$g^{\#}\psi$ - closed sets in topological spaces

Kanimozhi, K

(15PMA003)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore-641 043

In Partial Fulfilment of the Requirements for the Degree of

Master of Science in Mathematics

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10.4.17
Signature of the Head of the Department

N.Balamani
10.04.2017
Signature of the Supervisor

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CONTENT

CONTENT

INTRODUCTION

INTRODUCTION

An equation means nothing to me unless it expresses a thought of God.

-Srinivasa Ramanujan.

Topology is a major area of mathematics concerned with spatial properties that are preserved under continuous deformation of objects, such as deformations that involve stretching, but no tearing or gluing. Topology developed as a field of study out of geometry and set theory, through analysis of such concepts as space, dimension and transformation. It is the study of continuity and connectivity. The topological structures are modeled suitably in the fields of computer graphics, pattern recognition, artificial intelligence, data mining, rough set theory, information systems, quantum physics etc.

Closed sets are fundamental objects in topological spaces. In the study of topological spaces many concepts of topology have been generalized by introducing the concept of semi open sets due to Levine (1963) instead of open sets. Levine (1970) introduced the concept of generalized closed (briefly, g - closed) sets in topological spaces. Using this concept and Levine's idea many researchers have introduced and studied various types of generalized closed sets.

The notion of continuity is one of the most important concepts in mathematics. Many stronger and weaker forms of continuity have been introduced and investigated by several authors. Levine (1963) introduced semi continuous functions using semi-open sets. A weaker form of continuous functions called g – continuous function was introduced by Balachandran et.al (1991).

The deliberations in the research work include the following topics:

1) $g^{\#}\psi$ - closed sets in topological spaces
2) $g^{\#}\psi$ - continuous functions in topological spaces

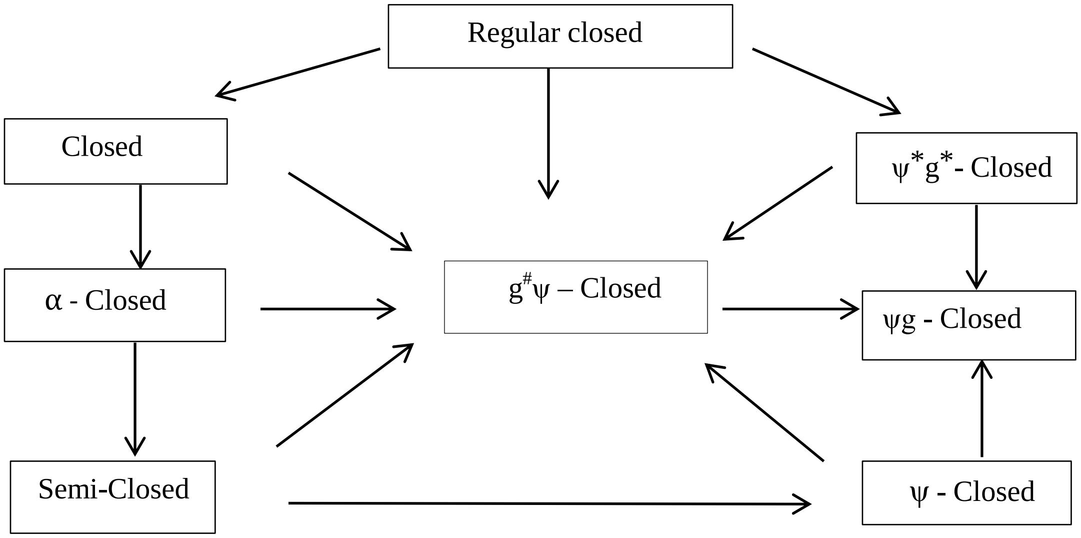
Chapter 1 deals with preliminary definitions that are needed for our study.

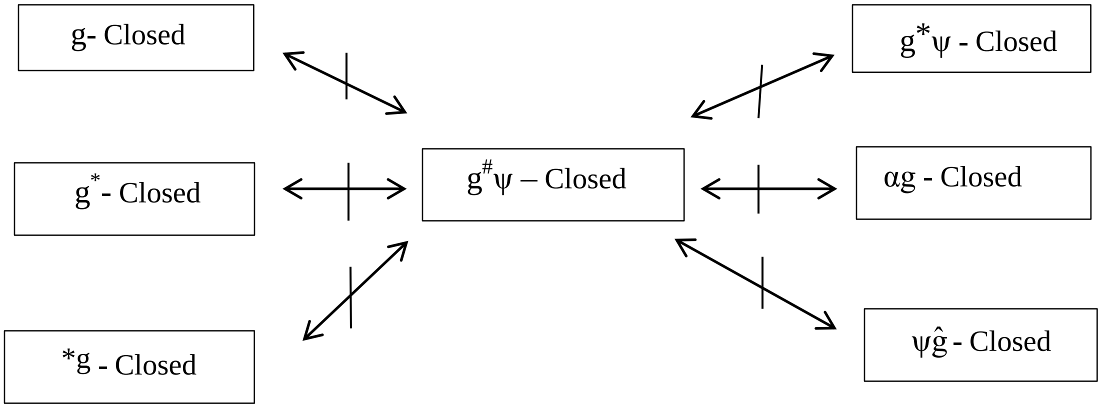
In Chapter 2, a new class of closed sets called $g^{\#}\psi$ - closed sets (briefly, $g^{\#}\psi$ - closed sets) which contains the class of ψ- closed sets and contained in the class ψg - closed sets is introduced in topological spaces. Properties and characterizations of $g^{\#}\psi$ - closed sets are discussed and comparative study between $g^{\#}\psi$ - closed sets and already existing various closed sets is carried out. As an application of $g^{\#}\psi$ -closed sets seven new spaces namely $g^{\#}\psi T_{rc}$ - space , $g^{\#}\psi T_{c}$ - space, $g^{\#}\psi T_{\alpha}$ - space, $g^{\#}\psi T_{sc}$ - space, $g^{\#}\psi T_{\psi}$ - space, $g^{\#}\psi T_{\psi^{*}g^{*}}$ - space and ${}_{\psi g}T_{g^{\#}\psi}$ - space are introduced and some of their properties are studied. Interrelations between these spaces and already existing spaces are analyzed.

A subset A of a topological space (X, τ) is called $g^{\#}\psi$ – closed if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ψ - open in (X, τ).

The class of all $g^{\#}\psi$ - closed sets in (X, τ) is denoted by $g^{\#}\psi\mathbf{C}(X, \tau)$

The following diagrams exhibit the relations between $g^{\#}\psi$ - closed sets with other existing closed sets.





where $A \longrightarrow B$ represents A implies B and $A \nleftrightarrow B$ represents A and B are independent.

Some of the interesting results obtained in this chapter are as follows:

- If A is a $g^{\#}\psi$- closed subset of (X, τ) and $A \subseteq B \subseteq \psi cl(A)$. Then B is also a $g^{\#}\psi$ - closed set in (X, τ).
- Let A be $g^{\#}\psi$ - closed set in (X, τ), then $\psi cl(A) – A$ contains no non - empty closed set.
- A set A is $g^{\#}\psi$ - closed in (X, τ) if and only if $\psi cl(A) – A$ contains no non - empty ψ - closed set.
- Let A be $g^{\#}\psi$- closed set in (X, τ). Then A is ψ - closed if and only if $\psi cl(A) – A$ is closed.
- Let A be any $g^{\#}\psi$ - closed set of (X, τ). Then A is ψ - closed if and only if $\psi cl(A) – A$ is ψ - closed.

A space (X, τ) is said to be a

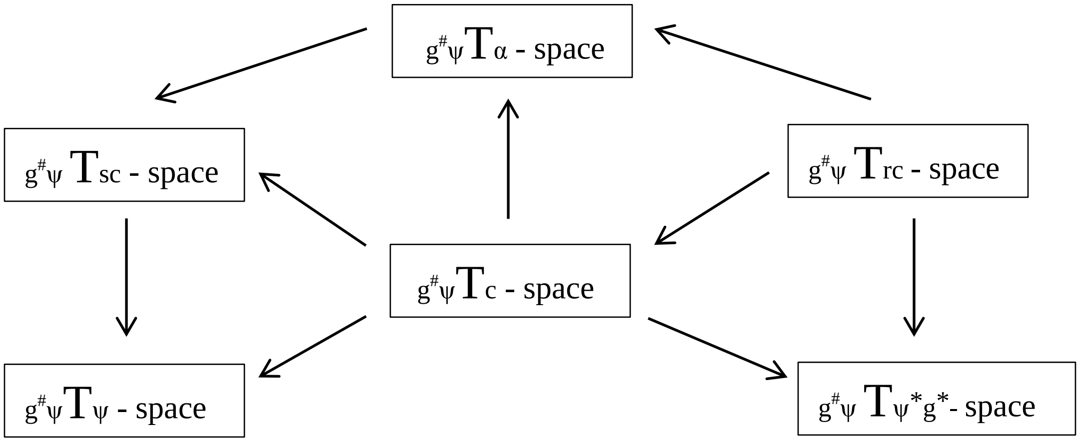
1. $\mathbf{g^{\#}\psi\, T_{rc}}$ - space if every $g^{\#}\psi$ - closed subset of (X, τ) is regular - closed in (X, τ).
2. $\mathbf{g^{\#}\psi\, T_{c}}$ - space if every $g^{\#}\psi$ - closed subset of (X, τ) is closed in (X, τ).
3. $\mathbf{g^{\#}\psi\, T_{\alpha}}$ - space if every $g^{\#}\psi$ - closed subset of (X, τ) is α - closed in (X, τ).
4. $\mathbf{g^{\#}\psi\, T_{sc}}$ - space if every $g^{\#}\psi$ - closed subset of (X, τ) is semi - closed in (X, τ).

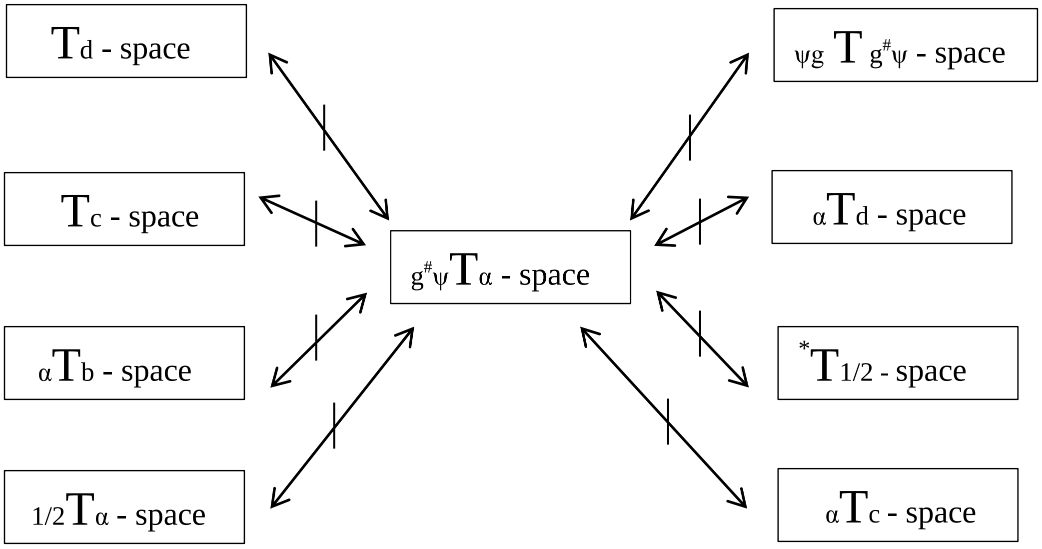
5. **$g^{\#}\psi T_{\psi}$** - space if every $g^{\#}\psi$ - closed subset of (X, τ) is ψ - closed in (X, τ).

