Metric Completeness and Order Completeness

BY

Selvi S.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE - 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY 1995

METRIC COMPLETENESS AND ORDER COMPLETENESS

By

SELVI.S.

A DISSERTATION SUBMITTED TO

AVINASHILINGAM INSTITUTE FOR HOME SCIENCE

AND HIGHER EDUCATION FOR WOMEN

(DEEMED UNIVERSITY)

COIMBATORE - 641 043.

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY 1995

CERTIFIED AS BONAFIDE RESEARCH WORK

K. Sivakamasundari

Signature of the Guide

K. N. Meenakshi

Signature of the
Head of the Department

9/5/95

Signature of the
Dean of the Faculty

9/5/95

ACKNOWLEDGEMENT

ACKNOWLEDGEMENT

The investigator is grateful to **PADMASHREE COLONEL** Dr.(Tmt.) **RAJAMMAL P.DEVADAS, M.A.,** Ph.D.(**OHIO STATE), D.Sc.** (MADRAS), Chancellor, Avinashilingam Institute for Home Science and Higher Education for women (Deemed University) Coimbatore, for providing the opportunity to conduct her investigation in this much esteemed university.

She expresses her gratitude to **Dr.(Tmt.) LAKSHMI SANTA** RAJAGOPAL, **M.S. (TENNESEE), Ph.D. (MADRAS)**, Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for women (Deemed University), Coimbatore, for providing facilities to carry out the study.

The author extends her deep sense of gratitude to Dr.(Tmt.) **SAROJA PRABAHAR, M.A.,** Dip.Ed **(MADRAS),** Ph.D.(**MOTHER** THERESA), Registrar and **Dr.(Tmt.) NIRMALA.K.MURTHY,B.Sc (Hons)** ANNAMALAI, **M.sc. (ICWA), Ph.D. (MADRAS),** Dean, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for women (Deemed University), Coimbatore, for the keen interest shown by them.

The author wishes to record her deepest sense of immense gratitude to Dr. **(MISS) K.N.MEENAKSHI, M.Sc., Ph.D. (MADRAS),** Professor and Head of the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for women (Deemed University), Coimbatore for helping in the preparation of his dissertation.

She wishes to express her profound gratitude to Mrs.K.SIVAKAMASUNDARAI, **M.Sc., M.Phil., (ANNAMALAI), D.H.Ed.** (MADRAS), Lecturer, Avinashilingam Insititute for Home Science and Higher Education for women (Deemed University), Coimbatore, for the inspiring guidance and valuable advice which enabled the author to bring this dissertation in its present and final form.

The author wishes to thank **(Tmt.) A.PARVATHY, M.Sc., M.Phil., Dip. Ed. (MADRAS),** Lecturer, Avinashilingam Institute for Home Science and Higher Education for women (Deemed University), Coimbatore, for her pearless guidance, valuable council and keen interest she has rendereed to carryout this study.

She records her sincere thanks to her parents, brother and her friends for their kind help and encouragement throughout the study.

INTRODUCTION

INTRODUCTION

This thesis is an attempt to discuss the concept of basic structures introduced by J.C. MARGON [8] in 1991. It is interesting to note that every metric structure on a set induces a basic system and a totally ordered structure on a set induces a basic system.

The main results proved here are as follows.

1. A metric space is complete iff the associated basic system is complete.

2. An order system is Dedekind complete iff the associated basic system is complete.

3. Let (X,d) be a metric space and let (X,c,ψ) be the associated basic system. Let (X^*,c^*,ψ^*) be the completion of (X,c,ψ). In X^*, we can introduce a metric d^* such that (X^*,d^*) is the completeness of (X,d).

4. Let $(X,<)$ be a totally ordered set and let (X,c,ψ) be the associated basic system Let (X^*,c^*,ψ^*) be the basic completion of (X,c,ψ). In X^* we can introduce an order $<^*$ such that $(X^*,<^*)$ is the completion of $(X,<)$.

Chapter I deals with the preliminaries on metric completeness and Dedekind completeness.

In chapter II the methods of associating basic systems with metric spaces and ordered spaces are studied in detail.

In the first section of chapter III, the metric completion of a metric space is constructed via the associated basic system.

In section 2 of chapter III an order completion of an ordered space is constructed via the associated basic system.

CHAPTER I

In the first section of this chapter we deal metric spaces, complete metric spaces and completion of metric spaces. In the second we collect the preliminary definitions and results on ordered spaces.

SECTION:1

To begin with we shall state the preliminary definitions and constructions leading to completeness.

METRIC SPACES

A metric space is a non-empty set equipped with a concept of distance which is suitable for the treatment of convergent sequences in the set and continuous functions defined on the set.

DEFINITION : 1:1:1

Let X be a non-empty set. A metric on X is a real valued function d of ordered pairs of elements of X which satisfies the following three conditions.

(i) $d(x,y)\geq 0$, and $d(x,y)=0$ iff $x=y$

(ii) $d(x,y)=d(y,x)$ (symmetry)

(iii) $d(x,y)\leq d(x,z)+d(z,y)$ (the triangle inequality)

The function d assigns to each pair (x,y) of elements of X a non-negative real number d(x,y), which by symmetry does not depend on the order of the elements; d(x,y) is called the distance between x and y.

EXAMPLE: 1.1.2.

Let X be an arbitraty non-empty set and define d by

$$d(x,y) = \begin{cases} 0 \text{ if } x=y \\ 1 \text{ if } x\neq y \end{cases}$$

we can easily verify the above axioms for a metric space.

EXAMPLE: 1.1.3

Consider the real line R and the real function $|x|$ defined on R.

We now define a metric on R by $d(x,y)=|x-y|$.This is called the usual metric on R, and the real line as a metric space, is always understood to have this as its metric.

CAUCHY SEQUENCES

DEFINITION: 1.1.4

Let X be a metric space. A sequence $(a_1,a_2,\ldots..)$ in X is a cauchy sequence iff for every $\epsilon>0$ there exists $n_0 \in N$ such that $d(a_n, a_m)<\epsilon$ whenever $n,m\geq n_0$

EXAMPLE: 1.1.5

Let (a_n) be a convergent sequence; say $a_n \longrightarrow P$. Then (a_n) is necessarily a cauchy sequence.

REMARK: 1.1.6

Every convergent sequence in a metric space is a cauchy sequence.

The converse of the above remark is not true, as seen in the next example.

EXAMPLE: 1.1.7

Let $X=(0,1)$ with the usual metric. Then $\{1/2, 1/3 \ldots\}$ is a sequence in X which is cauchy but which does not converge in X.

COMPLETE METRIC SPACES

DEFINITION: 1.1.8

A metric space (X,d) is complete if every cauchy sequence (a_n) in X converges to a point $p \in X$.

EXAMPLE: 1.1.9

The real line R with the usual metric is complete.

EXAMPLE OF NON-COMPLETE METRIC SPACE

EXAMPLE: 1,1,10

An example of a non-complete metric space is the space Q of rational numbers in the usual metric $d(x,y)=|x-y|$ properties of complete metric spaces.

(i) A closed subset A of a complete metric space is complete with respect to the restricted metric.

(ii) If X is complete under the metric d then it is complete under the standard bounded metric $\bar{d}$ defined by

$$\bar{d}(x,y)=\frac{1}{1+d(x,y)}$$

(iii) X is a complete metric space iff every cauchy sequence has a convergent subsequence.

(iv) Cantor's Intersection Theorem:

Let X be a complete metric space and let $\{F_n\}$ be a decreasing sequence of non-empty closed subset of X such that $d(F_n) \longrightarrow 0$ Then $F=\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

(v) Baire's Category Theorem:

Every complete metric space X is of second category.

The completion of metric spaces

For a metric space X it is possible to find a complete metric space X^* such that X is isometric to a dense subspace of X^* (not just uniformely isomorphic).

THEOREM: 1.1.11

Each metric (or pseudo-metric) space can be mapped by a one-to-one isometry onto a dense subset of a complete metric (respectively pseudo-metric) space.

