim

$fgA \supset A$.

Proof

Let a be in A.

Then there exists $x_n \in fg^{\,n-1}X$ such that $fg(x_n) = a$, for every $n \in N$.

Then consider the sequence $\{x_n\}$.

This sequence has a cluster point $x \in X$. Since fg is continuous fgx is a cluster point of $\{fg(x_n)\}$.

But $fgx = a$.

If we prove $x \in A$, then $a \in fgA$.

Now consider $x_n \in fg^{m} X$ for all $n \geq m$ (1)

Since X is compact and fg is continuous. And $fg^{m}X$ is closed. So any cluster point x of $\{x_n\}$ belongs to $fg^{m}X$ for all m and hence belongs to A.

Thus $x \in A$.

Hence $fgA \supset A$.

Hence the claim.

By (i) we have $fA \subset A$ and $gA \subset A$.

Thus $A = fgA \subset fA \subset A$.

So $A = fA$.

Similarly we can prove $gA = A$.

This completes the proof.

2. Unique common fixed point theorems

Theorem 2.2.1.

Let f, g be self mappings of (X, τ) such that fg is continuous and $fg = gf$. If $fx \neq gy$ implies $F(fx, gy) < \sup \{F(hu, bv) / u, v \in \{x, y\} \text{ and } h, b \in H_{fg}\}$ (2)

then f and g have a unique common fixed point a. In fact, $a = ha$ for all $h \in H_{fg}$.

Proof

Let $A = \bigcap_{n=1}^{\infty} (fg)^n X$: Then (i) & (ii) of the lemma hold.

Since F is continuous and A is compact; there exist two points $z, w \in A$ such that $F(z, w) = \sup \{F(x, y) / x, y \in A\}$ (since A is bounded)

By (ii), we can find $s, t \in A$ such that $z = fs$, $w = gt$.

Suppose $z \neq w$.

For $u, \nu \in \{s, t\} \subset A$, $h, b \in H_{fg}$, we have $hu, b\nu \in A$.

Using (2)

$F(z,w) = F(fs, gt)$

$< \sup\{f(hu, b\nu) / u, v \in \{s, t\} \text{ and } h, b \in H_{fg}\}$

$\leq \sup\{f(x,y) / x, y \in A\}$ (Since $H_{fg} \supset \{(fg)^n / n \in \{0\} \cup N\}$ and $A = \bigcap_{n=1}^{\infty} (fg)^n X$

$= F(z, w)$

$\Rightarrow F(z, w) = F(fs, gt) < F(z, w)$ a contradiction.

Hence $z = w$ and this implies A is a singleon. Say $A = \{a\}$.

Clearly $a = ha$ for all $h \in H_{fg}$; (Since $hA \subset A$, for every $h \in C_{fg}$ and $H_{fg} \supset C_{fg}$)

in particular, $a = fa = ga$.

Now if $c = fc = gc$, $gfc = c$ and thus $(gf)^n c = c$, for $n \in N$;

i.e., $c \in A = \{a\}$.

Thus, a is the only common fixed point of f and g.

This completes the proof.

Remark: 1

The above Theorem 2.2.1 extends Theorems 4.2 of ***Jungck*** [19] and Theorem 6 of ***Singh and Rao*** [34], which states that "Let (x, τ) be a non empty compact Hausdorff space, $F: X \times X \rightarrow R_+$ continuous and $F(x, x) = 0$ for all x in X.

Let f, g, s, T : X $\to$ X be such that fg = gf, s(fg) = (fg)s, T(fg) = (fg)T, fg continuous and, for fx $\neq$ gy,

$$[F(fx, gy)]^2 < \sup \{F(h_1u_1, h_2u_2)\, F(h_3u_3, h_4u_4) / u_1, u_2, u_3, u_4 \in \{x, y\}, h_1, h_2, h_3, h_4 \in \mathcal{F}\},$$

where $\mathcal{F}$ is the semi group of self-mappings of X generated by f, g, s and T. Then f, g, s and T have a unique common fixed point of f and g".

Theorem 2.2.2.

Let f and g be continuous self mappings of (X, τ). If fx $\neq$ gy implies

$$F(fx, gy) < \sup \{F(hx, by) / h \in H_f \text{ and } b \in H_g \} \qquad (3)$$

then f and g have a unique common fixed point a. In fact, a = ha = ba for all $h \in H_f$ and $b \in H_g$.

Proof

Let $A = \bigcap_{n=1}^{\infty} f^n X$ and $B = \bigcap_{n=1}^{\infty} g^n X$.