6. **$g^{\#}\psi\ T_{\psi^{*}g^{*}}$**- space if every $g^{\#}\psi$ - closed subset of (X, τ) is $\psi^{*}g^{*}$- closed in (X, τ).

7. **$\psi g\ T_{g^{\#}\psi}$** - space if every ψg - closed subset of (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

The following diagrams show the relationship between the newly defined spaces and already existing spaces:



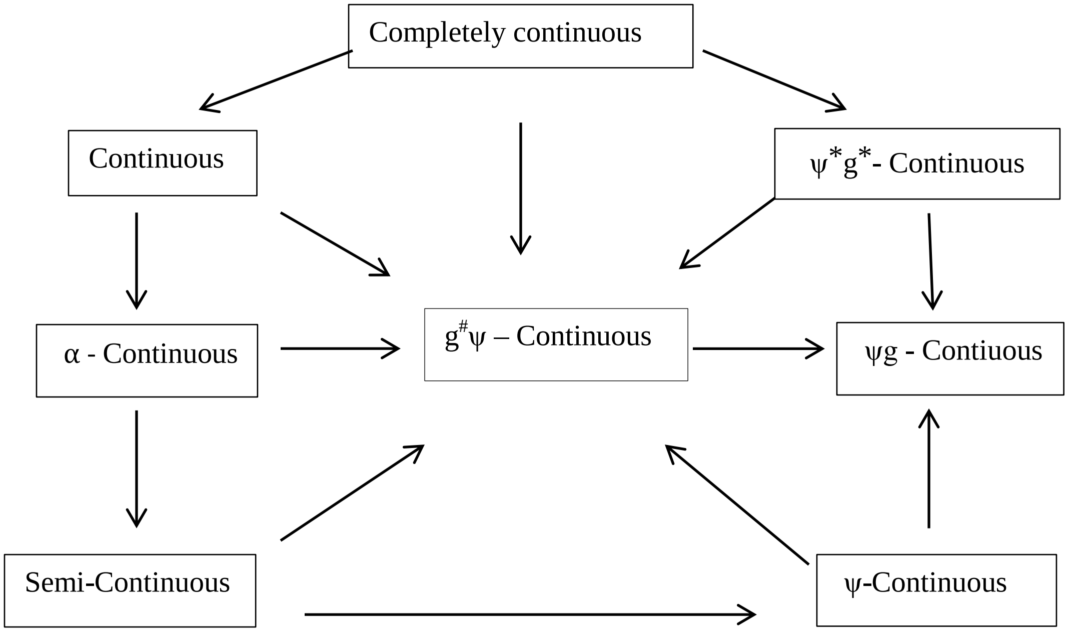


where $A \longrightarrow B$ represents A implies B and $A \not\longleftrightarrow B$ represents A and B are independent.

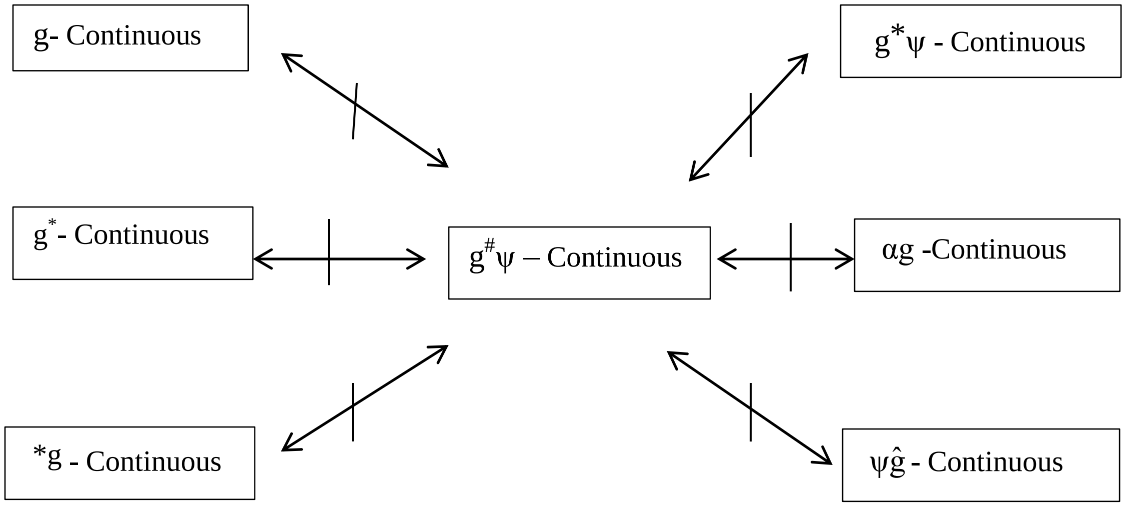
Chapter 3 is devoted to the study of $g^{\#}\psi$ - continuous functions and the relationship between $g^{\#}\psi$ - continuous functions with the existing continuous functions. Properties and characterization are obtained.

A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be **$g^{\#}\psi$ -continuous** if the inverse image of every closed set in (Y,σ) is $g^{\#}\psi$ - closed in (X,τ).

The following diagrams depict the relations of $g^{\#}\psi$ - continuous function with already existing continuous functions:



Where A → B represents A implies B.



where A $\not\longleftrightarrow$ B represents A and B are independent.

- A function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}\psi$ - continuous if and only if the inverse image of every open set in (Y, σ) is $g^{\#}\psi$ - open in (X, τ).
- If $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}\psi$ - continuous then $f\,(g^{\#}\psi cl(V)) \subseteq cl(f(V))$ for every subset V of (X, τ).
- The composition of two $g^{\#}\psi$ - continuous function need not be $g^{\#}\psi$ - continuous function as seen from the following example.
- Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z,\eta)$ be $g^{\#}\psi$ - continuous functions. Then $g\circ f : (X, \tau) \to (Z, \eta)$ is also a $g^{\#}\psi$ - continuous function, if (Y, σ) is a ${}_{g\#\psi}T_c$ -space .
- If $f : (X, \tau) \to (Y, \sigma)$ is $g^{\#}\psi$ - continuous and $g : (Y, \sigma) \to (Z,\eta)$ is continuous then $g\circ f : (X, \tau) \to (Z,\eta)$ is $g^{\#}\psi$ - continuous.
- Let $f : (X, \tau) \to (Y, \sigma)$ be ψg - continuous and $g : (Y, \sigma) \to (Z,\eta)$ be continuous.
- Then $g\circ f : (X, \tau) \to (Z, \eta)$ is a $g^{\#}\psi$ - continuous function, if (X, τ) is a ${}_{\psi g}T_{g\#\psi}$ – space.
- If $f : (X, \tau) \to (Y, \sigma)$ is α - irresolute and $g : (Y, \sigma) \to (Z,\eta)$ is continuous then $g\circ f : (X, \tau) \to (Z, \eta)$ is $g^{\#}\psi$ - continuous.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

Topology has experienced rapid growth during the past fifty years and nowadays its language and concepts pervade much of modern day mathematics. Topological structures on the collection of data's are suitable mathematical models for mathematizing not only quantitative data but also qualitative ones.

Initially the topological spaces were characterized by open sets Stone (1937) introduced regular openness which is stronger than openness. In (1963) Levine introduced the notion of semi openness which is weaker than the notion of openness. Levine (1970) introduced the notion of generalized closed (briefly g-closed) sets in topological spaces. The concept of α-closed sets was introduced by Njastad (1965) and using this concept Maki et.al (1993, 1994) introduced the concepts of generalized α-closed sets and α-generalized closed sets and investigated the properties of these two sets.

Veerakumar (2000) introduced ψ-closed sets in topological spaces. Further Veerakumar (2000, 2002) introduced g^*-closed sets and $\hat{g}$-closed sets in topological spaces and studied their properties. Veerakumar (2005, 2006), $g^*\psi$-closed sets and ${}^*g$-closed sets in topological spaces. Ramya and Parvathi (2011) introduced ψg-closed and $\psi\hat{g}$-closed sets in topological spaces and studied their properties. Balamani and Parvathi (2015) introduced ψ^*g^*-closed sets in topological spaces.

Continuous functions are an important notion in the study of mathematical sciences. Levine (1970) introduced continuous functions. Arya and Gupta (1974) introduced completely continuous, Balachandran et.al (1991) introduced g-continuous functions, Devi et.al, (1997) introduced α **- generalized continuous and generalized α-continuous, Ramya and Parvathi (2012) introduced ψg-continuous and $\psi\hat{g}$-continuous fuctions in topological spaces.

CHAPTER-I

CHAPTER 1

PRELIMINARIES

Definition 1.1[9]

A subset A of a topological space (X,τ) is called **semi - open** if A⊆ cl(int(A)) and **semi - closed** if int(cl(A)) ⊆A.

Definition 1.2[19]

A subset A of a topological space (X,τ) is called **regular open** if A=int(cl(A)) and **regular - closed** if cl(int(A))=A.

Definition 1.3[16]

A subset A of a topological space (X,τ) is called **α - open** if A⊆ int(cl(int(A))) and **α - closed** if cl(int(cl(A)))⊆A.

Definition 1.4[10]

A subset A of a topological space (X,τ) is called **generalized closed** (briefly g - closed) if cl(A) ⊆U whenever A⊆U and U is open in (X, τ).

Definition 1.5[14]

A subset A of a topological space (X,τ) is called **α - generalized closed** (briefly αg - closed) if αcl(A) ⊆U whenever A⊆U and U is open set in (X, τ).

Definition 1.6[2]

A subset A of a topological space (X,τ) is called **generalized semi - closed** (briefly gs - closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in (X, τ).

Definition 1.7[5]

A subset A of a topological space (X,τ) is called **semi - generalized closed** (briefly sg - closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).

Definition 1.8[23]

A subset A of a topological space (X,τ) is called $\hat{g}$ **-closed** if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).

Definition 1.9[21]

A subset A of a topological space (X,τ) is called $\mathbf{g^*}$ **- closed** if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).

Definition 1.10[25]

A subset A of a topological space (X,τ) is called $\mathbf{{}^*g}$ **- closed** if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ -open in (X, τ).

Definition 1.11[22]

A subset A of a topological space (X,τ) is called ψ **- closed** if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ).

Definition 1.12[24]

A subset A of a topological space (X,τ) is called **$g^*\psi$ - closed** if $\psi cl(A)\subseteq U$ whenever $A\subseteq U$ and U is g-open in (X, τ).

Definition 1.13[17]

A subset A of a topological space (X,τ) is called **ψg - closed** if $\psi cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X, τ).

Definition 1.14[17]

A subset A of a topological space (X,τ) is called **$\psi\hat{g}$ - closed** if $\psi cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $\hat{g}$ -open in (X, τ).

Definition 1.15[4]

A subset A of a topological space (X,τ) is called **ψ^*g^* - closed** if $\psi cl(A)\subseteq U$ whenever $A\subseteq U$ and U is ψg - open in (X, τ).

Definition 1.16[6]

A subset A of a topological space (X,τ) is called **T_d - space** if every gs - closed subset of (X, τ) is g - closed in (X, τ).

Definition 1.17[21]

A subset A of a topological space (X,τ) is called **T_c - space** if every gs - closed subset of (X, τ) is g^*- closed in (X, τ).

Definition 1.18[8]

A subset A of a topological space (X,τ) is called **αT_b - space** if every αg - closed subset of (X, τ) is closed in (X, τ).

Definition 1.19[13]

A subset A of a topological space (X,τ) is called **$_{1/2}T_\alpha$** - space if every αg - closed subset of (X, τ) is α - closed in (X, τ).