PROOF:

It is only necessary to prove the theorem for a pseudometric space (X,d)

Let X^* be the class of all cauchy sequences in X, and for members S and T of X^*,

Let us define a metric d* such that d* (S,T) be the limit of $d(S_m,T_m)$ as m becomes large (formally, the limit of $[d(S_m,T_m), m \in w]$)

It is easy to verify that d* is a pseudometric for X^*

Let F be the map which carries each point x of X into the sequence which is constantly equal to x; that is, $F(x_n)_n = x$ for all n

Evidently F is a one-to-one isometry and it remains to prove that $F[\mathcal{X}]$ is dense in $\mathcal{X}^*$ and $\mathcal{X}^*$ is complete.

The first of these statements is almost self evident; if $S \in \mathcal{X}^*$ and n is large, then $F(S_n)$ is near S.

To show $\mathcal{X}^*$ complete, first observe that it is sufficient to show that each cauchy sequence in $F[\mathcal{X}]$ converges to a point of $\mathcal{X}^*$ because $F[\mathcal{X}]$ is dense in $\mathcal{X}^*$.

Finally each cauchy sequence in $F[\mathcal{X}]$ is of the form $F \circ S = \{F(S_n), n \in w\}$, where S is a cauchy sequence in $\mathcal{X}$. and $F \circ S$ converges in $\mathcal{X}^*$ to the member S of $\mathcal{X}^*$.

SECTION: 2

ORDERED SPACE

DEFINITION: 1.2.1

Let P be a non-empty set. An order relationin P is a relation which is symbolized by $\leq$ and assumed to have the following properties.

(i) $x \leq x$ for every x(reflexivity)

(ii) $x \leq y$ and $y \leq x \implies x=y$ (antisymmetry)

(iii) $x \leq y$ and $y \leq z \implies x \leq z$ (transitivity)

(iv) Any two elements are comparable

ie, For every x and y is in P for which $x \neq y$, either $x<y$ or $y<x$

Then P is called ordered space.

EXAMPLE: 1.2.2

Consider the relation on the real line consisting of all pairs (x,y) of real numbers such that $x<y$. It is an order relation called the "Usual order relation", On the real line. A less familiar order relation on the real line is the following:

Define $x<y$ if $x^2<y^2$, or if $x^2<y^2$ and $x<y$.,It is easily checked that this is an order relation.

BOUNDED AND UNBOUNDED SETS: SUPREMUM, INFIMUM

DEFINITION: 1.2.3

A set S of numbers is said to be bounded above if there exist a number K such that every memeber of S is less than or equal to K. ie, $x \leq k \ \forall \ x \in S$.

The number K is called an upper bound of S. If no such member K exists, the set is said to be unbounded above or not bounded above.

DEFINITION: 1.2.4

The set S is said to be bounded below if there exists a number K such that every member of S is greater than or equal to K. ie, $K \leq x \ \ \forall \ x \in S$

The number K is called a lower bound of S. If no such number K exists the set is said to be unbounded below or not bounded below.

DEFINITION: 1.2.5

A set is said to be bounded if it is bounded above as well as bounded below.

DEFINITION: 1.2.6

A member G of a set S is called the greatest number of S if every member S is less than or equal to G.

ie, (i) $G \in S$

(ii) $x \leq G \quad \forall\, x \in S$

Similarly, a member g of the set is its smallest (or the least) member if every memeber of the set is greater than or equal to g.

DEFINITION: 1.2.7

If the set of all upper bounds of a set S has the smallest member, say M, the M is called the least upper bound (l.u.b.) or the supremum of S.

COMPLETE ORDERED SPACE

DEFINITION: 1.2.8

An ordered space P satisfying the following axiom is called complete ordered space.

(i) Every non-empty subset of P which is bounded above has the supremum.

EXAMPLE: 1.2.9

The real line R is a complete ordered space.

EXAMPLE: 1.2.10

The rational field Q is not a complete ordered space.

CHAPTER II

In this chapter we shall discuss the paper on "ON COMPLETENESS" by JOHN.C.MORGAN [8], the author introduces the concept of basic structure. It is proved that every metric structure on a set induces a basic system.

Theorem 2.4 gives a characterization for a metric space to be complete. Theorem 2.7 derives a characterization for an ordered space to be complete. Theorem 2.9 states that "A metric space is complete iff the associated basic system is complete." Theorem 2.10 states that "An ordered space is complete iff the associated basic system is complete." Theorem 2.16 gives a property of a complete system. Theorem 2.35 and 2.37 lead to the construction of a completion of a basic system.

DEFINITION: 2.1 Regions and subregions

Let X be a set and C be a class of subsets of X. Every subset of C, except empty set, is called a Region.

If A is a region, a subset of A is also a region and that region is said to be subregion of A.

DEFINITION: 2.2 Basic system

Let $\psi = [\psi_n : n \in N]$ be a sequence of mappings from C to C

having the property that, for every region A and every $n \in N$, $\psi_n(A)$ is a subregion of A. (In the case that $\emptyset \in C$ we take $\psi_n(\emptyset)=\emptyset$ for every n) Then the triple $\mathcal{X}=(X,C,\psi)$ is called a basic system

EXAMPLE: 2.3

Let (X,d) be a metric space and let C be the family of all opensets in X. (If $A \in C$ then A is open We know
$dia(A)=max \{d(a,b) : a,b \in A\}$

For each region A and each $n \in N$ we define

$\psi_n(A)= A$ if $dia(A) \leq 1/n$

$\psi_n(A)= B$ if $dia(A) > 1/n$

where B is the first subregion of A for which $dia(B) \leq 1/n$

Here either A or B is a subregion of A satisfies the definition of a basic system.

(X,C,ψ) is a basic system

We note that completeness (in the sense of cauchy) of a metric (or psuedo-metric) space X has the following equivalent characrerization [13].

THEOREM: 2.4

X is complete iff every descending sequence of closed spheres in X whose diameters converge to 0 has a non-empty intersection.

PROOF:

NECESSARY CONDITION:

Let us suppose that X is complete and consider $A_1 \supset A_2 \supset$ ..... be a descending sequence of closed spheres of X such that $\lim_{n\to\infty}$ dia (A_n) tends to zero.

To prove $\bigcap A_n \neq \varphi$

Since each Ai is non-empty, we can choose a sequence $(a_1, a_2, \ldots)$ such that $a_1 \in A_1, a_2 \in A_2, \ldots$

We claimthat (a_n) is a cauchy sequence

Let $\epsilon > 0$, since $\lim_{n\to\infty} d(A_n) = 0$ there exists $n_o \in N$ such that $d(A_{n_o}) < \epsilon$

But $A_1 \supset A_2 \supset \ldots$; hence $n, m > n_o \Longrightarrow A_n, A_m \subset A_{n_o}$

$\Longrightarrow$ the corresponding $a_n, a_m \in A_{n_o}$

$\Longrightarrow d(a_n, a_m) < d(A_{n_o}) < \epsilon$

Thus (a_n) is cauchy

Since X is complete (a_n) converges to say $P \in X$

We claim that $P \in \bigcap_n A_n$

Suppose not, ie, suppose $\exists K \in N$ such that $P \notin A_k$

Since A_k is a closed set, $d(P, A_k) = \S > 0$ there exists an open sphere

$S = S(P, \S/2)$

Hence $S \cap A_k = \varphi$

Hence $n>k \implies a_n \in A_k \implies a_n \notin S(P, \S/2)$, this is impossible since $a_n \longrightarrow P$

Hence $P \in \bigcap_n A_n$ and so $\bigcap_n A_n$ is non-empty .

SUFFICIENT CONDITION:

If every decending sequence of closed spheres in X whose diameters converge to 0 has a nonempty intersection then to prove that X is complete.

Let $P_1, P_2, \ldots\ldots P_k$ be a infinite series of points of X satisfying cauchy's condition.

