Chapter 3

CHAPTER – III

NON-MARKOVIAN BATCH ARRIVAL QUEUES WITH A SECOND OPTIONAL SERVICE CHANNEL UNDER BI-LEVEL CONTROL POLICY AND SERVER VACATIONS

INTRODUCTION

This chapter also analyses batch arrival queueing models along with server single and multiple vacations under bi-level threshold policy. The models considered in this chapter are more general than the models of Chapter II for two reasons.

(i) The server provides two phases of heterogeneous service facility to the arriving customers. Each customer who finishes the first phase of service (called first essential service) may leave the system with probability $(1 - r)$ or may immediately opt for a second optional service with probability r.

(ii) The random variables (setup time, service time and vacation time) are assumed to follow general distributions with finite moments.

One of the important characteristics of queueing system is service process. Several queueing models belong to a class of systems where the service discipline involves more than one service and these models have been receiving a lot of attention recently. Various scenarios have been considered in the literature thus giving rise to various denominations for this kind of systems. Some of the scenarios often used in literature are the following :

(1) The server provides two types of heterogeneous service and customer may choose either type of service.

(2) Each customer receives two phases of heterogeneous service one after the other and then departs from the system.

(3) Each customer chooses one of the two types of heterogeneous service in the first phase and then he has the option to repeat or leave the system.

(4) The server provides the First Essential Service (FES) to all the arriving customers. At the end of the FES, each customer may either choose the Second Optional Service (SOS) or depart from the system.

These scenarios have been combined with many of different features known in queueing theory such as bulk arrival, server vacations, break downs and some control policies such as N-policy and D-policy etc. The first study of queueing system with SOS was done by Madan (2000). He cited some important applications of the model in many real life situations.Choudhury and Madan (2004) later studied a batch arrival queueing system, where the server provides two phases of heterogenous service one after the other to the arriving batches under Bernoulli schedule vacation. Then Madan *et. al.,* (2005) analysed a batch arrival queue with a single server providing two kinds of general heterogeneous service and each customer may choose one of the services. Choudhury and Paul (2006) considered an $M^X/G/1$ queueing system with SOS channel under N-policy and obtained queue size distribution at random epoch as well as at departure epoch and derived a procedure to obtain optimal stationary policy under a suitable linear cost structure.

To the best of our knowledge, batch arrival queueing models with two types of heterogeneous service, combined with double threshold policy and server vacation are not analysed in literature.

The systems being studied in this chapter are among the most general Poisson batch arrival systems with double threshold policy, server vacation and second optional service facility and these models include many previous works as special cases. Section (3.1) deals with single vacation policy. The multiple vacation policy is considered in section (3.2).

SECTION: 3.1

SINGLE VACATION QUEUEING SYSTEM

3.1.1 Mathematical Analysis of the System

I Model Description

This section considers an $M^X/G/1$ queueing system where the arrivals occur in batches according to a compound Poisson process with random batch size X, group arrival rate λ and probability distribution $g_k = Pr(X=k)$ k=1,2,3... Arriving customers form a single waiting line and the service is done one by one.

The server is turned off and leaves the system for a single vacation of random length V, each time when the system becomes empty. After returning from the vacation, if the server finds less than **m** customers in the system, then the server remains idle (build up period) in the system, until the queue length reaches atleast **m** and then starts a setup operation of random length D. On the other hand if the server finds **m** or more customers in the system at the end of the vacation then he immediately starts a setup operation. At the end of the setup operation if the queue length is $\geq$ N, then the server begins to serve the customers one by one exhaustively. Otherwise, the server remains dormant in the system waiting for the queue length to reach or exceed N before starting a service. Here the idle period of the server is made up of vacation period, buildup period, setup period and dormant period. The vacation time (V), setup time (D) assumed to follow general distributions V(t) and D(t) with finite moments.

The service pattern followed in this section is different from that of Chapter II.