Definition 1.20[8]

A subset A of a topological space (X,τ) is called **αT_d - space** if every αg - closed subset of (X, τ) is g- closed in (X, τ).

Definition 1.21[21]

A subset A of a topological space (X,τ) is called **αT_c - space** if every αg - closed subset of (X, τ) is g^*- closed in (X, τ).

Definition 1.22[21]

A subset A of a topological space (X,τ) is called **$^*T_{1/2}$ - space** if every g - closed subset of (X, τ) is g^*- closed in (X, τ).

Definition 1.23[16]

A subset A of a topological space (X,τ) is called **α – space** if every α – closed subset of (X, τ) is closed in (X, τ).

Definition 1.24[10]

A function f: (X, τ) →(Y, σ) is said to be **continuous** if $f^{-1}(V)$ is closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.25[9]

A function f: (X, τ) →(Y, σ) is said to be **semi - continuous** if $f^{-1}(V)$ is semi-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.26[1]

A function f: (X, τ) →(Y, σ) is said to be **completely continuous** if $f^{-1}(V)$ is regular - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.27[15]

A function f: (X, τ) →(Y, σ) is said to be **α - continuous** if $f^{-1}(V)$ is α - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.28[3]

A function f: (X, τ) →(Y, σ) is said to be **generalized continuous** (briefly g - continuous) if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.29[7]

A function f: (X, τ) →(Y, σ) is said to be **α - generalized continuous** (briefly αg - closed) if $f^{-1}(V)$ is αg-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.30[20]

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be **semi - generalized continuous** (briefly sg - continuous) if $f^{-1}(V)$ is sg-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.31[21]

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be **g^* - continuous** if $f^{-1}(V)$ is g^*-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.32[25]

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be **${}^*g$ - continuous** if $f^{-1}(V)$ is ${}^*g$-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.33[22]

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be **ψ - continuous** if $f^{-1}(V)$ is ψ-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.34[24]

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be **$g^*\psi$ - continuous** if $f^{-1}(V)$ is $g^*\psi$ -closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.35[18]

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be **ψg - continuous** if $f^{-1}(V)$ is ψg -closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.36[18]

A function f: (X, τ) →(Y, σ) is said to be **ψĝ - continuous** if $f^{-1}(V)$ is ψĝ-closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.37[12]

A function f: (X, τ) →(Y, σ) is said to be **α - irresolute** if $f^{-1}(V)$ is α- closed in (X, τ) for every α - closed set V of (Y, σ).

CHAPTER-II

CHAPTER 2

$g^{\#}\psi$ – CLOSED SETS IN TOPOLOGICAL SPACES

2.1 Introduction

Levine (1970) introduced generalized closed sets in topological space. Levine (1963) defined the concepts semi-open sets in topological space. Veerakumar (2000) defined ψ-closed sets in topological spaces.

In this chapter we introduce and study a new class of sets called $g^{\#}\psi$ -closed sets which contains the class of ψ-closed sets and contained in the class of ψg – closed sets in topological spaces. As an application of $g^{\#}\psi$ -closed sets seven new spaces namely $g^{\#}\psi\,T_{rc}$ -space , $g^{\#}\psi\,T_{c}$-space, $g^{\#}\psi T_{\alpha}$-space, $g^{\#}\psi\,T_{sc}$-space, $g^{\#}\psi T_{\psi}$ –space, $g^{\#}\psi\,T_{\psi^{*}g^{*}}$ space and $\psi g T_{g^{\#}\psi}$ -space are introduced and some of their properties are studied.

2.2 $g^{\#}\psi$ - closed sets

In this section, a new class of generalized closed sets called $g^{\#}\psi$ -closed sets is defined and some relations between $g^{\#}\psi$ – closed sets and other existing closed sets are analyzed.

Definition 2.2.1

A subset A of a topological space (X, τ) is called $g^{\#}\psi$ – closed, if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ψ - open in (X, τ).

The class of all $g^{\#}\psi$ - closed sets in (X, τ) is denoted by $g^{\#}\psi\mathbf{C}(X, \tau)$

Example 2.2.2

Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. Then the subsets X, ϕ, $\{b\}$, $\{c\}$ and $\{b, c\}$ are $g^{\#}\psi$ - closed.

Proposition 2.2.3

Every closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proof:

Let A be a closed set in (X, τ) and U be any ψ - open set containing A in (X, τ). Since A is closed, $cl(A) = A$. For every subset A of X $\psi cl(A) \subseteq cl(A) = A \subseteq U$ and so we have $\psi cl(A) \subseteq U$. Hence A is $g^{\#}\psi$ - closed.

The converse of the above proposition need not be true as seen from the following example.

Example 2.2.4

Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{b\}$ is $g^{\#}\psi$ - closed but not closed.

Proposition 2.2.5

Every regular - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proof:

The proof follows from the result that every regular - closed set is closed and by **Proposition 2.2.3**.

Example 2.2.6

Let X = {a, b, c} with topology τ ={X, ϕ, {a}}. Then the subset {b} is $g^{\#}\psi$ - closed but not regular - closed set.

Proposition 2.2.7

Every α- closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proof:

Let A be an α - closed set and U be any ψ - open set containing A. Since A is α - closed $\alpha cl(A) = A$. For every subset A of X $\psi cl(A) \subseteq \alpha cl(A) = A \subseteq U$ and so we have $\psi cl(A) \subseteq U$. Hence A is $g^{\#}$ ψ - closed.

The converse of the above proposition need not be true as seen from the following example.

Example 2.2.8

Let X = {a, b, c} with topology τ = {X, ϕ, {a}, {b}, {a, b}}. Then the subset {b} is $g^{\#}\psi$ - closed but not α - closed.

Proposition 2.2.9

Every semi - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proof:

Let A be a semi - closed set and U be any ψ - open set containing A. Since A is semi - closed scl(A) = A For every subset A of X $\psi cl(A) \subseteq scl(A) = A \subseteq U$ and so we have $\psi cl(A) \subseteq U$. Hence A is $g^{\#}\psi$ - closed.

The converse of the above proposition need not be true as seen from the following example.

Example 2.2.10

Let X = {a, b, c} with topology τ = {X, ϕ, {a, b}}. Then the subset {a, c} is $g^{\#}\psi$ - closed but not semi - closed.

Proposition 2.2.11

Every ψ - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proof:

Let A be a ψ - closed set and U be any ψ - open set containing A. Since A is ψ - closed, $\psi cl(A) = A \subseteq U$. Hence A is $g^{\#}\psi$ - closed.

The converse of the above proposition need not be true as seen from the following example.

Example 2.2.12

Let X = {a, b, c} with topology τ = {X, ϕ, {a}, {b, c}}. Then the subset {b} is $g^{\#}\psi$- closed but not ψ - closed.

Proposition 2.2.13

Every $\psi^{*}g^{*}$ - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proof:

Let A be a $\psi^{*}g^{*}$- closed set and U be any ψ - open set containing A. Since every ψ - open set is ψg – open and A is $\psi^{*}g^{*}$- closed, $\psi cl(A) \subseteq U$. Hence A is $g^{\#}\psi$ – closed.

The converse of the above proposition need not be true as seen from the following example.

Example 2.2.14

Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the subset $\{a, b\}$ is $g^{\#}\psi$ - closed but not $\psi^{*}g^{*}$- closed.

Proposition 2.2.15

Every $g^{\#}\psi$ - closed set in (X, τ) is ψg - closed in (X, τ).

Proof:

Let A be a $g^{\#}\psi$ - closed set and U be any open set containing A. Since every open set is ψ – open and A is $g^{\#}\psi$ – closed, $\psi cl(A) \subseteq U$. Hence A is ψg – closed.

The converse of the above proposition need not be true as seen from the following example.

Example 2.2.16

Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{a, b\}$ is ψg - closed but not $g^{\#}\psi$- closed.

Remark 2.2.17

The following examples show that $g^{\#}\psi$ - closedness is independent from g^{*} - closedness.

Example 2.2.18

Let X = {a, b, c} with τ = {X, ϕ, {a}}. In this topology the subset {c} is $g^{\#}\psi$ - closed but not g^{*} - closed.

Example 2.2.19

Let X = {a, b, c} with τ = {X, ϕ, {a}, {a, b}}. In this topology the subset {a,c} is g^{*} - closed but not $g^{\#}\psi$ - closed.

Remark 2.2.20

The following examples show that $g^{\#}\psi$ - closedness is independent from αg - closedness.

Example 2.2.21

Let X = {a, b, c} with τ = {X, ϕ, {a},{b}, {a, b}}. In this topology the subset {a} is $g^{\#}\psi$ - closed but not αg - closed.

Example 2.2.22

Let X = {a, b, c} with τ = {X, ϕ, {a},{a, b}}. In this topology the subset {a,c} is αg - closed but not $g^{\#}\psi$ - closed.

Remark 2.2.23

The following examples show that $g^{\#}\psi$ - closedness is independent from $g^{*}\psi$ - closedness.

Example 2.2.24

Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$.In this topology the subset $\{a,c\}$ is $g^*\psi$ - closed, but not $g^\#\psi$ - closed.

Example 2.2.25

Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b,c\}\}$.In this topology the subset $\{b\}$is $g^\#\psi$ - closed but not $g^*\psi$ - closed.

Remark 2.2.26

The following examples show that $g^\#\psi$ - closedness is independent from $\psi\hat{g}$ - closedness and $^*g$ - closedness.

Example 2.2.27

Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$.In this topology the subset $\{a, b\}$ is $\psi\hat{g}$ - closed and $^*g$ - closed but not $g^\#\psi$ - closed.

Example 2.2.28

Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$.In this topology the subset $\{a, b\}$ is $g^\#\psi$ - closed but not $\psi\hat{g}$ – closed and not $^*g$ - closed.

Remark 2.2.29

The following examples show that $g^\#\psi$ - closedness is independent from g - closedness.

Example 2.2.30

Let X = {a, b, c} with τ = {X, ϕ, {a}, {b}, {a, b}}. In this topology the subset {b} is $g^{\#}\psi$ - closed but not g - closed.

Example 2.2.31

In X = {a,b,c} with τ = {X, ϕ, {a}}. In this topology the subset {a, c} is g - closed but not $g^{\#}\psi$ - closed.

Remark 2.2.32

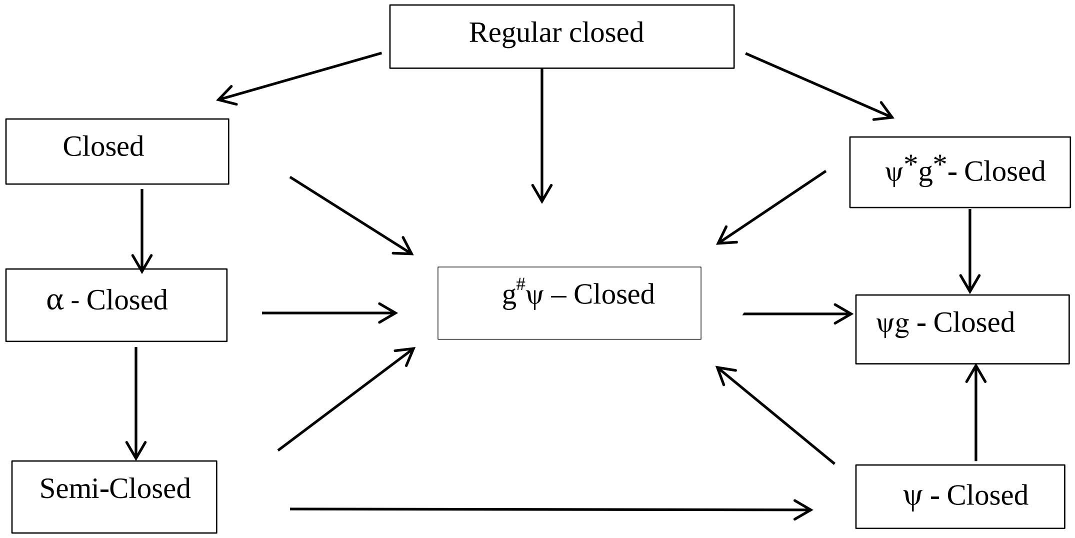
Union of two $g^{\#}\psi$ - closed sets need not be $g^{\#}\psi$ - closed sets as seen from the following example.