$\therefore$ there exists for every natural number K, a natural number n_k such that

(i) $d(P_{n_k}, P_n) < 2^{-k}$ for $n \geq n_k$

$\therefore$ We have $n_{k+1} \geq n_k$ for $k=1,2,3,\ldots\ldots$

(ii) $S_k = S(P_{n_k}, 2^{-k+1})$ for $k=1,2,3,\ldots$

$\therefore S_{k+1} = S(P_{n_{k+1}}, 2^{-k})$

there exists q in S_{k+1} such that $d(q, P_{n_{k+1}}) \leq 2^{-k}$ which is a radius of S_{k+1}

Using triangle inequality,

$$d(q, P_{n_k}) \leq d(q, P_{n_{k+1}}) + d(P_{n_{k+1}}, P_{n_k})$$
$$< 2^{-k} + 2^{k}$$
$$< 2^{-k+1}$$

so $q \in S_k$ for $q \in S_{k+1}$

The result is that $S_{k+1} \subset S_k$ for k=1,2,... implies that $S_1 \supset S_2 \supset \cdots$ is a family of closed spheres in X whose diameters converge to zero ($\lim_{k\to\infty} 2(2^{-k})=0$)

Hence $\cap S_i$ is nonempty by the given condition there exists $P \in \cap S_i$

According to (ii)

$d(P, P_{n_k}) \leq 2^{-k+1}$

According to (i), for $n \geq n_k$ applying triangle in equality

$$d(P, P_n) \leq d(P, P_{n_k}) + d(P_{n_k}, P_n)$$
$$< 2^{-k+1} + 2^{-k}$$
$$< 2^{-k+2} \text{ where } \lim_{n\to\infty} P_n = P$$

Hence P is a limit of the cauchy sequence $P_1, P_2 \ldots$

Hence the space X is complete.

EXAMPLE: 2.5

Let (Y,<) be an ordered set having no smallest or largest elements and containing a denumerable everyhere dense set $Q=\{r_n : n \in N\}$ and let $\{I_m : m \in N\}$ be an enumeration of all open intervals of Y with end points in Q [ie, $(a,b)= I_m$ where $a,b \in Q$]

Let X be a subset of Y. Let $\mathcal{C}=\{c_m : m \in N\}$ where $C_m = I_m \cap X$ From the definition of generalized interval I_m is generalized interval in Y and since X is a subset of Y. $I_m \cap X$ is a generalized interval in X.

In particular $C_m = I_m \cap X$ are all generalized intervals in X and hence region of X.

Hence, for every $n \in N$

C_m's are all regions of X.

Define $\psi_n(C_m) = C_k$ where k is the smallest index greater than or equal to m such that

C_k is a subregion of C_m

and I_k is a subinterval of I_m

Which contains atmost one of the elements $r_1, r_2 \ldots\ldots r_{n+1}$ and none of these elements is an end point of I_k.

Here (X, C, ψ) is a basic system.

DEFINITION: 2.6 Generalized interval

A subset C of an ordered set Z is called a generalized interval in Z if it satisfies the condition.

For all elements $x, y \in C$, $z \in Z$, if $x \leq z < y$ then $z \in C$

Here in the above example 2.5 satisfies the conditions for a generalized interval in an ordered set Y and X is a subset of Y then $I \cap x$ is a generalized interval in X.

In particular the sets C_m above are generalized intervals in X.

We note that completeness (in the sense of Dedekind) of the ordered set X above has the equivalent characterzation.

THEOREM: 2.7

X is complete iff every descending sequence of bounded, closed intervals in X has a nonempty intersection.

PROOF:

NECESSARY CONDITION:

Let us suppose that X is complete. Hence every set bounded above has a supremum.

Consider $A_i=[a_i,b_i]$ $\forall$ $i \in N$ such that $A_1 \supset A_2 \supset \ldots$
Therefore $a_1>a_2>\ldots\ldots>b_2>b_1$

The set of points $a_1, a_2 \ldots$ is bounded above by b_1 and hence has a supremum α such that

$\alpha \in [a_1, b_1]$

ie, $a_1 \leq \alpha \leq b_1$

But $a_n \leq b_k$ for all $n \not< k$

$\alpha \geq$ every a_n [since α is supremum of a_n's]

$\therefore \alpha \in [a_i, b_i]$, $\forall_i$

Hence every descending sequence of bounded closed intervals have non-empty intersection.

SUFFICIENT CONDITION:

Let us suppose that every descending sequence of bounded closed intervals has a non empty intersection.

To prove X is complete. ie, To prove every set bounded above has a supremum. Y contains a denumerable everywhere dense set $Q=\{r_n : n \in N\}$

Consider a set $A=\{a_1, a_2 \ldots\ldots\}$ which bounded above say the bounded being b. Between a_1 & a_2 there exist a point $r_{n_1} \in Q$ If b is not a supremum there exists a point S_{n_1} such that $a_i < S_{n_1}, \forall_i$. Consider a_2, a_3 there exists a point r_{n_2} such that $a_2 \leq r_{n_2} \leq a_3$

If S_{n_1} is not a supremum there exists a point S_{n_2} such that $a_i < S_{n_2}, \forall_i$. Continue this process and collect the descending sequence of bounded,closed intervals $[r_{n_i}, S_{n_i}]_{i \in N}$

By the condition there exists a point $\alpha \in \bigcap [r_{n_i}, S_{n_i}]$

Claim: α is the supA $\forall \epsilon > 0$

Between $\alpha - \epsilon$ and α $\exists$ $r_{ni} \in Q$ such that

$$\alpha - \epsilon < r_{n_i} < a_{i+1} < r_{n_{i+1}} < \alpha$$

Hence $\alpha - \epsilon < a_{i+1} < \alpha$

Hence the calim. $\therefore$ Every set bounded above has a supremum. $\therefore$ X is complete.

The analogous characterzations theorem 2.4 and theorem 2.7 suggest that a general notion of completeness, based on descending sequences of sets and non empty intersections, will effect a unification of the aforementioned metric and order completion results. It is of some interest to note that our approach to the unification of metrical and order theoretic analogies and our approach to the unificaion of topological and measure theorectic analogies have a game of Mazur as their common origin. { [9], [10], [12] }

DEFINITION: 2.8

A basic system is called a complete system if it satisfies the condition:

Every sequences $(A_n)_{n \in N}$ of regions, for which the sequence $(\Psi_n(A_n))_{n \in N}$ is descending has a non empty intersection.

THEOREM: 2.9

A metric space is complete iff the associated basic system is complete.

PROOF:

Let (X,d) be a metric space. let C be a family of all closed spheres in X and for each region $A \in C$ and each $n \in N$

Let Ψ_n (A) be the first sub region B of A with dia(B) $\leq$ 1/n. Then $(\Psi_n(A_n))_{n \in N}$ is a descending sequence of closed spheres in X whose diameters converge to 0.

By the theorem 2.4 X is complete iff every descending sequence of closed spheres in X whose diameters converge to 0 has a nonempty intersection if $(A_n)_{n \in N}$ has a nonempty intersection then X is complete. Hence (X,C,Ψ) is a complete system.

THEOREM: 2.10

An ordered space is complete iff the associated basic system is complete.

PROOF:

Let (Y,<) be an ordered set with no smallest or largest elements and containing a denumerable everywhere dense set. Consider the basic sytem associated with Y constructed as in example 2.5 and let Ψ_n be the identity mapping for each $n \in N$

Theorem 2.7 states that "every descending sequence of bounded, closed intervals in X has a nonempty intersection iff X is complete,"

Here Ψ_n being the identity mapping if $(\Psi_n(A_n))_{n \in N}$ is descending $(A_n)_{n \in N}$ is the same as $(\Psi_n(A_n))_{n \in N}$ is also descending and hence if $(A_n)_{n \in N}$ has non empty intersection then X is complete by the theorem 2.7. This is equialent to the definition 2.8.

EXAMPLE : 2.11

Let $(X,\mathcal{T})$ be a Topology, Let C be the family of all closed sets, and let ψ_n be the identity mapping for each $n \in N$. Then the definition 2.8 is equivalent to the Topology being countably compact.

DEFINITION: 2.12 Singular set

A set is a singular set if every region has a subregion disjoint from the set.