Service Pattern

During busy period, the server provides to each unit, two stages of heterogeneous services of which one is optional (i.e.) the server provides First Essential Service (FES) for all the units. After the completion of FES of a unit, the customer may leave the system with probability (1 – r) or may opt for a Second Optional Service (SOS) in an additional channel by the same server with probability r ($0 \leq r \leq 1$). The server continues this type of service until the system becomes empty (i.e.) the server is turned off the system only when the system becomes again empty and leaves the system for vacation. Thus the busy period ends and the cycle is completed. The system will be turned on again for setup, when the server is present in the system and the system contains at least **m** customers. The service times S_1 and S_2 of two channels are assumed to be mutually independent of each other having general law of distribution with distribution function $S_i(x)$ and Laplace-Stieltjes Transform (LST) $S_i^*(\theta)$, i = 1, 2 denoting FES and SOS channels respectively. Here the same server handles both the channels.

The system is denoted by **($M^X_{(m,N)}$ / G_{SOS} / 1 / SV)**

In this section, using supplementary variable technique the steady state system state equations under the steady state condition are analysed, and the PGF of the system size is obtained in a closed form so that various performance measures of the model can be derived from it. The following notations are used to discuss the model.

N(t)	:	The system size at time t
λ	:	Group arrival rate
X	:	Group size random variable
g_k	:	Pr (X = k), k = 1, 2, 3, . . .
X(z)	:	Probability generating function of X.

The notations of Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments are listed as below:

	RV	CDF	Pdf	LST	k^{th} moment
Vacation time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$
Setup time	D	D(x)	d(x)	$D^*(\theta)$	$E(D^k)$
FES	S_1	$S_1(x)$	$s_1(x)$	$S_1^*(\theta)$	$E(S_1^k)$
SOS	S_2	$S_2(x)$	$s_2(x)$	$S_2^*(\theta)$	$E(S_2^k)$

where $F^*(\theta) = \int_0^\infty e^{-\theta x} f(x)\,dx = \int_0^\infty e^{-\theta x} d(F(x))$

Let $V^0(t)$, $D^0(t)$, $S_1^0(t)$ and $S_2^0(t)$ denote the remaining vacation time, setup time, first essential service time and second optional service time respectively at time t.Further the server states are denoted by the RV Y(t) at time t, (i.e.) Y(t) = 0, 1, 2, 3, 4 and 5 denote the server is in vacation, buildup, setup, dormant and in busy with FES and SOS state respectively. The supplementary variables are introduced in order to obtain a bivariate Markov process {N(t),δ(t)} where N(t) denotes the system size random variable and $\delta(t) = V^0(t),\ 0,\ D^0(t),\ 0,\ S_i^0(t), i=1,2$ according as Y(t) = 0,1,2,3,4,5 respectively.

Let $Q_n(x, t)\,dt = \Pr\{N(t) = n, x \le V^0(t) \le x + dt, Y(t) = 0\}, \quad n \ge 0$

$R_n(t) = \Pr\{N(t) = n, Y(t) = 1\}, \quad 0 \le n \le m-1$

$D_n(x, t)\,dt = \Pr\{N(t) = n, x \le D^0(t) \le x + dt, Y(t) = 2\}, \quad n \ge m$

$U_n(t) = \Pr\{N(t) = n, Y(t) = 3\}, \quad m \le n \le N-1$

$P_{n,1}(x, t)\,dt = \Pr\{N(t) = n, x \le S_1^0(t) \le x+dt, Y(t) = 4\}, \quad n \ge 1$

$P_{n,2}(x, t)\,dt = \Pr\{N(t) = n, x \le S_2^0(t) \le x+dt, Y(t) = 5\}, \quad n \ge 1$

Thus $R_n(t)$ and $U_n(t)$ respectively denote the probability that there are n customers in the system at time t, when the system is in buildup and dormant states. $Q_n(x, t)$, $D_n(x, t)$ and $P_{n,i}(x, t)$ i = 1, 2 denote the probability that there are n customers in the system at arbitrary epoch with the remaining vacation time, setup time and service time lie in the interval $[x, x + \Delta t]$.

Further $Q_n(0)$, $D_n(0)$, $P_{n,i}(0)$ i=1,2 denote the probability that there are n customers in the system at the termination of vacation period, setup period and service time respectively.