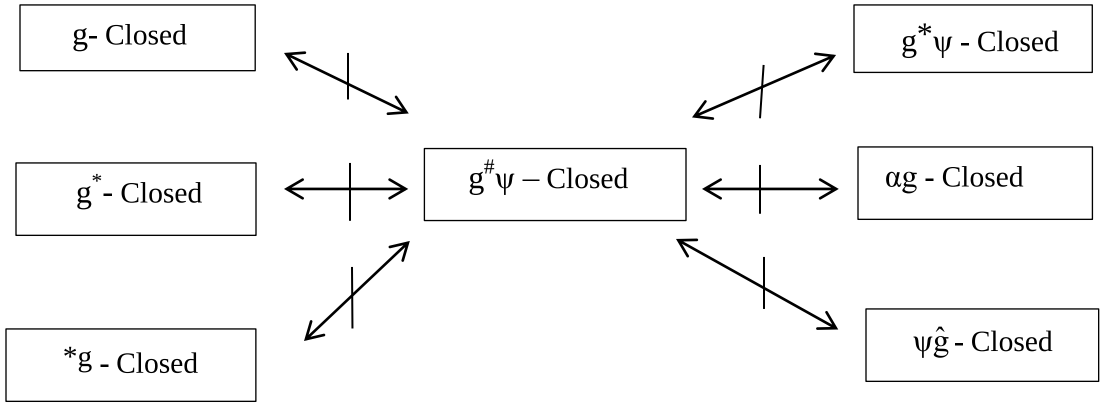
Example 2.2.33

Let X = {a, b, c} with τ = {X, ϕ, {a}, {b}, {a, b}}.Then the subsets X, ϕ, {a},{b},{c}, {a, c} and {b, c} are $g^{\#}\psi$ - closed but {a} U {b}={a, b} is not $g^{\#}\psi$ - closed.

Remark 2.2.34

The following diagrams show the relationship between $g^{\#}\psi$ – closed sets with already existing closed sets.





where $A \longrightarrow B$ represents A implies B and $A \not\longleftrightarrow B$ represents A and B are independent.

Theorem 2.2.35

If A is a $g^{\#}\psi$- closed subset of (X, τ) and $A \subseteq B \subseteq \psi cl(A)$. Then B is also a $g^{\#}\psi$ - closed set in (X, τ).

Proof:

Let U be any ψ- open set in (X, τ) such that $B \subseteq U$. Then $A \subseteq U$, Since A is $g^{\#}\psi$ -

closed, $\psi cl(A) \subseteq U$. Also since $B \subseteq \psi cl(A)$, $\psi cl(B) \subseteq \psi cl(\psi cl(A)) = \psi cl(A)$. Hence $\psi cl(B) \subseteq U$. Therefore B is also a $g^{\#}\psi$- closed set in (X, τ).

Theorem 2.2.36

Let A be $g^{\#}\psi$ - closed set in (X, τ), then $\psi cl(A) – A$ contains no non - empty closed set.

Proof:

Suppose that A is $g^{\#}\psi$ - closed in (X, τ). Let F be a closed subset of $\psi cl(A) – A$, Then F^c is open and hence ψ – open such that $A \subseteq F^c$. Since A is $g^{\#}\psi$- closed, $\psi cl(A) \subseteq F^c$. Thus $F \subseteq (\psi cl(A))^c$. Since every closed set is ψ - closed, F is ψ - closed. Hence $F \subseteq \psi cl(A) –A$. Therefore $F \subseteq \psi cl(A) \cap (\psi cl(A))^c = \phi$. Hence $F = \phi$.

Theorem 2.2.37

A set A is $g^{\#}\psi$ - closed in (X, τ) if and only if $\psi cl(A) – A$ contains no non - empty ψ - closed set.

Proof: (Necessity)

Let A be $g^{\#}\psi$ - closed subset of X. Let F be a ψ - closed set contained in $\psi cl(A) – A$. Since F^c is ψ - open with $A \subseteq F^c$ and A is $g^{\#}\psi$ – closed set in X, $\psi cl(A) \subseteq F^c$, Then $F \subseteq (\psi cl(A))^c$. Also $F \subseteq \psi cl(A) – A$ .Therefore $F \subseteq (\psi cl(A))^c \cap \psi\ cl(A) = \phi$. Hence $F = \phi$.

Sufficiency:

Let $\psi cl(A) – A$ contains no non - empty ψ - closed set. Let $A \subseteq G$ and G be ψ -open. If $\psi cl(A)$ is not a subset of G the $\psi cl(A) \cap G^c$ is a non - empty ψ - closed subset of $\psi cl(A) – A$, which is a contradiction. Therefore $\psi cl(A) \subseteq G$ and hence A is $g^{\#}\psi$ - closed.

Proposition 2.2.38

If a set A is ψ - open and $g^{\#}\psi$ - closed in (X, τ). Then A is a ψ - closed set of (X, τ).

Proof:

Since A is ψ - open, $g^{\#}\psi$ - closed, $\psi cl(A) \subseteq A$. Hence A is ψ- closed.

Theorem 2.2.39

If a set A is $g^{\#}\psi$ - closed and ψ - open and F is ψ- closed in (X, τ), then $A \cap F$ is ψ- closed.

Proof:

Since A is $g^{\#}\psi$ - closed and ψ - open, A is ψ - closed (by **Proposition 2.2.38**). Since F is ψ - closed in X, $A \cap F$ is ψ - closed in (X, τ).

Theorem 2.2.40

For each $x \in X$ either $\{x\}$ is ψ - closed or X - $\{x\}$ is a $g^{\#}\psi$ - closed set in (X, τ).

Proof:

Let $x \in X$ and suppose that $\{x\}$ is not ψ - closed in X. Then X - $\{x\}$ is not ψ - open in X. Hence X is the only ψ - open set containing X-$\{x\}$. That is $(X-\{x\}) \subseteq X$. Therefore $\psi cl(X - \{x\}) \subseteq X$ which implies that X -$\{x\}$ is $g^{\#}\psi$ - closed set in (X, τ).

Theorem 2.2.41

Let A be $g^{\#}\psi$- closed set in (X, τ). Then A is ψ - closed if and only if $\psi cl(A) – A$ is closed.

Proof: (Necessity)

Let A be an any ψ - closed subset of X. Then ψ cl(A) = A and so ψcl(A) – A = ϕ, Which is closed.

Sufficiency:

Let ψcl(A) – A be a closed set. Since A is $g^{\#}\psi$ - closed by **theorem 2.2.36**. ψcl(A) – A contains no non – empty closed set which implies ψcl(A) – A = ϕ. That is ψcl(A) = A. Hence A is ψ - closed.

Theorem 2.2.42

Let A be any $g^{\#}\psi$ - closed set of (X, τ). Then A is ψ - closed if and only if ψcl(A) – A is ψ - closed.

Proof: (Necessity)

Let A be any ψ - closed subset of X. Then ψ cl(A) = A and so ψcl(A) – A =ϕ, Which is ψ - closed in (X, τ).

Sufficiency:

Let ψcl(A) – A be a ψ- closed sets, Since A is $g^{\#}\psi$- closed by **theorem 2.2.37** ψcl(A) – A contains no non - empty ψ - closed set which implies ψcl(A) – A = ϕ. That is ψ cl(A) = A. Hence A is ψ - closed.

Theorem 2.2.43

Let $A \subseteq B \subseteq$ X and suppose that A is ψ - closed set in X then A is $g^{\#}\psi$ - closed set relative to Y.

Proof:

Let A be a $g^{\#}\psi$- closed set in X, Let A $\subseteq$ Y $\cap$ U, where U is ψ- open set in X. Since A is $g^{\#}\psi$- closed, ψ cl(A) $\subseteq$U. That is Y $\cap$ ψ cl(A) $\subseteq$ Y$\cap$U, where Y $\cap$ ψ cl(A) is ψ closure of A in Y. Hence $\psi cl_Y (A) \subseteq Y \cap U$. Thus A is $g^{\#}\psi$ - closed set relative toY.

2.3 $g^{\#}\psi$ – open sets

In this section we defined a new class of generalized open sets called $g^{\#}\psi$ – open sets in topological spaces and studied some of their properties.

Definition 2.3.1

A subset A of a topological space (X, τ) is called $g^{\#}\psi$ – open if its complement A^c is $g^{\#}\psi$ – closed in (X, τ).

The class of all $g^{\#}\psi$ - open sets in (X, τ) is denoted by $g^{\#}\psi O(X, \tau)$

Example 2.3.2

Let X = {a, b, c} with topology τ ={X, ϕ, {a}}. Then the subsets X, ϕ, {a}, {a, b} and {a, c} are $g^{\#}$ ψ - open.

Proposition 2.3.3

Every open set in (X, τ) is $g^{\#}\psi$ – open in (X, τ).

Proof:

The converse of the above proposition need not be true as seen from the following example.

Example 2.3.4

Let X = {a, b, c} with topology τ ={X, ϕ, {a}}. Then the subset {a, c} is $g^{\#}\psi$ - open but not open.

Proposition 2.3.5

1) Every regular - open set in (X, τ) is $g^{\#}\psi$ - open in (X, τ).
2) Every α- open set in (X, τ) is $g^{\#}\psi$ - open in (X, τ).
3) Every semi - open set in (X, τ) is $g^{\#}\psi$ -openin (X, τ).
4) Every ψ - open set in (X, τ) is $g^{\#}\psi$ - open in (X, τ).
5) Every $\psi^{*}g^{*}$ - open set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ).
6) Every $g^{\#}\psi$ - open set in (X, τ) is ψg - open in (X, τ).

The converses of the above statements in the above proposition are not true in general as can be seen from the **examples 2.2.6, 2.2.8, 2.2.10, 2.2.12, 2.2.14, 2.2.16**

Theorem 2.3.6

A subset A of a topological space (X, τ) is said to be $g^{\#}\psi$ -open if and only if $U \subseteq \psi int(A)$ whenever $U \subseteq A$ and U is ψ -closed.

Proof:

Assume that A is $g^{\#}\psi$ - open set in (X, τ). Then A^c is $g^{\#}\psi$ -closed. Let U be a ψ - closed set in (X, τ) contained in A. Then U^c is a ψ - open set in (X, τ) containing A^c. Since A^c is $g^{\#}\psi$ - closed, $\psi cl(A^c) \subseteq U^c$ equivalently $U \subseteq \psi\ int(A)$.

Conversely assume that U is contained in $\psi int(A)$ whenever U is contained in A and U is ψ - closed in (X, τ). Let A^c be contained in U, where U is ψ - open. Then U^c is contained in A. By criteria, $U^c \subseteq \psi int(A)$. This implies $(\psi int(A))^c \subseteq U$ that is $\psi cl(A^c) \subseteq U$. Therefore A^c is $g^{\#}\psi$ - closed. Hence A is $g^{\#}\psi$ - open in (X, τ).

Proposition 2.3.7

If $\psi int(A) \subseteq B \subseteq A$ and A is $g^{\#}\psi$ – open, then B is $g^{\#}\psi$ – open.