DEFINITION: 2.13 Meager set

A set is a Meager set if it representable as a countable union of singular Sets.

DEFINITION: 2.14 Abundant set

A set which is not a meager is called abundant set.

In the case that (X,C) is a Topology, the singular, Meager, and Abundant sets concide with the no where dense, first category and second category sets.

DEFINITION: 2.15 Baire's family

Generalizing the notion of a Baire topology we define (X,C) is a Baire family if every region is an abundant set.

The importance of complete systems stems the following set theorectical formulation of the Baire category theorem ([12] pp. 71)

THEOREM: 2.16

If X is a complete system then (X,C) is a Baire family

PROOF:

We know in Topology X is a Baire space iff every open set of X is of second category.

The abundant sets in the basic system coincide with the second category sets in a topological system.

Hence the Baire family in the basic system coincides with the Baire space in the topological system.

By Baire category theorem X is a complete metric space then it is a Baire space equivalently if X is complete system then (X,C) is a Baire family in the basic system.

Note:

This theorem encompasses numerous topological versions of the classical Baire category theorem which states that

" If X is a compact Housdorff space or a complete metric space then X is a Baire space."

EXAMPLE: 2.17

(X,d) is a complete metric space and C is the family of all opensets in X. For each region A and each $n \in N$. We take $\Psi_n(A)$ to be the first region B in C whose closure is contained in A and waith diam (B) $\leq 1/n$

EXAMPLE: 2.18

(X,C) is a countably compact, regular topology ([2]). For each region A and each $n \in N$, we use the assumed regularity to define $\Psi_n(A)$ to be the first region in C whose closure is contained in A. The satisfaction of the definition 2.8 results from applying the assumed countable compactness to the sequence of closures of the sets $\Psi_n(A_n)$

EXAMPLE: 2.19

(X,C) is a locally compact, regular topology ([2]). For each region A and each $n \in N$, we define $\Psi_n(A)$ to be the first region in C whose closure is s compact subset of A.

EXAMPLE: 2.20

(X,C) is a smirnov's deleted sequence topology; ie, X=R and C consists of all sets A representable in the form A=G-E, where G is an open set in the usual topology for R and $E \subset \{1/m: m \in N\}$. For each such region A and each $n \in N$. We define Ψ_n (A) to be the first open interval B whose closure is contained in A with length $l(B) \leq 1/n$. We note that this topology is not regular, not countably compact not locally compact, etc ([14])

EXAMPLE: 2.21

(X,C) is the countable completement topology for an uncountable set X and all the mappings Ψ_n are the identity mapping ([8])

we now show how a given basic system $X = (X,C,\Psi)$ generates a second basic system $\mathfrak{X}^* = (X^*,C^*,\Psi^*)$ by means of a purely set-theoretical construction.

DEFINITION: 2.22 **Regualr sequence**

A sequence $(E_m)_{m \in N}$ of regions is called a regular sequence if it satisfies the condition.

$E_{m+1} \subset \Psi_m(E_m)$

for every $m \in N$

LEMMA: 2.23

Every regular sequence is necessarily a descending sequence.

PROOF:

Let $(E_m)_{m \in N}$ be a regular sequence then $E_{m+1} \subset \Psi_m(E_m)$ for every $m \in N$. But we know $\Psi_m(E_m) \subset E_m$

$\therefore$ $E_{m+1} \subset E_m$

$\Longrightarrow$ It is a descending sequence.

$\therefore$ Every reqular sequence is necessarily a descending sequence.

LEMMA: 2.24

Every region A contains a regular sequence

EXAMPLE: 2.25

Take $E_1 = A$

$E_2 = \Psi_2\Psi_1(A)$

$E_3 = \Psi_3\ \Psi_2\ \Psi_2\ \Psi_1\ (A)$, and

in general $E_m = \Psi_m\Psi_{m-1}\ \Psi_{m-1}\ldots\Psi_2\Psi_2\Psi_1(A)$ for $m>2$

To prove $E_2 \subset \Psi_1(E_1)$

$$E_2=\Psi_2\Psi_1(A)=\Psi_2(\Psi_1(A)) \subset \Psi_1(A)\ [\because \Psi_n(A) \subset A]$$

$$= \Psi_1(E_1) [\because A=E_1]$$

$\Longrightarrow E_2 \subset \Psi_1(E_1)$

Next to prove $E_3 \subset \Psi_2(E_2)$

$$E_3= \Psi_3\Psi_2\Psi_2\Psi_1(A) = \Psi_3(\Psi_2\Psi_2\Psi_1(A))$$

$$\subset \Psi_2\Psi_2\Psi_1(A)$$

$$= \Psi_2(\Psi_2\Psi_1(A))$$

$$\subset \Psi_2(E_2)\ [\because E_2=\Psi_2\Psi_1(A)]$$

$\Longrightarrow E_3 \subset \Psi_2(E_2)$

Proceeding like this, finally we get

$E_{m+1} \subset \Psi_m(E_m)$

$\therefore$ Every region contains a regular sequence.

DEFINITION: 2.26

Sequences of sets $(E_m)_{m \in N}$ and $(F_m)_{m \in N}$ are said to be interlaced

if $(\forall n)\ (\exists m)\ (E_m \subset F_n)$ and

$(\forall m)\ (\exists k)\ (F_k \subset E_m)$

The equivalence class containing a given regular sequence $(E_m)_{m \in N}$ will be denoted by $[(E_m)]$

[$[(E_m)]$ is an equivalence class which contain all (F_m) which are interlaced with (E_m)]

The set of all such equivalence class is denoted by X^*

ie, $X^* = \{[(E_m)] / (E_m)$ is a regular sequence $\}$, for each $A \in C$ we define

$A^* = \{[(E_m)] \in X^* / \exists m \in N \ni E_m \subset A\}$

ie, A^* consists of all equivalence classes $x^* \in X^*$ having the property that for any representative element $(E_m)_{m \in N}$ in x^*, $\exists$ an index m such that $E_m \subset A$ and $C^* = \{A^*/A \in C\}$

For each $A \in C$ and each $n \in N$ we define

$\psi_n^*: C^* \longrightarrow C^*$ such that

$\psi_n^*(A^*) = (\psi_n(A))^*$ where

$\psi^* = \{\psi_n^* / n \in N\}$

We then have a well-defined system

$\mathcal{X}^* = (X^*, C^*, \psi^*)$

To prove It is a basic system

ie, to prove $\psi_n^*(A^*) \subseteq A^*$

From definition $\psi_n^*(A^*) = (\psi_n(A))^*$ But we know $\psi_n(A) \subseteq A$

Claim: $B \subseteq A$ then $B^* \subset A^*$

Let $[(E_m)] \in B^*$ then $\exists m \ni E_m \subseteq B \subseteq A$

$\Longrightarrow [(E_m)] \in A^*$ [$\because A^* = \{[(E_m)] \in x^* / \exists m \in N \ni E_m \subset A\}$

$\Longrightarrow B^* \subset A^*$

Hence the claim

$(\Psi_m(A))^* \subseteq A^*$

Hence (X*, C*, A*) is a basic system

Note:

The converse is true only for point regular set. It is proved in lemma (2.31) later.

DEFINITION: 2.27 limit

A regular sequence $(E_m)_{m \in N}$ of regions in $\mathcal{C}$ is said to converge to a point $x \in X$ if its intersection is $\{x\}$, in which case x is called the limit of the given sequence ie, $\bigcap_m E_m = \{x\}$.

DEFINITION: 2.28 **Point regular system**

A basic system $\mathfrak{X}$ is called a point regular system if the following conditions are satisfied.

(i) Every point is the limit of a regular sequence of regions.

(ii) If $(E_m)_{m \in N}$ is any regular sequence of regions converging to a point x and A is any region containing x then there exists an index m $\ni E_m \subset A$

LEMMA: 2.29

Any two regular sequences converging to the same point are interlaced.