Assuming that at steady state, probabilities are independent of time t, we have

$$\underset{t\to\infty}{\mathrm{lt}} \frac{\partial}{\partial x} Q_n(x, t) = \frac{d}{dx} Q_n(x)$$

$$\underset{t\to\infty}{\mathrm{lt}} \frac{\partial}{\partial x} D_n(x, t) = \frac{d}{dx} D_n(x)$$

$$\underset{t\to\infty}{\mathrm{lt}} \frac{\partial}{\partial x} P_{n,i}(x, t) = \frac{d}{dx} P_{n,i}(x) \qquad i = 1, 2$$

$$\underset{t\to\infty}{\mathrm{lt}} \left(\frac{\partial}{\partial t} P_{n,i}(x, t) = \frac{\partial}{\partial t} D_n(x, t) = \frac{\partial}{\partial t} Q_n(x, t) \right) = 0 \text{ and}$$

$$\underset{t\to\infty}{\mathrm{lt}} R_n(t) = R_n \text{ and } \underset{t\to\infty}{\mathrm{lt}} U_n(t) = U_n$$

II The System Size Distribution

The following steady state equations are obtained for the queueing system using supplementary variable technique, and following the argument of Cox (1955).

$$-\frac{d}{dx} Q_0(x) = -\lambda Q_0(x) + (P_{1,1}(0)(1-r) + P_{1,2}(0))\, v(x)$$

$$-\frac{d}{dx} Q_n(x) = -\lambda Q_n(x) + \lambda \sum_{k=1}^{n} Q_{n-k}(x)\, g_k \qquad n \geq 1$$

$$\lambda R_0 = Q_0(0)$$

$$\lambda R_n = Q_n(0) + \lambda \sum_{k=1}^{n} R_{n-k}\, g_k \qquad 1 \leq n \leq m-1$$

$$-\frac{d}{dx} D_m(x) = -\lambda D_m(x) + Q_m(0)\, d(x) + \lambda \sum_{k=1}^{m} R_{m-k}\, g_k\, d(x)$$

$$-\frac{d}{dx} D_n(x) = -\lambda D_n(x) + Q_n(0)\, d(x) + \lambda \sum_{k=n-m+1}^{n} R_{n-k}\, g_k\, d(x) + \lambda \sum_{k=1}^{n-m} D_{n-k}(x)\, g_k, \quad n \geq m+1$$

$$\lambda U_m = D_m(0)$$

$$\lambda U_n = D_n(0) + \lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k, \quad m+1 \leq n \leq N-1$$

$$-\frac{d}{dx} P_{1,1}(x) = -\lambda P_{1,1}(x) + (1-r)\, P_{2,1}(0)\, s_1(x) + P_{2,2}(0)\, s_1(x)$$

$$-\frac{d}{dx} P_{n,1}(x) = -\lambda P_{n,1}(x) + (1-r)\, P_{n+1,1}(0)\, s_1(x) + \lambda \sum_{k=1}^{n-1} P_{n-k,1}(x)\, g_k + P_{n+1,2}(0)\, s_1(x), \quad 2 \leq n \leq N-1$$

$$-\frac{d}{dx} P_{n,1}(x) = -\lambda P_{n,1}(x) + (1-r)\, P_{n+1,1}(0)\, s_1(x) + \lambda \sum_{k=1}^{n-1} P_{n-k,1}(x)\, g_k + P_{n+1,2}(0)\, s_1(x) + \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\, s_1(x) + D_n(0)\, s_1(x), \quad n \geq N$$

$$-\frac{d}{dx} P_{1,2}(x) = -\lambda P_{1,2}(x) + r\, P_{1,1}(0)\, s_2(x)$$

$$-\frac{d}{dx} P_{n,2}(x) = -\lambda P_{n,2}(x) + \lambda \sum_{k=1}^{n-1} P_{n-k,2}(x)\, g_k + r\, P_{n,1}(0)\, s_2(x), \quad n \geq 1$$

The LST of the above equations are obtained by using the definition of Laplace-Stieltjes transformation and its properties. The LST of the density functions are defined in previous table and the remaining notations of the LST are listed below.