Proof:

ψint(A) $\subseteq$ B $\subseteq$ A implies $A^c \subseteq B^c \subseteq (\psi int(A))^c$ i.e.$A^c \subseteq B^c \subseteq (\psi int(A^c))$ and A^c is $g^{\#}\psi$ - closed. By **theorem 2.2.35** B^c is $g^{\#}\psi$ - closed. Hence B is $g^{\#}\psi$ - open.

Theorem 2.3.8

If A is $g^{\#}\psi$ – open in (X, τ) if and only if G = X whenever G is ψ - open and ψint(A) $\cup$ $A^c \subseteq$G.

Proof :(Necessity)

Let A be $g^{\#}\psi$ **-** open and G is ψ - open and ψint(A) $\cup$ $A^c \subseteq$ G. This gives $G^c \subseteq (\psi int(A) \cup A^c)^c = (\psi int(A))^c \cap A = (\psi int(A))^c - A^c = \psi cl(A^c)- A^c$. Since A^c $g^{\#}\psi$ – closed and G^c is ψ - closed by **theorem 2.2.37**, it follows that $G^c = \phi$. Therefore G=X.

(**Sufficiency)**

Suppose that F is ψ - closed and F$\subseteq$A. Then ψint(A) $\cup$ $A^c \subseteq$ ψint(A) $\cup$ F^c. As open implies ψ - open,we get ψint(A) is ψ - open and F^c is ψ - open. Hence ψint(A) $\cup$ F^c is ψ - open .It follows by the hypothesis that ψint(A) $\cup$ F^c = X and hence F $\subseteq$ ψint(A). Therefore by **theorem 2.3.6,** A is $g^{\#}\psi$ - open in X.

Definition 2.3.9

The intersection of all ψ-open subsets of (X,τ) containing A is called ψ- kernal of A and is denoted by ψ-ker(A).

i.e ψ-ker(A)=$\cap$\{U / U is ψ-open in (X,τ) and A $\subseteq$U\}

Theorem 2.3.10

A subset A of (X,τ) is $g^{\#}\psi$ -closed in (X,τ) if and only if $\psi\,cl(A) \subseteq \psi$-ker(A).

Proof:(Necessity)

Suppose that A is $g^{\#}\psi$ -closed set in (X,τ). Let $x \in \psi\,cl(A)$. If $x \notin \psi$-ker(A), then there exists a ψ- open set U in (X,τ) such that $A \subseteq U$ and $x \notin U$. Since U is ψ - open set containing A and A is $g^{\#}\psi$ –closed, we have $\psi cl(A) \subseteq U$, which is a contradiction to $x \in \psi cl(A)$ and $x \notin U$.

Sufficiency:

Suppose that $\psi\,cl(A) \subseteq \psi$-ker(A). If U is any ψ -open set containing A, then $\psi cl(A) \subseteq \psi$-ker(A) so we have $\psi cl(A) \subseteq U$. Hence A is $g^{\#}\psi$ - closed.

2.4 Applications of $g^{\#}\psi$ – closed sets

As an application of $g^{\#}\psi$ -closed sets seven new spaces namely $g^{\#}\psi\,T_{rc}$ -space , $g^{\#}\psi\,T_{c}$-space, $g^{\#}\psi T_{\alpha}$-space, $g^{\#}\psi\,T_{sc}$-space, $g^{\#}\psi T_{\psi}$ –space, $g^{\#}\psi\,T_{\psi^*g^*}$ space and $\psi g T_{g^{\#}\psi}$ -space are introduced and some of their properties are studied.

Definition 2.4.1

A space (X, τ) is said to be a

1) **$g^{\#}\psi\,T_{rc}$** - space if every $g^{\#}\psi$ - closed subset of (X, τ) is regular - closed in (X, τ).
2) **$g^{\#}\psi\,T_{c}$** - space if every $g^{\#}\psi$ - closed subset of (X, τ) is closed in (X, τ).
3) **$g^{\#}\psi T_{\alpha}$** - space if every $g^{\#}\psi$ - closed subset of (X, τ) is α - closed in (X, τ).
4) **$g^{\#}\psi\,T_{sc}$** - space if every $g^{\#}\psi$ - closed subset of (X, τ) is semi - closed in (X, τ).

5) **$g^{\#}\psi T_{\psi}$** - space if every $g^{\#}\psi$ - closed subset of (X, τ) is ψ - closed in (X, τ).

6) **$g^{\#}\psi\ T_{\psi^* g^*}$**- space if every $g^{\#}\psi$ - closed subset of (X, τ) is $\psi^* g^*$- closed in (X, τ).

7) **$\psi g\ T\ g^{\#}\psi$** - space if every ψg - closed subset of (X, τ) is $g^{\#}\psi$ - closed in (X, τ).

Proposition 2.4.2

Every $g^{\#}\psi T_{rc}$ - space is a $g^{\#}\psi T_c$ – space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{rc}$ - space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is regular closed in (X, τ). Since every regular closed set is closed, A is closed in(X, τ). Therefore (X, τ) is a $g^{\#}\psi T_c$ – space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.3

Let $X= \{a, b, c\}$ with $\tau=\{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Then (X, τ) is a $g^{\#}\psi\ T_c$ - space but not $g^{\#}\psi T_{rc}$ – space, since the subset $\{c\}$ is $g^{\#}\psi$ - closed but not regular closed in (X, τ).

Proposition 2.4.4

Every $g^{\#}\psi T_{rc}$ - space is a $g^{\#}\psi T_{\alpha}$ – space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{rc}$ - space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is regular closed in (X, τ). Since every regular closed set is α - closed, A is α - closed in (X, τ) .Therefore (X, τ) is a $g^{\#}\psi T_{\alpha}$ –space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.5

Let X= {a, b, c} with τ={X, ϕ, {a}, {b}, {a,b}, {a,c}}. Then (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space but not $g^{\#}\psi T_{rc}$ – space, since the subset {c} is $g^{\#}\psi$ - closed but not regular closed in (X, τ).

Proposition 2.4.6

Every $g^{\#}\psi T_{c}$ - space is a $g^{\#}\psi T_{\alpha}$ - space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{c}$ – space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is closed in (X, τ). Since every closed set is α - closed, A is α - closed in (X, τ). Therefore (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.7

Let X= {a, b, c} with τ={X, ϕ, {a}}. Then (X, τ) is a $g^{\#}\psi T_{\alpha}$ -space but not $g^{\#}\psi T_{c}$ –space, since the subset {b} is $g^{\#}\psi$ - closed but not closed in (X, τ).

Proposition 2.4.8

Every $g^{\#}\psi T_{c}$ - space is a $g^{\#}\psi T_{\psi}$ - space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{c}$ –space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is closed in (X, τ). Since every closed set is ψ - closed, A is ψ - closed in (X, τ). Therefore (X, τ) is a $g^{\#}\psi T_{\psi}$ - space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.9

Let X= {a, b, c} with τ={X, ϕ, {a}, {b}, {a, b}}. Then (X, τ) is a $g^{\#}\psi T_{\psi}$ - space but not $g^{\#}\psi T_{c}$ – space, since the subset {a} is $g^{\#}\psi$ - closed but not closed in (X, τ).

Proposition 2.4.10

Every $g^{\#}\psi T_{c}$ - space is a $g^{\#}\psi T_{sc}$ - space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{c}$ – space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is closed in (X, τ). Since every closed set is semi - closed, A is semi - closed in (X, τ).Therefore (X, τ) is a $g^{\#}\psi T_{sc}$ - space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.11

Let $X= \{a, b, c\}$ with $\tau=\{X, \phi, \{a\}\}$. Then (X, τ) is a $g^{\#}\psi T_{sc}$ -space but not $g^{\#}\psi T_{c}$ – space, since the subset $\{b\}$ is $g^{\#}\psi$ - closed but not closed in (X, τ).

Proposition 2.4.12

Every $g^{\#}\psi T_{c}$ - space is a $g^{\#}\psi T_{\psi^{*}g^{*}}$- space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{c}$ - space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is closed in (X, τ). Since every closed set is $\psi^{*}g^{*}$ - closed, A is $\psi^{*}g^{*}$- closed in (X, τ). Therefore (X, τ) is a $g^{\#}\psi T_{\psi^{*}g^{*}}$ – space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.13

Let $X= \{a, b, c\}$ with $\tau=\{X, \phi,\{a\}\}$. Then (X, τ) is a $g^{\#}\psi T_{\psi^{*}g^{*}}$- space but not

$g^{\#}\psi$ T_c - space, since the subset {b} is $g^{\#}\psi$ - closed but not closed in (X, τ).

Proposition 2.4.14

Every $g^{\#}\psi T_\alpha$ - space is a $g^{\#}\psi T_{sc}$ - space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_\alpha$ –space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is α-closed in (X, τ). Since every α - closed set is semi-closed, A is semi-closed in (X, τ).Therefore (X, τ) is a $g^{\#}\psi T_{sc}$ - space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.15

Let X= {a, b, c} with τ={X, ϕ, {a}, {b}, {a,b}}. Then (X, τ) is a $g^{\#}\psi T_{sc}$ - space but not $g^{\#}\psi T_\alpha$ – space, since the subset {b} is $g^{\#}\psi$ - closed but not α - closed in (X, τ).

Proposition 2.4.16

Every $g^{\#}\psi T_\alpha$ - space is a $g^{\#}\psi T_\psi$ – space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_\alpha$ – space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is α-closed in (X, τ). Since every α - closed set is ψ - closed, A is ψ - closed in (X, τ).Therefore (X, τ) is a $g^{\#}\psi T_\psi$ - space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.17

Let X= {a, b, c} with τ={X, ϕ, {a, b}}. Then (X, τ) is a $g^{\#}\psi T_{\psi}$ - space but not $g^{\#}\psi T_{\alpha}$ – space, since the subset {a, c} is $g^{\#}\psi$ - closed but not α - closed in (X, τ).

Proposition 2.4.18

Every $g^{\#}\psi T_{sc}$ -space is a $g^{\#}\psi T_{\psi}$ – space.

Proof:

Assume that (X, τ) is $g^{\#}\psi T_{sc}$ - space. Let A be a $g^{\#}\psi$ - closed set in (X, τ). Then A is semi - closed in (X, τ). Since every semi - closed set is ψ - closed, A is ψ - closed in (X, τ). .Therefore (X, τ) is a $g^{\#}\psi T_{\psi}$ – space.

The converse of the above proposition need not be true as seen from the following example.

Example 2.4.19

Let X= {a, b, c} with τ={X, ϕ, {a,b}}. Then (X, τ) is a $g^{\#}\psi T_{\psi}$ - space but not $g^{\#}\psi T_{sc}$ – space, since the subset {a,c} is $g^{\#}\psi$ - closed but not semi - closed in (X, τ).

Remark 2.4.20

The following examples show that $g^{\#}\psi T_{\alpha}$ - space is independent from $\psi g T_{g^{\#}\psi}$ - space.

Example 2.4.21

Let X={a,b,c} with topology τ={X, ϕ, {a},{b},{a,b}} then (X, τ) is a $\psi g T_{g^{\#}\psi}$ - space but not $g^{\#}\psi T_{\alpha}$ – space, since the subset {a} is $g^{\#}\psi$ - closed but not α - closed.

Example 2.4.22

Let X={a,b,c} with topology τ={X, ϕ,{a}} then (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space but not $\psi g T_{g^{\#}\psi}$ - space, since the subset {a,b} is ψg - closed but not $g^{\#}\psi$ - closed.

Remark 2.4.23

The following examples show that $g^{\#}\psi T_{\alpha}$ - space is independent from T_d - space and T_c - space .