As in the proof of the Lemma, we can show that A and B are nonempty compact subsets of X, fA = A, gB = B, $hA \subset A$ for $h \in H_f$ and $bB \subset B$ for $b \in H_g$.

Since A and B are compact and F is continuous, there exist $z \in A$ and $w \in B$ such that $F(z, w) = \sup \{F(x, y) / x \in A,\ y \in B\}$

From fA = A and gB = B we can find $s \in A$, $t \in B$ such that fs = z, gt = w.

Suppose z $\neq$ w, using (3),

F(z,w) = F(fs, gt)

$< \sup\{f(hs, bt) / h \in H_f, b \in H_g\}$

$\leq \sup\{f(x,y) / x \in A, y \in B\}$

$= F(z, w)$.

A contradiction.

Therefore z = w and this implies that A = B and that A is a singleton say, $A = \{a\}$.

It is easy to see that a = ha = ba for all $h \in H_f$ and $b \in H_g$; in particular , a = fa = ga.

Now if c = fc = gc.

Then $f^n c = g^n c = c$ for $n \in N$;

i.e., $c \in A = B = \{a\}$.

Thus, a is the only common fixed point of f and g.

This completes the proof.

Remark : 2

Theorem 7 of ***singh and Rao*** [34], which states that "Let (X, τ) be a nonempty compact Hausdorff space, $F : X \times X \to R_+$ continuous and F(x, x) = 0 for all x in X. Let f, g, s, T : X → X be such that f and g are continuous fs = sf, gT = Tg and, for fx ≠ gy,

$$[F(fx, gy)]^2 < \sup \{F(h_1x, h_2y) F(h_3x, h_4y) / h_1, h_3 \in \mathcal{F}, h_2, h_4 \in y\},$$

where $\mathcal{F}$ is the semigroup of self-mappings of X generated by f and s semigroup of self- and y is the semigroup of self-mappings of X generated by g and T. Then f, g, s and T have a unique common fixed point which is the only common fixed point of f and s and of g and T". This theorem is a special case of the above theorem 2.2.2.

3. Fixed Point Theorems

Theorem : 2.2.3

Let f, g be continuous self mappings of (X,τ) satisfying $f(\bigcap_{n=1}^{\infty}(gf)^n X) = \bigcap_{n=1}^{\infty}(gf)^n X$, F be a symmetric function. If $fx \neq gy$ implies

$$F(fx, gy) > \inf\{F(x, fx), F(y, fy), F(x, gx), F(y, gy), F(hx, hy) \,/\, h \in C_f \cap C_g\} \qquad (4)$$

then at least one of f and g has a fixed point.

Proof

Let $A = \bigcap_{n=1}^{\infty}(gf)^n X$.

Then A is compact and $A = fA = gA = gfA \neq \phi$ and

$$hA \subset \bigcap_{n=1}^{\infty} h(gf)^n X \subset \bigcap_{n=1}^{\infty}(gf)^n hX \subset \bigcap_{n=1}^{\infty}(gf)^n X = A \qquad (5)$$

for $h \in C_f \cap Cg$. (Since $h : X \to X$)

Since f and g are continuous and A is compact, there exist $a, b \in A$ such that

$$\left.\begin{aligned} F(a, fa) &= \inf\{F(x, fx) \,/\, x \in A\}, \\ F(b, fb) &= \inf\{F(x, gx) \,/\, x \in A\}, \end{aligned}\right\} \text{(Since A is bounded)} \qquad (6)$$

without loss of generality we assume that

$$F(a, fa) \leq f(b, gb) \qquad (7)$$

By $gA = A$, there exists some $d \in A$ such that $gd = a$.

Suppose $fgd \neq gd$, i.e., $fa \neq a$.

From (4), (5), (6), (7) we obtain

$$F(fgd, gd) > \inf\{F(gd, fgd), F(d, fd), F(gd, ggd), F(d, gd), F(hgd, hd) \,/\, h \in C_f \cap C_g\}$$

$$= \inf\{F(a, fa), F(d, fd), F(a, ga), F(d, gd), F(hgd, hd) \,/\, h \in C_f \cap C_g\}$$

Put $d = b$

Therefore we have

$$F(fgd, gd) > \inf\{F(a, fa), F(b, fb), F(a, ga), F(b, gb), F(hdg, hd) / h \in C_f \cap C_g\}$$

$$\geq \inf\{F(a, fa), F(b, gb), F(hdg, hd) / h \in C_f \cap C_g\} \qquad \text{(from (7))}$$

$$\geq F(a, fa)$$

which implies

$$F(a, fa) = F(gd, fgd) > F(a, fa)$$

a contradiction.