PROOF:

Let (E_m) and (F_m) converge to x and say $x \in A$ then $\exists\, E_m \subset A$ and $F_n \subset A$

By definition $x = \cap E_m = \cap F_m$

$x \in E_m \implies \exists n \ni F_n \leq E_m$

Similarly $x \in F_n \implies \exists K \ni E_k \leq F_n$

$\therefore$ Two regular sequences converging to the same point are interlaced. Hence the lemma.

LEMMA: 2.30

Example 2.3 is point regular with each point $x \in X$ being the limit of the regular sequence $(E_{m,x})_{m \in N}$ of open spheres

$$E_{m,x} = \{ y \in X : d(x,y) < 1/2m \}$$

PROOF:

Let $x \in X$

Consider $E_{m,x} = \{y \in X / d(x,y) < 1/2m \}$

Claim: 1 $(E_{m,x})$ is a regular sequence. ie, To prove $E_{m+1,x} \subseteq \Psi_m (E_{m,x})$

Claim: 2 $\Psi_m(E_{m,x}) = E_{m,x}$ ie, $dia(E_{m,x}) < 1/m$

$y_1, y_2 \in E_{m,x}$

$d(y_1,y_2) \leq d(y_1,x) + d(x,y_2)$

$< 1/2m + 1/2m = 1/m$

$\implies dia(E_{m,x}) < 1/m \implies \Psi_m (E_{m,x}) = E_{m,x}$

Hence the claim:2

$E_{m+1,x} \subset E_{m,x}$ as

$y_1 \in E_{m+1,x} \implies d(y_1,x) = 1/2(m+1) < 1/2m$

$\implies y_1 \in E_{m,x}$

Hence the calim:1

Claim: 3 x is the limit of $(E_{m,x})_{m \in N}$

$y \in \bigcap_{m \in N} E_{m,x}$

then $d(y,x) < 1/2m \ \forall m$

as $m \longrightarrow \infty$, $1/2m \longrightarrow 0 \Longrightarrow d(y,x)=0$

$\Longrightarrow y=x$

Hence $\bigcap E_{m,x} = \{x\}$

Hence the claim:3

Similarly

Example 2.5 is point regular with each point $x \in X$ being the limit of the regular sequence $(E_{m,x})_{m \in N}$ of regions

$E_{m,x} = (a_{m,x}, b_{m,x}) \cap X$

defined inductively as follows

$a_{1,x}$ is the first element r_{j_1} in the enumeration of Q with index $j_1 > 2$ satisfying $r_{j_1} < x$ and

$b_{1,x}$ is the first element r_{k_1} of Q with index $k_1 > 2$ satisfying $x < r_{k_1}$; for $m>1$, $a_{m,x}$ is the first element r_{j_m} with index $j_m > j_{m-1}$ satisfying

$a_{m-1,x} < r_{jm} < x$ and

$b_{m,x}$ is the first element r_{km} with index $k_m > k_{m-1}$ satisfyijng $x < r_{k_m} < b_{m-1,x}$ we then have $\psi_m(E_{m,x}) = E_{m,x}$ for all $m \in N$ and all $x \in X$ [$\because \psi_m$ is an identity]

$a_{1,x} = r_{j_1} < a_{2,x} = r_{j_2} < x < r_{k_2} = b_{2,x} < r_{k_1} = b_{1,x}$

$(a_{m,x}, b_{m,x},)$ are descending sequence.

ie, $(a_{1,x}, b_{1,x}) \supset (a_{2,x}, b_{2,x}) \supset \cdots$

$\therefore x \in (a_{m,x}, b_{m,x})$

x is the limit of $(E_{m,x})$

LEMMA: 2.31

If $\mathfrak{X}$ is point regular, A* and B* are regions in C* and $A^* \subset B^*$ then $A \subset B$

PROOF:

$A^* = \{ [(E_m)] / \exists\, m \in N \ni E_m \subset A \}$

Given $A^* \subset B^*$

ie, $\forall\ [(E_m)] \in A^*$ is also in B^*

ie, $\exists\ m \in N \ni E_m \subset A$

$\exists\ m' \in N \ni E_{m'} \subset B$

If $x \in A$, To prove $x \in B$

By condition (i) of point regular

$\exists\ (E_m)_{m \in N} \longrightarrow x$

By the definition of limit, $x \in \cap E_m$ If $x \in A$, by condition (ii) of point regular definition and $\exists\, m \ni E_m \subset A$

But we know $[(E_m)] \in A^*$ by definition of A^*

Also we know $A^* \subset B^*$

$[(E_m)] \subset B^*$

$\Longrightarrow \exists\, m' \in N \ni E_{m'} \subset B$

But $x \in \cap E_m \Longrightarrow x \in E_m \subset B$

$\Longrightarrow x \in B$

$\therefore\ A \subset B$

DEFINITION: 2.32

A basic system X is called idempotent if each mapping Ψ_n is idempotent

ie, if $\Psi_n \Psi_n = \Psi_n$ for every $n \in N$

LEMMA: 2.33

Example 2.3 is idempotent

$\Psi_m(A) = A$ (if diam$_\circ$ (A) $< 1/n$)

$= B$ (if diam (A) $\geq 1/m$

where diam (B) $< 1/n$)

If diam (A) $< 1/n$; $\Psi_m \Psi_m(A) = \Psi_m(A)$

$\Longrightarrow \Psi_m^2 = \Psi_m$

If diam (A) $\geq 1/n$; $\Psi_m (\Psi_m(A)) = \Psi_m (B) = B$

[$\because$ dia (B) $< 1/n$]

But $\Psi_m(A) = B$

$\Longrightarrow \Psi_m^2 = \Psi_m$

LEMMA: 2.34

Example 2.5 is idempotent

Ψ_n is taken to be identity

$\Longrightarrow \Psi_n \Psi_n$ = identity. identitiy

= identity

$= \Psi_n$

THEOREM: 2.35

If $\mathcal{X}$ is a point-regular ; idempotent system then $\mathcal{X}^*$ is a complete system.

PROOF:

Suppose $(A_n^*)_{n \in N}$ is a sequence of regions in C^* for which the sequence $(\Psi_n^*(A_n^*))_{n \in N}$ is descending.

To prove $(A_n^*)_{n \in N}$ has a non empty intersection. By the Lemma 2.31, Whenever $A^* \supset B^*$ then $A \supset B$ in a point regular system we have $(\psi_m(A_m))_{m \in N}$ is a descending sequence of regions in C for

$$\psi_n^* (A_n^*) \supset \psi_{n+1}^* (A_{n+1}^*), \forall n$$

$$\Longrightarrow \psi_n(A_n) \supset \psi_{n+1}(A_{n+1})$$

According to the idempotence assumption, this sequence is a regualr sequence for $\psi_{n+1}(A_{n+1}) \subset \psi_n(A_n)$ [By the definition of descending sequence]

$$\text{ie, } \psi_{n+1}(A_{n+1}) \subset \psi_n^2 (A_n) \quad [\ \psi_n^2 = \psi_n\ ]$$

$$\Longrightarrow \psi_{n+1}(A_{n+1}) \subseteq \psi_n(\psi_n(A_n))$$

$$\Longrightarrow \{ \psi_n(A_n)\} \text{ is a regular sequence}$$

$$[\because E_{m+1} \subset \psi_n(E_m) \text{ for a regular sequence}]$$

If follows that $\exists\, m \in N$ such that

$$\therefore [\ (\psi_m(A_m)\ ] \in \psi_n^* (A_n^*) \text{ for every } n \in N$$

But $\psi_n^*(A_n^*) \subseteq A_n^*$ by the definition of ψ_n^*

$$[\ (\ \psi_m(A_m)] \in A_n^*, \forall n$$

This implies the intersection of the sets A_n^* is non empty. Hence $\mathfrak{X}^*$ is a complete system by the definition 2.8

DEFINITION: 2.36

A complete system (Y, D, φ) is called a completion of a basic system (X, C, ψ) if there exists a one to one function $\tau : X \longrightarrow Y$ having the property that for every region $B \in D$ there exists a region $A \in C$ such that $\tau(A) \subset B$

THEOREM: 2.37

If $\mathfrak{X}$ is a point regular idempotent system then $\mathfrak{X}^*$ is a completion of $\mathfrak{X}$.