Probability Distribution	**LST**
$G(x)$	$G^*(\theta) = \int_0^\infty e^{-\theta x} G(x)\, dx$
$Q_n(x)$	$Q_n^*(\theta)$
$D_n(x)$	$D_n^*(\theta)$
$P_{n,i}(x)$ I = 1,2	$P_{n,i}^*(\theta)$ i = 1, 2

Thus we have,

$$\theta\, Q_0^*(\theta) - Q_0(0) = \lambda\, Q_0^*(\theta) - [P_{1,1}(0)\,(1-r) + P_{1,2}(0)]\, V^*(\theta) \qquad (3.1.1)$$

$$\theta\, Q_n^*(\theta) - Q_n(0) = \lambda\, Q_n^*(\theta) - \lambda \sum_{k=1}^{n} Q_{n-k}^*(\theta)\, g_k, \qquad n \geq 1 \qquad (3.1.2)$$

$$\lambda\, R_0 = Q_0(0) \qquad (3.1.3)$$

$$\lambda\, R_n = Q_n(0) + \lambda \sum_{k=1}^{n} R_{n-k}\, g_k, \qquad 1 \leq n \leq m-1 \qquad (3.1.4)$$

$$\theta\, D_m^*(\theta) - D_m(0) = \lambda\, D_m^*(\theta) - Q_m(0)\, D^*(\theta) - \lambda \sum_{k=1}^{m} R_{m-k}\, g_k\, D^*(\theta) \qquad (3.1.5)$$

$$\theta\, D_n^*(\theta) - D_n(0) = \lambda\, D_n^*(\theta) - Q_n(0)\, D^*(\theta) - \lambda \sum_{k=n-m+1}^{n} R_{n-k}\, g_k\, D^*(\theta)$$

$$- \lambda \sum_{k=1}^{n-m} D_{n-k}^*(\theta)\, g_k, \qquad n \geq m+1 \qquad (3.1.6)$$

$$\lambda\, U_m = D_m(0) \qquad (3.1.7)$$

$$\lambda\, U_n = D_n(0) + \lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k, \qquad m+1 \leq n \leq N-1 \qquad (3.1.8)$$

$$\theta\, P_{1,1}^*(\theta) - P_{1,1}(0) = \lambda\, P_{1,1}^*(\theta) - (1-r)\, P_{2,1}(0)\, S_1^*(\theta) - P_{2,2}(0) S_1^*(\theta) \qquad (3.1.9)$$

$$\theta\, P_{n,1}^*(\theta) - P_{n,1}(0) = \lambda\, P_{n,1}^*(\theta) - (1-r)\, P_{n+1,1}(0)\, S_1^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,1}^*(\theta)\, g_k\, S_1^*(\theta)$$

$$- P_{n+1,2}(0) S_1^*(\theta) \qquad 2 \leq n \leq N-1 \qquad (3.1.10)$$

$$\theta\, P_{n,1}^*(\theta) - P_{n,1}(0) = \lambda\, P_{n,1}^*(\theta) - (1-r)\, P_{n+1,1}(0)\, S_1^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,1}^*(\theta)\, g_k\, S_1^*(\theta)$$

$$+ P_{n+1,2}(0)\, S_1^*(\theta) - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\, S_1^*(\theta) - D_n(0)\, S_1^*(\theta),$$

$$n \geq N \qquad (3.1.11)$$

$$\theta\, P_{1,2}^*(\theta) - P_{1,2}(0) = \lambda\, P_{1,2}^*(\theta) - r\, P_{1,1}(0)\, S_2^*(\theta) \qquad (3.1.12)$$

$$\theta\, P_{n,2}^*(\theta) - P_{n,2}(0) = \lambda\, P_{n,2}^*(\theta) - r\, P_{n,1}(0)\, S_2^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,2}^*(\theta)\, g_k,$$

$$n \geq 2 \qquad (3.1.13)$$

III The Probability Generating Functions

Now to obtain the partial PGFs of the number of customers in the system, the following PGFs are defined

$$R(z) = \sum_{n=0}^{m-1} R_n z^n ; \qquad U(z) = \sum_{n=m}^{N-1} U_n z^n$$

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta) z^n ; \qquad Q(z, 0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

$$D^*(z,\theta) = \sum_{n=m}^{\infty} D_n^*(\theta) z^n ; \qquad D(z, 0) = \sum_{n=m}^{\infty} D_n(0) z^n$$

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{n,i}^*(\theta) z^n \quad i = 1, 2 ; \quad P_i(z, 0) = \sum_{n=1}^{\infty} P_{n,i}(0) z^n \quad i = 1, 2$$