Example 2.4.24

Let X={a,b,c} with topology τ={X, ϕ,{a},{a,b}} then (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space but not T_d - space and not T_c – space, since the subset {b} is gs closed but not g - closed and not g^*- closed.

Example 2.4.25

Let X={a,b,c} with topology τ={X, ϕ,{a,b}} then (X, τ) is a T_d - space and a T_c - space but not $g^{\#}\psi T_\alpha$ - space, since the subset {a,c} is $g^{\#}\psi$ - closed but not α - closed.

Remark 2.4.26

The following examples show that $g^{\#}\psi T_\alpha$ - space is independent from $_\alpha T_b$ - space.

Example 2.4.27

Let X={a,b,c} with topology τ={X, ϕ,{a}} then (X, τ) is a $g^{\#}\psi T_\alpha$ space but not $_\alpha T_b$ – space, since the subset {b} is αg - closed but not closed.

Example 2.4.28

Let X={a,b,c} with topology τ={X, ϕ, {a}, {b}, {a,b}} then (X, τ) is a $_\alpha T_b$ - space but not $g^{\#}\psi T_\alpha$ – space, since the subset {a} is $g^{\#}\psi$ closed but not α - closed.

Remark 2.4.29

The following examples show that $g^{\#}\psi T_\alpha$ - space is independent from $_\alpha T_c$ - space.

Example 2.4.30

Let X={a,b,c} with topology τ={X, ϕ, {a},{a,b}} then (X, τ) is a $g^{\#}\psi T_\alpha$ –

space but not ${}_{\alpha}T_{c}$ – space, since the subset {b} is αg - closed but not g^* - closed.

Example 2.4.31

Let X={a,b,c} with topology τ={X, ϕ,{a,b}} then (X, τ) is a ${}_{\alpha}T_{c}$ - space but not $g^{\#}\psi T_{\alpha}$ – space, since the subset {b,c} is $g^{\#}\psi$ - closed but not α - closed.

Remark 2.4.32

The following examples show that $g^{\#}\psi T_{\alpha}$ space is independent from ${}^{*}T_{1/2}$ space.

Example 2.4.33

Let X={a,b,c} with topology τ={X, ϕ,{a}} then (X, τ) is a $g^{\#}\psi T_{\alpha}$ space but not ${}^{*}T_{1/2}$ - space, since the subset {b} is g - closed but not g^* - closed.

Example 2.4.34

Let X={a,b,c} with topology τ={X, ϕ,{a,b}} then (X, τ) is a ${}^{*}T_{1/2}$ - space but not $g^{\#}\psi T_{\alpha}$ - space, since the subset {a,c} is $g^{\#}\psi$ - closed but not α - closed.

Remark 2.4.35

The following examples show that $g^{\#}\psi T_{\alpha}$ - space is independent from ${}_{1/2}T_{\alpha}$ - space.

Example 2.4.36

Let X={a,b,c} with topology τ={X, ϕ,{a}} then (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space but not $_{1/2}T_{\alpha}$ – space, since the subset {a,b} is αg - closed but not α - closed.

Example 2.4.37

Let X={a,b,c} with topology τ={X, ϕ, {a}, {b}, {a,b}} then (X, τ) is a $_{1/2}T_{\alpha}$ - space but not $g^{\#}\psi T_{\alpha}$ - space, since the subset {a} is $g^{\#}\psi$ - closed but not α – closed.

Remark 2.4.38

The following examples show that $g^{\#}\psi T_{\alpha}$ - space is independent from $_{\alpha}T_{d}$ - space.

Example 2.4.39

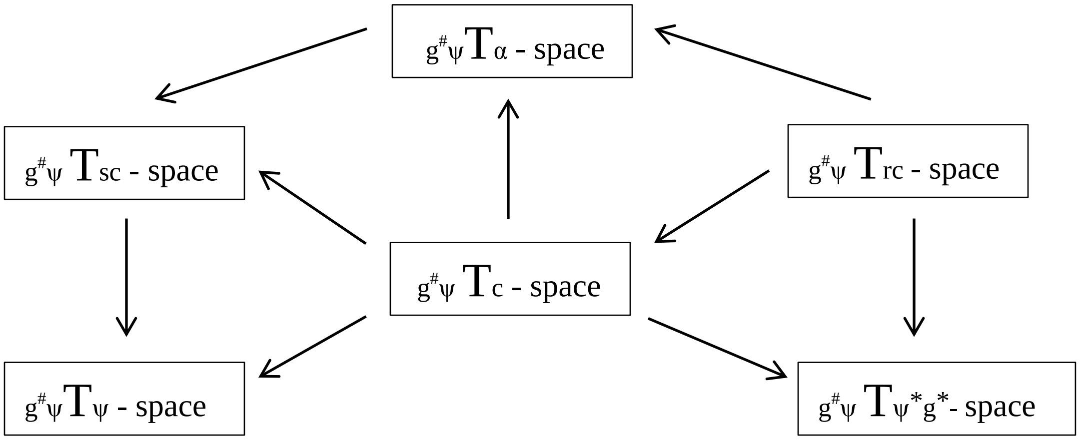
Let X={a,b,c} with topology τ={X, ϕ, {a}, {a,b}} then (X, τ) is $g^{\#}\psi T_{\alpha}$ - space but not $_{\alpha}T_{d}$ – space, since the subset {b} is αg - closed but not g - closed.

Example 2.4.40

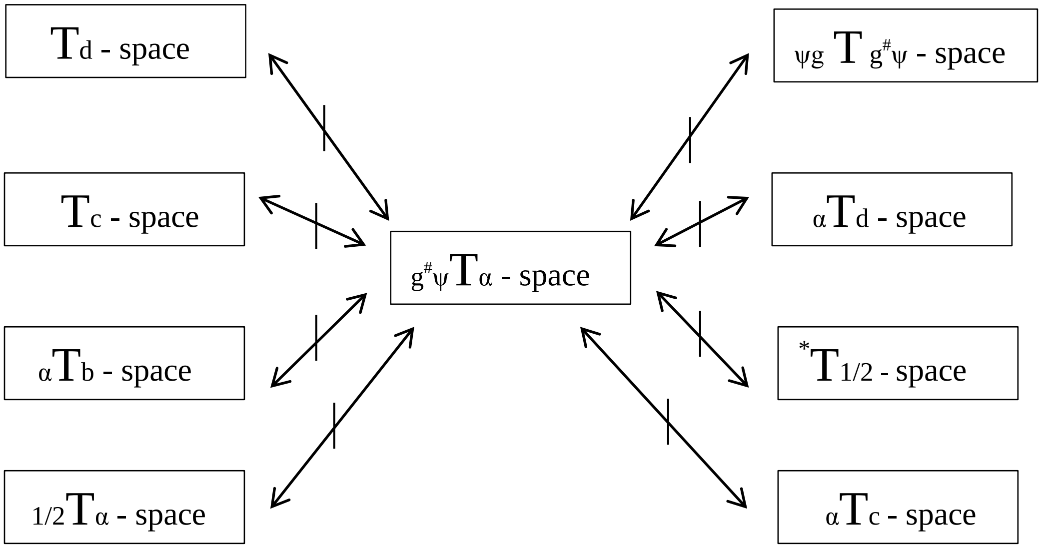
Let X={a,b,c} with topology τ={X, ϕ,{a,b}} then (X, τ) is a $_{\alpha}T_{d}$ - space but not $g^{\#}\psi T_{\alpha}$ – space, since the subset {a,c} is $g^{\#}\psi$ - closed but not α - closed.

Remark 2.4.41

The following diagrams show the relationship between the newly defined spaces and already existing spaces.



where $A \longrightarrow B$ represents A implies B



where $A \not\longleftrightarrow B$ represents A and B are independent.

Theorem 2.4.42

If (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space and an α - space then it is a $g^{\#}\psi T_{c}$ - space.

Proof:

Let A be a $g^{\#}\psi$ - closed set in (X, τ). Since (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space, A is α - closed in (X, τ). Since (X, τ) is an α - space, A is closed in (X, τ). Therefore (X, τ) is a $g^{\#}\psi T_{c}$ - space.

Theorem 2.4.43

If (X, τ) is a $g^{\#}\psi T_{c}$ - space, then for each $x \in X$ either {x} is ψ - closed or open.

Proof:

Let $x \in X$ and suppose {x} is not ψ - closed in (X, τ).Then X – {x} is not ψ - open. Hence X is the only ψ - open set containing X – {x}. So X – {x} is a $g^{\#}\psi$ - closed set in (X, τ). Since (X, τ) is a $g^{\#}\psi T_{c}$ – space, X – {x} is closed in (X, τ) or equivalently {x} is open in (X, τ).

Theorem 2.4.44

If (X, τ) is a $g^{\#}\psi T_{c}$ - space (resp $g^{\#}\psi T_{\alpha}$ - space) then $g^{\#}\psi cl(B) = cl(B)$ (resp. $\alpha cl(B)$) for each subset B of (X, τ).

Proof:

Since (X, τ) is a $g^{\#}\psi T_{c}$ - space (resp. $g^{\#}\psi T_{\alpha}$ - space). Since every closed (resp α – closed) set is $g^{\#}\psi$ - closed in (X, τ), $g^{\#}\psi C((X, \tau) = C(X, \tau)$ (resp $\alpha C(X, \tau)$).
Hence $g^{\#}\psi cl(B) = cl(B)$ (resp. $\alpha cl(B)$) for each subset B of (X, τ).

Theorem 2.4.45

For a space (X, τ) the following conditions are equivalent

i. (X, τ) is a $g^{\#}\psi T_{\alpha}$ - space.

ii. For each $x \in X$, $\{x\}$ is either α - open or ψ - closed.

Proof:

(i) $\Rightarrow$(ii)

Let $x \in X$ and suppose $\{x\}$ is not ψ - closed in (X, τ). Then X-{x} is not ψ - open. Hence X is the only ψ - open set containing X-{x}. So X-{x} is a $g^{\#}\psi$ – closed set in (X, τ). Since (X, τ) ia a $g^{\#}\psi T_{\alpha}$ – space, X-{x} ia an α – closed set in (X, τ) or equivalently {x} is an α –open set in (X, τ).

(ii) $\Rightarrow$(i)

Let A be a $g^{\#}\psi$ – closed set in (X, τ) and $x \in \psi cl(A)$. We show that $x \in A$ for the following two cases.

Case 1:

Assume that {x} is α – open. Then X-{x} is α –closed. If $x \notin A$, then $A \subseteq X-\{x\}$. Since $x \in \psi cl(A)$, we have $x \in X-\{x\}$, which is the contradiction. Hence $x \in A$.

Case 2:

Assume that {x} is ψ–closed and $x \notin A$. Then ψcl(A)-A contains a ψ – closed set {x}.This contradicts above **theorem 2.2.37**. Therefore $x \in A$.

CHAPTER-III

CHAPTER 3

$g^{\#}\psi$ - CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

3.1 Introduction

Continuity is a most important concept in mathematics and over the years many types of continuous functions have been analyzed. In this chapter we introduce $g^{\#}\psi$-continuous functions in topological spaces and study their properties.

3.2 $g^{\#}\psi$ – continuous function

In this section $g^{\#}\psi$ – continuous functions in topological spaces are introduced and discussed some of their basic properties. The composition of two $g^{\#}\psi$ – continuous function need not be $g^{\#}\psi$ – continuous is proved by by counter example.

Definition 3.2.1

A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called **$g^{\#}\psi$ - continuous** if $f^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ) for each closed set V in (Y, σ).

Example 3.2.2

Let X ={a, b, c}=Y with τ = {X, ϕ, {a}} and σ = {Y, ϕ, {a}, {a, b}}.Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous.