PROOF:

For each point $x \in X$

We choose a regular sequence $(E_{m,x})_{m \in N}$ converging to x and define $\tau(x) = [(E_{m,x})]$

The function τ is well defined one to one mapping of X into X*.

First to show τ is well defined

$x=y$

$[(E_{m,x})] = [(E_{m,y})]$

Suppose $[(E_{m,x})] \neq [(E_{m,y})]$

$\forall m \; \exists$ no k such that $E_{m,x} \notin E_{m,y}$

ie, $E_{k,x}$ is disjoint from $E_{m,y}$. But $x = \cap E_{k,x} = y = \cap E_{m,y}$

Which is a contrudiction

$\therefore [(E_{k,x})] = [(E_{m,y})]$

$\therefore \tau$ is well defined

Next to show τ is one to one

$x \neq y$

$[(E_{m,x})] \neq [(E_{m,y})]$

$\cap E_{m,x} \neq \cap E_{m,y}$

$\forall m \; \exists$ no k such that $E_{k,x} \notin E_{m,y}$

$\Longrightarrow [(E_{m,x})] \neq [(E_{m,y})]$

Hence τ is one to one

finally we show $\tau(A) \subset A^*$

By condition (ii) in the definition of a point-regular system.

If $(E_{m,x})_{m \in N}$ converging to x and if $x \in A$ then $\exists$ m such that $E_{m,x} \subset A$ But the definition of A* states that

" $A^* = \{[(E_{m,x})] \ / \ m \in \mathcal{N}$ such that $E_{m,x} \subseteq A\}$

Hence we get $[\ (E_{m,x})\ ] \in A^*$

Also we know $\tau(x) = [(E_{m,x})] \in A^*$

$\Longrightarrow$ $\tau(A) \subset A^*$ for every region $A \in C$

CHAPTER III

This chapter deals with the applications of the results studied in the previous chapter. Application 1 gives Hoausdorff's completion of a metric space and Application 2 gives Dedekind's completion of the rational numbers.

In the first section of this chapter we shall start with the metric space (X,d) and construct the basic system associated with this. If X* denotes the completion of this basic system, we can introduce a metric d* in X* and show that (X*,d*) is the completion of (X,d)

In section 2, we start with the ordered set (X,<) and consider the associated basic system. If X* denotes the completion of this basic system then we can introduce an order relation <* in X* and show that (X*,<*) is the completion of (X,<)

APPLICATION:1 **Hausdorff's completion of a metric space**

SECTION: 1

Let (X,d) be a given metric space. From example 2.3 as we have already proved (X,C,ψ) is a basic system and we proved the existence of (X*,C*,ψ*) a second basic system.

For elements $x^* = [\ (E_m)\ ]$, $y^* = [(F_m)]$

where $E_m = \{\ z \mid d(x,z) < 1/m\ \}$ $F_m = \{\ z \mid d(y,z) < 1/m\}$

we define $d^*\ (x^*,y^*) = \lim_{m\to\infty} d(E_m,F_m)$

where $d(E_m,F_m) = \text{g.l.b}\ \{d(z_1,z_2)/z_1 \in E_m, z_2 \in F_m\}$

LEMMA: 3.1.1

(X^*,d^*) is a pseudometric space.

PROOF:

The sequence $(d(E_m,F_m))_{m \in N}$ is a sequence of non-negative real numbers

As (E_m) and (F_m) are regular sequences, $E_{m+1} \subset E_m$ for every m and $F_{m+1} \subset F_m$ for every m

Hence $d(E_{m+1}, F_{m+1}) \geq d(E_m,F_m)$ for every m

ie, $(d(E_m,F_m))_{m \in N}$ is increasing.

The sequence $(d(E_m,F_m))_{m \in N}$ bounded above by $d(x_2,y_2)+2$ where $x_2 \in E_2$ and $y_2 \in F_2$

$d(E_m,F_m) = \text{g.l.b}\ \{d(x_m,y_m)/x_m \in E_m \text{ and } y_m \in F_m\}$

$$
\begin{aligned}
& x_m \in E_m \implies d(x,x_m) < 1/m \\
& y_m \in F_m \implies d(y,y_m) < 1/m \\
& d(x_m,y_m) \leq d(x_m, x_2) + d(x_2,y_2) + d(y_2,y_m) \\
& d(x_m,x_2) \leq d(x_m,x) + d(x,x_2) \\
& \qquad \leq 1/m + 1/2 \\
& d(y_m,y_2) \leq d(y_m,y) + d(y,y_2) \\
& \qquad \leq 1/m + 1/2 \\
& \therefore d(x_m,y_m) \leq 1/m + 1/2 + d(x_2,y_2) + 1/m + 1/2 \\
& \qquad \leq 2/m + 1 + d(x_2,y_2)
\end{aligned}
$$

$m > 1,\ 2/m < 1$

$\therefore d(x_m, y_m) \leq 1+1+d(x_2, y_2)$

$< 2+d(x_2, y_2)$

$\therefore d(x_m, y_m) \leq d(x_2, y_2) + 2$

$\therefore d(E_m, F_m) \leq d(x_2, y_2) + 2$

Hence the sequence $\{d(E_m, F_m)\}$ is bounded above by $d(x_2, y_2) + 2$

We know any set of real numbers which is bounded above has a supremum

Thus the limit $d^*(x^*, y^*) = \lim_{m\to\infty} d(E_m, F_m)$ exists and, because of the interlacing, the limit is independent of the representative sequence $(E_m)_{m\in N}$ and $(F_m)_{m\in N}$ selected we obviously have

$d^*(x^*, x^*) = \lim_{m\to\infty} d(E_m, E_m) = 0$

and $d^*(x^*, y^*) = \lim_{m\to\infty} d(E_m, F_m)$

$= \lim_{m\to\infty} d(F_m, E_m)$

$= d^*(y^*, x^*)$

From the consequence of inequality

$$d(E_m, F_m) \leq d(E_m, G_m) + d(G_m, F_m) + \text{diam}\ (G_m)$$

we have the triangle inequality

$$d^*(x^*, y^*) \leq d(x^*, z^*) + d(z^*, y^*)$$

Hence d^* is a pseudo-metric

DEFINITION: 3.1.2 Isometric

If x and y are metric spaces with metrices d_1 and d_2, a mapping f on X onto Y is called isometric if

$$d_1(x, x') = d_2(f(x), f(x'))$$

LEMMA: 3.1.3

The mapping $\tau : X \longrightarrow X^*$ is an isometry.

PROOF:

Suppose $(E_m)_{m \in N}$ and $(F_m)_{m \in N}$ are regular sequences of regions in $\mathcal{C}$ converging to points $x, y \in X$ respectively

For each index $m \geq 1$ and any points $u \in E_m$, $v \in F_m$ we have

$$d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$$

$$\leq 1/m + d(u,v) + 1/m$$

$$\leq 2/m + d(u,v)$$

$$\Longrightarrow d(x,y) - 2/m \leq d(u,v)$$

$$\Longrightarrow d(x,y) - 2/m \leq d(E_m, F_m) \leq d(x,y)$$

Since $x \in E_m$ and $y \in F_m$

Taking limit as $m \longrightarrow \infty$

$$d(x,y) \leq \lim_{m \to \infty} d(E_m, F_m) \leq d(x,y)$$

$$d(x,y) \leq d^* (x^*, y^*) \leq d(x,y)$$

$$d^* (x^*, y^*) = d(x,y)$$

$$\therefore\ d^* (\tau(x), \tau(y)) = d(x,y)$$

LEMMA: 3.1.4

If S is an open sphere in X* then there is a region $A \in C$ such that $A^* = S$

PROOF:

Let $S = \{y^* \in X^* : d^*(x^*, y^*) < r\}$ be an open sphere in X^* with centre $x^* = [(E_m)]$ in X^*.