Equations (3.1.1) and (3.1.2) imply

$$(\theta - w_X(z))\, Q^*(z,\theta) = Q(z, 0) - P_1(0)\, V^*(\theta)$$

where $P_1(0) = (1-r)\, P_{1,1}(0) + P_{1,2}(0)$ (3.1.14)

and $w_X(z) = \lambda\,(1 - X(z))$

By evaluating Q(z, 0) at $\theta = w_X(z)$, the generating function of the system size at vacation terminating epoch and at arbitrary epoch, when the server is on vacation are given by

$$Q(z, 0) = P_1(0)\, V^*(w_X(z)) \tag{3.1.15}$$

and
$$Q^*(z,\theta) = P_1(0) \frac{\left[V^*(w_X(z)) - V^*(\theta)\right]}{(\theta - w_X(z))} \tag{3.1.16}$$

If α_n denotes the probability that n customers arrive during vacation time V, then $V^*(w_X(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$ (Gross and Harris 1985)

Thus the equation (3.1.15) implies

$$Q_n(0) = P_1(0)\, \alpha_n \qquad n \geq 0 \tag{3.1.17}$$

Next equations (3.1.5) and (3.1.6) imply

$$(\theta - w_X(z))\, D^*(z,\theta) = D(z,0) - D^*(\theta) \sum_{n=m}^{\infty} Q_n(0)\, z^n - \lambda\, D^*(\theta) \sum_{n=m}^{\infty} z^n \sum_{k=n-m+1}^{n} R_{n-k}\, g_k \qquad (3.1.18)$$

and equations (3.1.3) and (3.1.4) give

$$\lambda\, R(z) = \sum_{n=0}^{m-1} Q_n(0)\, z^n + \lambda \sum_{n=1}^{m-1} z^n \sum_{k=1}^{n} R_{n-k}\, g_k \qquad (3.1.19)$$

Multiplying equation (3.1.19) by $(-D^*(\theta))$ and then adding with (3.1.18), the PGF of the system size at setup termination epoch and arbitrary epoch when the server is in setup state are given by

$$D(z,0) = D^*(w_X(z))\, [P_1(0)\ V^*(w_X(z)) - R(z)\, w_X(z)] \qquad (3.1.20)$$

and $$D^*(z,\theta) = \frac{\left[D^*(w_X(z)) - D^*(\theta)\right]}{(\theta - w_X(z))} [P_1(0)\ V^*(w_X(z)) - R(z)\, w_X(z)] \qquad (3.1.21)$$

To obtain R(z), the following theorem which is analogous to theorem 2.1.1 is used.

Theorem : 3.1

Let $\pi_0 = 1$, $\pi_n = \sum_{i=1}^{n} g_i\, \pi_{n-i}$, $\psi_0 = \alpha_0$, $\psi_n = \sum_{i=0}^{n} \alpha_i\, \pi_{n-i}$ $\quad 0 \le n \le m-1.$

Then $R(z) = P_1(0)\, \psi(z)$ where $\psi(z) = \sum_{n=0}^{m-1} \frac{\psi_n z^n}{\lambda}$

Proof

Equations (3.1.3) and (3.1.4) recursively imply

$$\lambda\, R_n = \sum_{i=0}^{n} Q_k(0)\, \pi_{n-k} \qquad 0 \le n \le m-1$$

By substituting $Q_k(0)$ from (3.1.17)

$$\lambda R_n = P_1(0) \sum_{k=0}^{n} \alpha_k \pi_{n-k} = P_1(0) \psi_n$$

Hence $R(z) = P_1(0) \psi(z)$, where $\psi(z) = \sum_{n=0}^{m-1} \frac{\psi_n z^n}{\lambda}$ (3.1.22)

Substituting for R(z) the equations (3.1.20) and (3.1.21) lead to

$$D(z, 0) = P_1(0) D^*(w_X(z)) [V^*(w_X(z)) - w_X(z) \psi(z)] \quad (3.1.23)$$

and $$D^*(z, \theta) = P_1(0) \left[\frac{D^*(w_X(z)) - D^*(\theta)}{\theta - w_X(z)} \right] [V^*(w_X(z)) - w_X(z) \psi(z)] \quad (3.1.24)$$

Next, by using the equations (3.1.12) and (3.1.13) the generating functions of the system size when the server is busy with SOS, at the service completion epoch and at arbitrary epoch are obtained as

$$P_2(z, 0) = r S_2^*(w_X(z)) P_1(z, 0) \quad (3.1.25)$$

and $$P_2^*(z, \theta) = r P_1(z, 0) \frac{(S_2^*(w_X(z)) - S_2^*(\theta))}{(\theta - w_X(z))} \quad (3.1.26)$$