Proposition 3.2.3

Every continuous function is a $g^{\#}\psi$ - continuous function.

Proof:

Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a continuous function. Let V be any closed set in (Y,σ). Since f is continuous, $f^{-1}(V)$ is closed in (X, τ). Since every closed set is $g^{\#}\psi$ - closed, $f^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ). Hence f is $g^{\#}\psi$ - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.4

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}} and σ = {Y, ϕ, {a, b}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous but not continuous, since for the closed set {c} in (Y,σ), f^{-1}({c}) = {c} is not closed in (X, τ).

Proposition 3.2.5

Every completely - continuous function is a $g^{\#}\psi$ - continuous function.

Proof:

Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a completely - continuous function. Let V be any closed set in (Y,σ). Since f is completely - continuous, $f^{-1}(V)$ is regular - closed in (X, τ). Since every regular - closed set is $g^{\#}\psi$ - closed, $f^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ). Hence f is $g^{\#}\psi$ - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.6

Let X = {a, b, c} = Y with τ={X, ϕ, {a}} and σ={Y, ϕ, {a}, {a, b}}. Let

$f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous but not completely continuous, since for the closed set {b, c} in (Y,σ), $f^{-1}(\{b, c\}) = \{b, c\}$ is not regular closed in (X, τ).

Proposition 3.2.7

Every α - continuous function is a $g^{\#}\psi$ - continuous function.

Proof:

Let $f:(X,\tau)\to(Y,\sigma)$ be a α - continuous function. Let V be any closed set in (Y,σ). Since f is α - continuous, $f^{-1}(V)$ is α - closed in (X, τ). Since every α - closed set is $g^{\#}\psi$ - closed, $f^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ). Hence f is $g^{\#}\psi$ - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.8

Let X = {a, b, c} = Y with $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous but not α - continuous, since for the closed set {a, c} in (Y,σ), $f^{-1}(\{a, c\}) = \{a, c\}$ is not α - closed in (X, τ).

Proposition 3.2.9

Every ψ - continuous function is a $g^{\#}\psi$ - continuous function.

Proof:

Let $f:(X,\tau)\to(Y,\sigma)$ be a ψ - continuous function. Let V be any closed set in

(Y,σ). Since f is ψ - continuous, $f^{-1}(V)$ is ψ - closed (X, τ). Since every ψ - closed set is $g^{\#}\psi$ - closed, $f^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ). Hence f is $g^{\#}\psi$ - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.10

Let $X = \{a, b, c\} = Y$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by f(a) = b, f(b) = c, f(c) = a. Then f is $g^{\#}\psi$ - continuous but not ψ - continuous, since for the closed set {b, c} in (Y,σ), $f^{-1}(\{b, c\}) = \{a, b\}$ is not ψ - closed in (X, τ).

Proposition 3.2.11

Every semi - continuous function is a $g^{\#}\psi$ - continuous function.

Proof:

Let $f : (X, \tau) \to (Y, \sigma)$ be a semi - continuous function. Let V be any closed set in (Y,σ). Since f is semi - continuous, $f^{-1}(V)$ is semi - closed in (X, τ). Since every semi - closed set is $g^{\#}\psi$ - closed, $f^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ). Hence f is $g^{\#}\psi$ - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.12

Let $X = \{a, b, c\} = Y$ with $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous but not semi - continuous, since for the closed set {b, c} in (Y,σ), $f^{-1}(\{b, c\}) = \{b, c\}$ is not semi - closed in (X, τ).

Proposition 3.2.13

Every ψ^*g^* - continuous function is a $g^\#\psi$ - continuous function.

Proof:

Let $f:(X,\tau)\to(Y,\sigma)$ be a ψ^*g^* - continuous function. Let V be any closed set in (Y,σ). Since f is ψ^*g^* - continuous, $f^{-1}(V)$ is ψ^*g^* - closed in (X, τ). Since every ψ^*g^* - closed set is $g^\#\psi$ - closed, $f^{-1}(V)$ is $g^\#\psi$ - closed in (X, τ). Hence f is $g^\#\psi$ - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.14

Let X = {a, b, c} = Y with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = b, f(b) = c, f(c) = a. Then f is $g^\#\psi$ - continuous but not ψ^*g^* - continuous, since for the closed set {b, c} in (Y,σ), $f^{-1}(\{b, c\}) =\{a, b\}$ is not ψ^*g^* - closed in (X, τ).

Proposition 3.2.15

Every $g^\#\psi$ - continuous function is a ψg - continuous function.

Proof:

Let $f:(X,\tau)\to(Y,\sigma)$ be a $g^\#\psi$ - continuous function. Let V be any closed set in (Y,σ). Since f is $g^\#\psi$ - continuous, $f^{-1}(V)$ is $g^\#\psi$ - closed in (X, τ). Since every $g^\#\psi$ – closed set is ψg – closed, $f^{-1}(V)$ is ψg - closed in (X, τ). Hence f is ψg - continuous.

The converse of the above proposition need not be true as seen from the following example.

Example 3.2.16

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}} and σ ={Y, ϕ, {a},{a, b}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = b, f(b) = c, f(c) = a. Then f is ψg - continuous but not $g^{\#}\psi$ - continuous, since for the closed set {b, c} in (Y,σ), f^{-1}({b, c}) = {a, b} is not ψg - closed in (X, τ).

Remark 3.2.17

The following examples show that $g^{\#}\psi$ - continuity is independent of αg – continuity.

Example 3.2.18

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}, {b}, {a, b}} and σ ={Y, ϕ, {a}, {b}, {a, b},{a, c}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ – continuous but not αg - continuous, since for the closed set {b} in (Y,σ), f^{-1}({b}) = {b} is $g^{\#}\psi$ - closed but not αg - closed in (X, τ).

Example 3.2.19

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}} and σ ={Y, ϕ, {a}, {b},{a, b}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is αg - continuous but not $g^{\#}\psi$ - continuous, since for the closed set {a, c} in (Y,σ), f^{-1}({a, c}) = {a, c} is αg – closed but not $g^{\#}\psi$ - closed in (X, τ).

Remark 3.2.20

The following examples show that $g^{\#}\psi$ -continuity is independent of g^{*} - continuity.

Example 3.2.21

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}} and σ ={Y, ϕ, {a, b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous but not g^{*} - continuous, since for the closed set {c} in (Y,σ), f^{-1}({c}) = {c} is $g^{\#}\psi$ - closed but not g^{*} - closed in (X, τ).

Example 3.2.22

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}, {a, b}} and σ ={Y, ϕ, {b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is g^{*} - continuous but not $g^{\#}\psi$ - continuous, since for the closed set {a, c} in (Y,σ), f^{-1}({a, c}) = {a, c} is g^{*} – closed but not $g^{\#}\psi$ - closed in (X, τ).

Remark 3.2.23

The following examples show that $g^{\#}\psi$ -continuity is independent of $g^{*}\psi$ – continuity.

Example 3.2.24

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}, {b, c}} and σ ={Y, ϕ, {a, b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ - continuous but not $g^{*}\psi$ - continuous, since for the closed set {c} in (Y,σ), f^{-1}({c}) = {c} is $g^{\#}\psi$ - closed but not $g^{*}\psi$ - closed in (X, τ).

Example 3.2.25

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}, {a, b}} and σ ={Y, ϕ, {a}, {b}, {a, b}, {a, c}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^*\psi$ - continuous but not $g^\#\psi$ - continuous, since for the closed set {a, c} in (Y,σ), f^{-1}({a, c}) = {a, c} is $g^*\psi$ – closed but not $g^\#\psi$ - closed in (X, τ).

Remark 3.2.26

The following examples show that $g^\#\psi$ -continuity is independent of g – continuity.

Example 3.2.27

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}, {b}, {a, b}} and σ ={Y, ϕ, {b, c}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^\#\psi$ – continuous but not g - continuous, since for the closed set {a} in (Y,σ), f^{-1}({a}) = {a} is $g^\#\psi$ - closed but not g - closed in (X, τ).

Example 3.2.28

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}}and σ ={Y, ϕ, {a}, {b}, {a, b}}. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is g - continuous but not $g^\#\psi$ - continuous, since for the closed set {a, c} in (Y,σ), f^{-1}({a, c}) = {a, c} is g – closed but not $g^\#\psi$ - closed in (X, τ).

Remark 3.2.29

The following examples show that $g^\#\psi$ -continuity is independent of $\psi\hat{g}$ - continuity.

Example 3.2.30

Let X = {a, b, c} = Y with τ = {X, ϕ, {a},{b, c}} and σ ={Y, ϕ, {a}, {b}, {a, b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ – continuous but not $\psi\hat{g}$ - continuous, since for the closed set {c} in (Y,σ), f^{-1}({c}) = {c} is $g^{\#}\psi$ - closed but not $\psi\hat{g}$ - closed in (X, τ).

Example 3.2.31

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}}and σ ={Y, ϕ, {b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $\psi\hat{g}$ continuous but not $g^{\#}\psi$ - continuous, since for the closed set {a, c} in (Y,σ), f^{-1}({a, c}) = {a, c} is $\psi\hat{g}$ - closed but not $g^{\#}\psi$ - closed in (X, τ).

Remark 3.2.32

The following examples show that $g^{\#}\psi$ - continuity is independent of $^{*}g$ - continuity.

Example 3.2.33

Let X = {a, b, c} = Y with τ = {X, ϕ, {a},{a, b}} and σ ={Y, ϕ, {a, c}}.Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $g^{\#}\psi$ – continuous but not and not $^{*}g$ - continuous, since for the closed set {b} in (Y,σ), f^{-1}({b}) = {b} is $g^{\#}\psi$ - closed but not $^{*}g$ - closed in (X, τ).

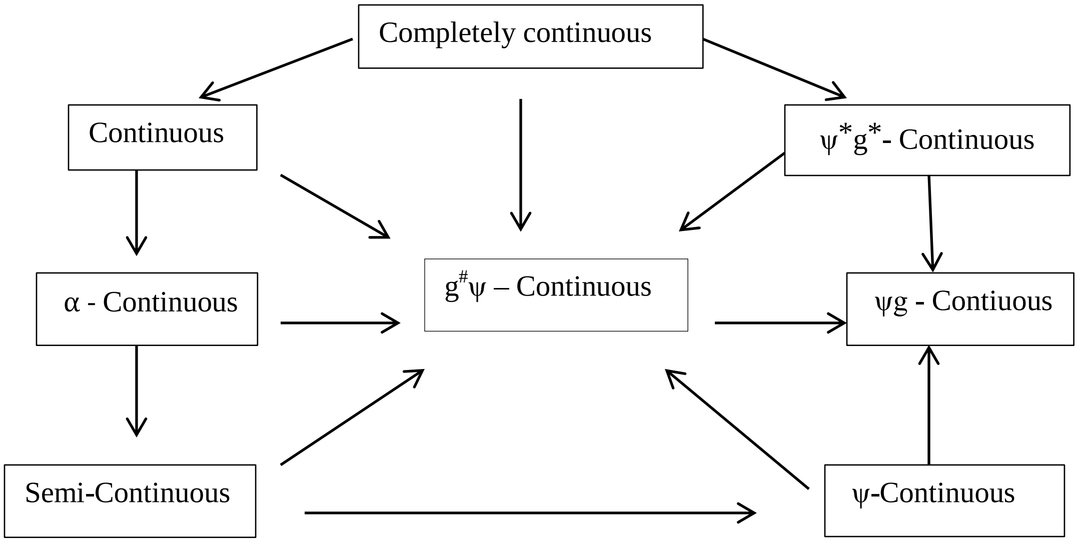
Example 3.2.34

Let X = {a, b, c} = Y with τ = {X, ϕ, {a}}and σ ={Y, ϕ, {c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be defined by f(a) = a, f(b) = b, f(c) = c. Then f is $^{*}g$ - continuity but not