Set $A = \cup \{ C \in \mathcal{C} : \tau(C) \subset S \}$

Since A is open set in X, we have only to show A is non-empty to establish that A is a region in $\mathcal{C}$

Choose $m \in N$ such that diam $(E_m) < r/2$

Suppose u is any point of E_m

Let $(G_n)_{n \in N}$ be a regular sequence of regions in $\mathcal{C}$ converging to u then by the condition (ii) of definition of point regular system.

$\exists\ n_\circ \in N \ni n_\circ \geq m$ and $G_n \subset E_m$. since (G_n) is a descending sequence

$\Longrightarrow G_n \subset E_m \quad \forall\ n \geq n_\circ$

since $E_n \subset E_m$ and $G_n \subset E_m$, $\forall\ n \geq n_\circ \geq m$

For $s \in E_n$ and $t \in G_n$ we have $s,t \in E_m \Longrightarrow d(s,t) < r/2$

$\Longrightarrow d(x,u) = d^* (\tau(x) , \tau(u))$

$$d^* ([(E_m)], [(G_n)]) = \sup_{n \geq n_\circ} d(E_m, G_n) < r/2 < r$$

$\Longrightarrow \tau(u)$ is an element of S by the definition of S.

Hence $\forall\ u \in E_m$, $\tau(u) \in S$

$\Longrightarrow \tau(E_m) \subseteq S$

Hence A is not empty

Hence A is a region in C

CLAIM:1 $A^* \subseteq S$

Assume $Z^* = [(F_m)] \in A^*$, then by the definition of

$A^* \ \exists m \in N$ such that $F_n \subset A$ and consequently by the definition of A, $\tau(F_n) \subset S$ for all indices $n \geq m$. For such n, choose a point $w \in F_n$

Let $(G_j)_{j \in N}$ be a regular sequence of regions in $\mathbb{C}$ converging to w then by the condition (ii) of definition of point-regular system $j_n > n$ such that $G_j \subset F_n$ for all $j \geq j_n$

Since $\tau(F_n) \subset S$, $\tau(w) \in S$

consider $\lim_{j \to \infty} d(E_j, G_j) = d^*(x^*, \tau(w)) < r$

[from the definition of S]

For each $n \geq m$ and every $j > j_n \geq n$

$\therefore d(E_m, F_n) \leq d(E_j, G_j)$ [$\because$ d is increasing]

from which we get

$\lim_{n \to \infty} d(E_m, F_n) \leq \lim_{j \to \infty} d(E_j, G_j) < r$

$\therefore \lim_{n \to \infty} d(E_n, F_n) < r$

$\Longrightarrow Z^* \in S$

Hence we have prove $A^* \subseteq S$

CLAIM : 2 $S \subseteq A^*$

$Z^* \in [(F_m)] \subseteq S$

$\Longrightarrow d^*(x^*, z^*) < r$

call $d^*(x^*, z^*) = q$

$\therefore q < r$

Choose $m_0 \in N$ so that diam $(F_{m_0}) < \frac{r-q}{3}$ Let $u \in F_{m_0}$ and let $(G_j)_{j \in N}$ be a regular sequence of regions in $\mathbb{C}$ converging to u. Then by condition (ii) of point-regular system $\exists$ an index $j_0 \geq m_0$ such that

$G_j \subset F_{m_0}$ for all $j \geq j_0$

Since $j \geq j_o \geq m_o$.

$G_j \subseteq F_{m_o}$ and $F_j \subseteq F_{m_o}$

$$d(F_j, G_j) \leq dia(F_j) < \frac{r-q}{3}$$

Hence for these indices j,

$$d(E_j, G_j) \leq d(E_j, F_j) + d(F_j, G_j) + diam(E_m)$$
$$< q + \frac{r-q}{3} + \frac{r-q}{3}$$
$$< \frac{3q+2r-2q}{3} = \frac{2r+q}{3} < r$$

Hence $d^*(x^*, \tau(u)) = \lim_{j\to\infty} d(E_j, G_j) < r$. Thus, we have $\tau(u) \in S$ for every point $u \in F_{m_o}$ This means $\tau(F_{m_o}) \subset S$ then $F_{m_o} \subset A$

$\therefore$ $Z^* \in A^*$

Hence from the claim 1 & 2 we conclude $A^* = S$

LEMMA: 3.1.5

(X^*, d^*) is complete.

PROOF:

Let $(B_p)_{p \in N}$ be a descending sequence of closed spheres in X^*. ie, $B_1 \supset B_2$ ... closed spheres $\subseteq X^*$ such that $\lim_{n\to\infty} B_n = 0$

Define $B_p = \{y^* \in X^* / d^*(x_p^*, y_p^*) \leq y_p\}$

$B_p\ (x_p^*, r_p)$ is the closed spheres

Choose $P_1 < P_2 < P_3 < \ldots . < P_n < \ldots$

is an increasing sequence of numbers.

$B_{p_1} \supset B_{p_2} \supset B_{p_3} \supset \ldots .$ is a descending sequence of closed spheres such that

diam $(B_p) < 1/n$ ie, $d(B_{p_1}) < 1$

$$d(B_{p_2}) < 1/2$$

$$d(B_{p_3}) < 1/3 \ \ldots$$

consider $S_n = \{y^* \in X^* / d^*(x_{p_n}^*, y^*) < r_p\}$ where $S_n(x_{p_n}^*, r_{p_n})$ is an open spheres, by Lemma 3.1.4 $\exists A_n \in \mathcal{C}$, $\forall n \ni A_n^* = S_n$ where A_n is a region in X. Then for any arbitrary $u, v \in A_n$ such that $\tau(u), \tau(v) \in S_n$

Here τ is an isometry

$\therefore d(u,v) = d^*(\tau(u), \tau(v)) < 1/n$

$[\because \tau(u), \tau(v) \in S_n \subset B_{p_n}$

$\therefore d^*(\tau(u), \tau(v)) \leq \text{diam}(B_{p_n}) < 1/n]$

$\therefore$ diam$(A_n) < 1/n$

$\Longrightarrow \Psi_n(A_n) = A_n$

$\therefore \{S_n\}_{n \in N}$ is a descending sequence of regions in $\mathcal{C}^*$ with $\Psi_n^*(S_n) = S_n$ for each $n \in N$

By virtue of completeness of $\mathfrak{X}^*$, (By theorem 2.9) and by definition 2.8 the intersection of S_n is non-empty.

$\therefore \exists$ an element of $X^* \in \cap S_n$

$\Longrightarrow$ that element is also contained in every set B_p by the definition of S_n and B_p.

Since $j \geq j_0 \geq m_0$

$G_j \subseteq F_{m_0}$ and $F_j \subseteq F_{m_0}$

$$d(F_j, G_j) \leq dia(F_j) < \frac{r-q}{3}$$

Hence for these indices j,

$$d(E_j, G_j) \leq d(E_j, F_j) + d(F_j, G_j) + diam(E_m)$$

$$< q + \frac{r-q}{3} + \frac{r-q}{3}$$

$$< \frac{3q+2r-2q}{3} = \frac{2r+q}{3} < r$$

Hence $d^*(x^*, \tau(u)) = \lim_{j\to\infty} d(E_j, G_j) < r$. Thus, we have $\tau(u) \in S$ for every point $u \in F_{m_0}$ This means $\tau(F_{m_0}) \subset S$ then $F_{m_0} \subset A$

$\therefore$ $Z^* \in A^*$

Hence from the claim 1 & 2 we conclude $A^* = S$

LEMMA: 3.1.5

(X^*, d^*) is complete.