Hence $a = fa$.

This completes the proof.

Remark : 3

Theorem 2.2.3. holds if one assumes the commutativity of f and g instead of $f(\bigcap_{n=1}^{\infty} (gf)^n X = \bigcap_{n=1}^{\infty} (gf)^n X$. Thus Theorem 4.4 of ***Jungck*** [19] is a special case of Theorem 2.2.3.

Theorem 2.2.4.

Let f be a continuous self mapping of (X, τ), F be a symmetric function, then the following statements are equivalent:

a) f has a fixed point ;

b) $fx \neq fy$ implies $F(fx, fy) > F(hx, hy)$ for some $h \in C_f$;

c) $fx \neq fy$ implies $F(fx, fy) > \inf\{F(x, fx), F(y, fy), F(hx, hy) / h \in C_f\}$.

Proof

(a) ⇒ (b)

Assume that (a) holds and w is a fixed point of f ⇒ fw = w.

Define hx = w for all $x \in X$.

Consider h(f(x)) = w (since $f(x) \in X$ and h(x) = w, for all $x \in X$)

and

Consider f(h(x)) = fw

= w.

Therefore hf = fh

i.e., $h \in C_f$

And $fx \neq fy$ implies

$F(fx, fy) > 0 = F(w, w)$

$= F(hx, hy)$, for some $h \in C_f$ and $x, y \in X$

(Since h(x) = w, for all $x \in X$).

$\Rightarrow F(fx, fy) > F(hx, hy)$ for some $h \in C_f$.

Therefore f satisfies (b).

Hence (a) implies (b).

(b) ⇒ (c)

$fx \neq fy$ implies

$F(fx, fy) > \inf \{F(x, fx), F(y, fy), F(x, fx), F(y, fy), F(h_1x, h_1y), F(h_2x, h_2y) \ldots\ldots \}$

$= \inf \{F(x, fx), F(y, fy), 0\}$ (since $F(h_ix, h_iy) = 0$ for some i)

$= 0.$

Therefore,

$fx \neq fy$ implies,

$$F(fx, fy) > \inf \{F(x, fx), F(y, fy), F(hx, hy) / h \in C_f \}$$

Hence (b) implies (c).

(c) ⇒ (a)

Now suppose that (c) holds.

Take f = g in Theorem 2.2.3 We have from (4).

$$F(fx, gx) > \inf \{F(x, fx), F(y, fy), F(x, fx), F(y, fy), F(hx, hy) / h \in C_f \cap C_g \}$$

$$= \inf \{F(x, fx), F(y, fy), F(hx, hy) / h \in C_f \cap C_g\}$$

Then f has a fixed point.

i.e., (c) implies (a).

This completes the proof.

Remark : 4

Theorem 2.2.4. extends the main Theorem 2.1.1. of our last section 2.1, which states that

"Let f be a continuous mapping of a complete metric space (x, d) into itself. Then f has a fixed point in X iff there exists $\alpha \in (0, 1)$ and a mapping $g : X \to X$ which commutes with f and satisfies

(*) $g(X) \subset f(X)$ and $d(g(x), g(y)) \leq \alpha\, d(f(x), f(y))$ for all $x, y \in X$.

Indeed, f and g have a unique common fixed point if (*) holds and corollary 2.3 of ***Jungck*** [18]."

Summary and Conclusion

SUMMARY AND CONCLUSION

The main aim of this thesis is to study some important fixed point theorems. This study is based mainly on the following papers :

1. "***Some results in the fixed point theory III***", by ***Milan R. Taskovic***[37].
2. "***Some fixed point theorems in metric spaces***" by ***Nikola Jotic***[24].
3. "***Commuting mappings and fixed points***", by ***Gerald Jungck***[17].
4. "***Some fixed point theorems in compact Hausdorff spaces***" by ***Zeqing Liu***[40].

In the first chapter we have discussed the first two papers mentioned above. Here the author has introduced localization monotone principle and developed fixed point theorems. Many of the fixed point theorems published earlier can be obtained as collories of the theorems proved in this paper.

The second paper deals with fixed point theorems of functions defined on metric spaces. Many generalizations of Banach contraction principle can be proved using these theorems.

The papers (3) and (4) deal with fixed point theorems on complete metric spaces and on compact Hausdorff spaces respectively. Four main theorems are established in this connection.

We hope that a deep study of these concepts will lead to many interesting open problems yielding a very good scope for further research.

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