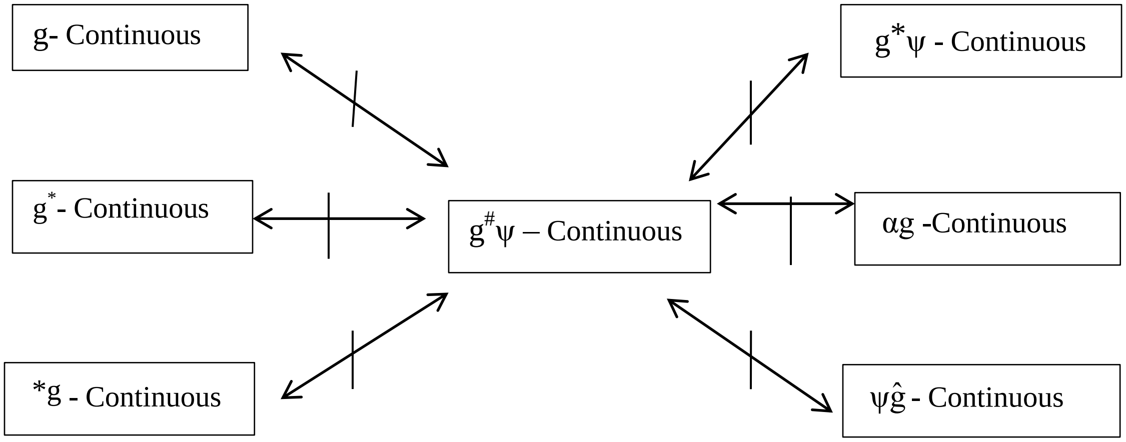
$g^{\#}\psi$ - continuous, since $\{a, b\}$ is closed in (Y,σ) but $f^{-1}(\{a, b\}) = \{a, b\}$ is ${}^{*}g$ - closed but not $g^{\#}\psi$ - closed in (X, τ).

Remark 3.2.35

The following diagrams show the relationship between $g^{\#}\psi$ – continuous functions with various continuous functions



Where A → B represents A implies B.



where A $\nleftrightarrow$ B represents A and B are independent.

Theorem 3.2.36

A function f : $(X, \tau) \to (Y, \sigma)$ is $g^{\#}\psi$ - continuous if and only if the inverse image of every open set in (Y, σ) is $g^{\#}\psi$ - open in (X, τ).

Proof: (Necessity)

Let F be an open set in (Y, σ). Then Y-F is closed in (Y, σ). Since f is $g^{\#}\psi$ - continuous, $f^{-1}(Y\text{-}F)= X\text{-} f^{-1}(F)$ is $g^{\#}\psi$ - closed in (X, τ). Hence f^{-1} (F) is $g^{\#}\psi$ - open in (X, τ).

(Sufficiency)

Assume that $f^{-1}(V)$ is $g^{\#}\psi$ - open in (X, τ) for each open set V in (Y, σ). Let V be any closed set in (Y, σ). Then Y-V is open in (Y, σ). By assumption, $f^{-1}(Y\text{-}V)= X\text{-} f^{-1}(V)$ is $g^{\#}\psi$ - open in (X, τ) which implies that $f^{-1}(V)$ is $g^{\#}\psi$ – closed in (X, τ). Hence f is $g^{\#}\psi$ - continuous.

Theorem 3.2.37

If f : $(X, \tau) \to (Y, \sigma)$ is $g^{\#}\psi$ - continuous then $f(g^{\#}\psi cl(V)) \subseteq cl(f(V))$ for every subset V of (X, τ).

Proof:

Let V be any subset of (X, τ).Then cl(f(V)) is closed in (Y, σ).Since f is $g^{\#}\psi$ - continuous, $f^{-1}(cl(f(V)))$ is $g^{\#}\psi$ - closed in (X, τ).Since $f(V) \subseteq cl(f(V))$, $V \subseteq f^{-1}(f(V)) \subseteq f^{-1}(cl(f(V)))$ and hence $f^{-1}(cl(f(V)))$ is a $g^{\#}\psi$ - closed set containing V. By definition of $g^{\#}\psi$ - closure, we have $g^{\#}\psi cl(V) \subseteq f^{-1}(cl(f(V)))$ which implies that $f(g^{\#}\psi cl(V)) \subseteq cl(f(V))$.

Remark 3.2.38

Let f : (X, τ) → (Y, σ) be a continuous function. Then for every subset V of (X, τ), $f(g^{\#}\psi cl(V)) \subseteq cl(f(V))$.

Proof:

Since every continuous function is $g^{\#}\psi$ -continuous and by **Theorem 3.2.37**, the result follows.

Remark 3.2.39

The composition of two $g^{\#}\psi$ - continuous function need not be $g^{\#}\psi$ - continuous function as seen from the following example.

Example 3.2.40

Let X = {a, b, c} = Y = Z with τ = {X, ϕ, {a}, {a, b}}, σ ={Y, ϕ, {a, b}} and η = {Z, ϕ, {a}}. Let $f:(X,\tau)\to(Y,\sigma)$ be a function defined by f(a) = a, f(b) = b, f(c) = c and $g:(Y,\sigma)\to(Z,\eta)$ be a function defined by g(a) = b, g(b) = a, g(c) = c. Then the functions f and g are $g^{\#}\psi$ – continuous but their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is not a $g^{\#}\psi$ -continuous function, since for the closed set {b, c} in (Z, η), $(g\circ f)^{-1}$({b, c) = {a, c} is not $g^{\#}\psi$ - closed in (X, τ).

Theorem 3.2.41

Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z,η) be $g^{\#}\psi$ - continuous functions. Then g∘f : (X, τ) → (Z, η) is also a $g^{\#}\psi$ - continuous function, if (Y, σ) is a $_{g\#\psi}T_c$ -space .

Proof:

Let V be any closed set in (Z,η). Since g is $g^{\#}\psi$ - continuous, $g^{-1}(V)$ is $g^{\#}\psi$ – closed in (Y, σ). Since (Y, σ) is a $_{g\#\psi}T_c$ -space, $g^{-1}(V)$ is closed in (Y, σ). Since f is $g^{\#}\psi$ - continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $g^{\#}\psi$ - closed in (X, τ). Hence g∘f is a $g^{\#}\psi$ - continuous function.

Theorem 3.2.42

If f : (X, τ) → (Y, σ) is $g^{\#}\psi$ - continuous and g : (Y, σ) → (Z,η) is continuous then g∘f : (X, τ) → (Z,η) is $g^{\#}\psi$ - continuous.

Proof:

Let V be any closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is $g^{\#}\psi$ - continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $g^{\#}\psi$ - closed in (X, τ). Therefore g∘f is $g^{\#}\psi$ - continuous.

Theorem 3.2.43

If f : (X, τ) → (Y, σ) is continuous (resp. α-continuous) and g : (Y, σ) → (Z,η) is continuous then g∘f : (X, τ) → (Z, η) is $g^{\#}\psi$ - continuous.

Proof:

Let V be any closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (Y,σ). Since f is continuous (resp. α-continuous), $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is closed (resp. α-closed) in (X, τ). Since every closed (resp. α-closed) set is $g^{\#}\psi$ – closed set, $(g\circ f)^{-1}(V)$ is $g^{\#}\psi$ - closed. Therefore g∘f is $g^{\#}\psi$ - continuous.

Theorem 3.2.44

If $f : (X, \tau) \to (Y, \sigma)$ is ψ - continuous and $g : (Y, \sigma) \to (Z,\eta)$ is continuous then $g \circ f : (X, \tau) \to (Z, \eta)$ is $g^{\#}\psi$ - continuous.

Proof:

Let V be any closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (Y,σ). Since f is ψ - continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is ψ - closed in (X, τ). Since every ψ - closed set is $g^{\#}\psi$ - closed, $(g \circ f)^{-1}(V)$ is $g^{\#}\psi$ - closed. Therefore $g \circ f$ is $g^{\#}\psi$ - continuous.

Theorem 3.2.45

If $f : (X, \tau) \to (Y, \sigma)$ is $\psi^{*}g^{*}$ - continuous and $g : (Y, \sigma) \to (Z,\eta)$ is continuous then $g \circ f : (X, \tau) \to (Z, \eta)$ is $g^{\#}\psi$ - continuous.

Proof:

Let V be any closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (Y,σ). Since f is $\psi^{*}g^{*}$ - continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\psi^{*}g^{*}$ - closed in (X, τ). Since every $\psi^{*}g^{*}$ - closed set is $g^{\#}\psi$ - closed, $(g \circ f)^{-1}(V)$ is $g^{\#}\psi$ - closed. Therefore $g \circ f$ is $g^{\#}\psi$ - continuous.

Theorem 3.2.46

Let $f : (X, \tau) \to (Y, \sigma)$ be ψg - continuous and $g : (Y, \sigma) \to (Z,\eta)$ be continuous. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a $g^{\#}\psi$ - continuous function, if (X, τ) is a ${}_{\psi g}T_{g\#\psi}$ – space.

Proof:

Let V be a closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is ψg-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is ψg- closed in (X, τ). Since (X, τ) is a ${}_{\psi g}T_{g\#\psi}$ - space, $(g \circ f)^{-1}(V)$ is $g^{\#}\psi$ - closed in (X, τ). Hence $g \circ f$ is a $g^{\#}\psi$ - continuous function.

Theorem 3.2.47

If $f : (X, \tau) \to (Y, \sigma)$ is α - irresolute and $g : (Y, \sigma) \to (Z,\eta)$ is continuous then $g\circ f : (X, \tau) \to (Z, \eta)$ is $g^{\#}\psi$ - continuous.

Proof:

Let V be any closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (Y,σ). Since every closed set is α - closed, $g^{-1}(V)$ is α - closed. Since f is α- irresolute, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is α - closed in (X, τ). Since every α - closed set is $g^{\#}\psi$ - closed, $(g\circ f)^{-1}(V)$ is $g^{\#}\psi$ – closed. Hence $g\circ f$ is $g^{\#}\psi$ - continuous.

Theorem 3.2.48

Let $f : (X, \tau) \to (Y, \sigma)$ be α - irresolute and $g : (Y, \sigma) \to (Z,\eta)$ be $g^{\#}\psi$ - continuous. If (Y, σ) is a ${}_{g\#\psi}T_{\alpha}$ – space, then $g\circ f : (X, \tau) \to (Z, \eta)$ is α - continuous.

Proof:

Let U be any closed set in (Z,η). Since g is $g^{\#}\psi$ - continuous, $g^{-1}(U)$ is $g^{\#}\psi$ - closed in (Y, σ). Since (Y, σ) is a ${}_{g\#\psi}T_{\alpha}$ - space, $g^{-1}(U)$ is α - closed in (Y, σ). Since f is α - irresolute, $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is α - closed in (X, τ). Hence $g\circ f$ is α - continuous.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

The dissertation is devoted to the study on $g^{\#}\psi$ - closed sets and $g^{\#}\psi$ - continuous functions in topological spaces.

Preliminary definition are given in **Chapter 1**.

In Chapter 2, $g^{\#}\psi$ - closed sets, $g^{\#}\psi$ - open sets in topological spaces are introduced and $g^{\#}\psi$ - closed sets are compared with various existing closed sets. As applications seven new spaces are introduced and their interrelations with various spaces are analyzed.

In Chapter 3, $g^{\#}\psi$ - continuous functions in topological spaces are introduced and its properties and characterizations are analyzed.

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PUBLICATIONS

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