PROOF:

Let $(B_p)_{p \in N}$ be a descending sequence of closed spheres in X^*. ie, $B_1 \supset B_2$ ... closed spheres $\subseteq X^*$ such that $\lim_{n\to\infty} B_n = 0$

Define $B_p = \{y^* \in X^* / d^*(x_p^*, y_p^*) \leq y_p\}$

$B_p(x_p^*, r_p)$ is the closed spheres

Hence, for every $n \in N \ \exists m \in N \ni E_m \subset F_n$ Similarly, for every $m \in N \ \exists K \in N \ni F_k \subset E_m$

We conclude $x^* = y^*$

LEMMA: 3.2.2

The mapping $\tau: X \longrightarrow X^*$ is an isomorphism

PROOF:

Suppose $x, y \in X$ and $x < y$

Let $\tau(x) = [(E_m)]$ and $\tau(y) = [(F_n)]$ where $(E_m)_{m \in N}$ and $(F_n)_{n \in N}$ are regular sequences of regions in C converging respectively to x and y

Choose $r, s, t \in Q$ so that $r < x < s < y < t$

Then $I = (r,s) \cap X$ and

$J = (s,t) \cap X$ are regions containing x and y

Accordingly, there exists $m, n \in N$ such that $E_m \subset I$ and $F_n \subset J$

$\exists m, \exists n$ such that $\forall u \in E_m, \forall v \in F_n, u \in E_m \subset I$ and $v \in F_n \subset J \Longrightarrow u < v$ $[\because I < J]$

By the definition $<^*$, $x^* <^* y^*$ ie, $\tau(x) <^* \tau(y)$

LEMMA: 3.2.3

The set $\tau(Q)$ is every where dense in X^*

PROOF:

Suppose $x^* = [(E_m)]$

$y^* = [(F_n)]$ are elements of X^* with $x^* <^* y^*$

Choose m_o, n_o < N such that for all $x \in E_{m_o}$ and all $y \in F_{n_o}$, we have $x < y$

The element $a = \sup E_{m_o}$ belongs to Q Hence, there is an element r_{m1} in the enumeration of Q such that

$r_{m1} = \sup E_{m_o} = a$

Let $m_2 = \max \{m_o, m_1\}$ and

Let $C = \sup \Psi_{m2} (E_{m2})$

$= \sup (I_{m2} \cap x)$

By definition $I_{m2} \cap x$, $\sup (I_{m2} \cap x) \in Q$

$E_{m_o} \supset E_{m2}$ $[\because m_2 \geq m_o]$

$\sup E_{m_o} > \sup E_{m2}$

$a > c$ (ie, $c < a$)

$\therefore$ We have $c \in Q$ & $c < a$

Choose $r \in Q$ satisfying $c < r < a$ & let $(G_k)_{k \in N}$ be a regular sequence of regions in C converging to r.

Then there is an index k_o such that $G_{k_o} \subset (c, a)$

For all $u \in E_{m_2+1}$ all $z \in G_{k_o}$ and all $v \in F_{m_o}$ we have $u<z<v$

This means $x^* <^* \tau(r) <^* y^*$

LEMMA: 3.2.4

If S is a non empty open interval in X* with end points in $\tau(Q)$ then $A=\tau^{-1}(S)$ is a region in C with A*=S

PROOF:

Let $S=(r^*, s^*)$ where $r^* = [(C_j)]$, $S^*=[(D_k)]$ are elements of $\tau(Q)$ with $r^* <^* S^*$

Let $r, s \in Q$ be such that $r^* = \tau(r)$ and $S^*=\tau(s)$

It being clear that A is a region in C, we have only to establish $A^*=S$. Suppose $x^*=[(E_m)]$ is an element of A^* then, there exists $m \in N$ for which $E_m \subset A$. Due to the regularity of the sequence $(E_m)_{m \in N}$ and the particular manner of defining the mapping ψ_n, We can find points $p,q \in Q$ and an index $n > m$ such that $r<p<q<s$ and $E_n \subset (p,q)$

Let $(a,b) \in Q$ satisfy $a<r<p<q<s<b$ since there are regular sequence of regions in the equivalance classes r^* and s^* converging to r and s respectively, there are indices $j,k \in N$ such that $C_j \subset (a,p)$ and $D_k=(q,b)$

From $E_n \subset (p,q)$ we then perceive $r^* <^* x^* <^* s^*$. So $x^* \in S$

Conversely

Suppose $x^* \in s$ then there exist $j,k,m \in N$ such that

$(\forall u \in C_j)\ (\forall w \in E_m)\ (u < w)$ and

$(\forall w \in E_m)\ (\forall v \in D_k)\ (w < v)$

In conjunction with the facts that $r \in C_j$ and $S \in D_k$, implies $E_m \subset A$

We conclude $x^* \in A^*$

LEMMA: 3.2.5

$(X^* <^*)$ is the completion of $(X,<)$

PROOF:

In view of cantor's theorem that any two complete ordered sets having neither smallest nor largest elements and containing everywhere dense, denumerable subsets are isomorphic, we have only to verify $(X^*, <^*)$ is complete

Suppose $(B_m)_{m \in N}$ is a descending sequence of closed intervals $B_m = [a_m^*, b_m^*]$ with $a_m^*, b_m^* \in X^*$

We show $\bigcap_{m=1}^{\infty} B_m \neq \varphi$

without loss of generality, we assume all elements a_m^* are different, all elements b_m^* are different, and none of the elements $r_k^* = \tau(r_k)$ belongs to every set B_m where $(r_k)_{r \in N}$ is a fixed enumeration of the set Q.

Choose $m_1 \in N$ so that $r_1^*, r_2^* \notin B_{m_1}$ continuing inductively, we choose $m_k \in N$ so that $m_k > m_{k-1}$ and $r_{k-1}^* \in B_{m_k}$, for each $k > 1$.

Choose sequences $(P_k^*)_{k \in N}$, $(q_k^*)_{k \in N}$ of elements of $\tau(Q)$ satisfying the relaionships.

$$a_{m_k}^* <^* P_k^* <^* a_{m_{k+1}}^* <^* b_{m_{k+1}}^* <^* q_k^* <^* b_{m_k}^*$$

We note that none of the elements $r_1^*, r_2^*, \ldots r_{k+1}^*$ belongs to the open interval $S_k = (P_k^*, q_k^*)$ nor is any one of these elements an endpoint of S_k then $(S_k)_{k \in N}$ is a descending sequence of regions in C^* with

$$\begin{aligned} \psi_k^* (S_k) &= (\psi_k; (\tau^{-1}(S_k)))^* \\ &= (\tau^{-1} (S_k))^* = S_k \end{aligned}$$

By completeness of X^*, there is an element of X^* which belongs to every set S_k and hence belongs to every set B_m.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. APOSTOL, T.M. — "Mathematical Analysis", ADDISON - WESLEY Publishing company, Inc. Ed.2, (1974).

2. CULLEN, H.F. — "Introduction to General Topology", D.C.Heath & Co.. Boston (1968).

3. DEDEKIND, R. — "Stetigkeit and irrationale Zahlen", F.Viewag & Sohn., Braunschweig(1872).

4. GEMIGNANI, M.C. — "Elementary Topology", Addison welsley publishing company, Ed. 2, (1972).

5. HAUSDORFF, F. — "Grundzuge der Mengenlehre", Veit & Comp., Leipzig (1914).

6. KURATOWSKI,C. — Topologie, I, Ed.2, pp 319-320.

7. KELLEY, J.L. — Convergence in topology, Duke Math. J. 17(1950),277-283.

8. MORGAN, J.C.II — Real Analysis Exchange Vol-16 (1990-91), Department of Mathematics, California state Polytechnic University, Pomona CA 91768, 526-536.

9. MORGAN, J.C. II — "Infinite games and singular sets", Colloq-Math 29 (1974), 7-17.

10. MORGAN, J.C.II — "Survey of point set theory", Lectures presented at the Stefan Banach International Mathematical center, (1989),51p.

11. MORGAN, J.C.II — "Baire category from an abstract viewpoint", Fundamenta Mathematical

12. MORGAN, J.C. II — "Point set Theory", pure and applied mathematics vol.131, Marcel Dekker. Inc. New York and Basel, (1990).

13. SIERPINSKI, W. — "sur une definition des espaces complete",Ganita 1 (1950),13-16.

14. STEEN, L.A. & SEEBACH, J.A.,Jr. — "Counter examples in Topology", springer-verlag. New York, Heidelberg, Berlin, (1978), Ed.2.

15. SIMMONS, G.F. — Introduction to Topology and Modern Analysis, McGraw Hill. Inc. (1963).