To derive the generating functions of the system size at arbitrary epoch and service completion epoch, when the server is busy with FES, the equations (3.1.9) to (3.1.11) are used and resulted in

$$(\theta - w_X(z)) P_1^*(z, \theta) = P_1(z, 0) \left(1 - \frac{(1-r) S_1^*(\theta)}{z} \right) - \frac{S_1^*(\theta)}{z} P_2(z, 0) + S_1^*(\theta) P_1(0)$$

$$- S_1^*(\theta) \sum_{n=N}^{\infty} D_n(0) z^n - \lambda S_1^*(\theta) \sum_{n=N}^{\infty} z^n \sum_{k=n-N+1}^{n-m} U_{n-k} g_k$$

(3.1.27)

For further simplification, equations (3.1.7) and (3.1.8) are used and they imply

$$\lambda U(z) = \sum_{n=m}^{N-1} D_n(0) z^n + \lambda \sum_{n=m+1}^{N-1} z^n \sum_{k=1}^{n-m} U_{n-k} g_k \quad (3.1.28)$$

Adding equation (3.1.27) with equation (3.1.28) multiplied by $(-S_1^*(\theta))$ and using equation (3.1.25) it is found that

$$(\theta - w_X(z))\, P_1^*(z,\theta) = P_1(z,0)\left[\frac{z - S_1^*(\theta)(1-r+r\,S_2^*(w_X(z)))}{z}\right] - S_1^*(\theta)\,[D(z,0) - P_1(0) - U(z)\, w_X(z)] \quad (3.1.29)$$

when $\theta = w_X(z)$ equation (3.1.29) becomes

$$P_1(z,0) = \frac{z\,S_1^*(w_X(z))\,[D(z,0) - P_1(0) - U(z)\,w_X(z)]}{(z - S_{sos}^*(w_X(z)))} \quad (3.1.30)$$

where $S_{sos}^*(w_X(z)) = S_1^*(w_X(z))\,((1-r) + r\,S_2^*(w_X(z)))$ (3.1.31)

Substituting for $P_1(z, 0)$ in the equation (3.1.29)

$$P_1^*(z,\theta) = \frac{z\,[S_1^*(w_X(z)) - S_1^*(\theta)]\,[D(z,0) - P_1(0) - U(z)\,w_X(z)]}{(\theta - w_X(z))\,(z - S_{sos}^*(w_X(z)))} \quad (3.1.32)$$

For further simplification, equations (3.1.7) and (3.1.8) are used, recursively.

Then, $\lambda\, U_n = \sum_{k=m}^{n} D_k(0)\,\pi_{n-k}$ $\quad m \le n \le N-1$ (3.1.33)

where $D_k(0)$ is the co-efficient of z^k in $D(z, 0)$. Then from equation (3.1.23),

$$D_k(0) = P_1(0) \text{ co-efficient of } z^k \text{ in } D^*(w_X(z))\,[V^*(w_X(z)) - w_X(z)\,\psi(z)]$$

By using the theorem (2.1.2 (ii)) it is found that

$$D^*(w_X(z))\,[V^*(w_X(z)) - w_X(z)\,\psi(z)] = \sum_{k=m}^{\infty} z^k \sum_{i=m}^{k} \xi_i\, h_{k-i}$$

where $\xi_k = \alpha_k + \sum_{i=0}^{m-1} \psi_i\, g_{k-i}$ $\quad k \ge m$

$$\text{Thus } D_k(0) = P_1(0) \sum_{i=m}^{k} \xi_i h_{k-i}$$

Substituting for $D_k(0)$ in equation (3.1.33) and rearranging the summation,

$$\lambda U_n = P_1(0) \sum_{k=m}^{n} \xi_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i}$$

$$\text{Thus, } U(z) = P_1(0) \sum_{n=m}^{N-1} \frac{\phi_n^s z^n}{\lambda} = P_1(0)\, \phi_S(z) \qquad (3.1.34)$$

$$\text{where } \phi_n^S = \sum_{k=m}^{n} \xi_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i} \qquad (3.1.35)$$

Substituting for D(z, 0) and U(z), it is found that

$$D(z, 0) - P_1(0) - U(z)\, w_X(z) = -w_X(z)\, P_1(0)\, I_S(z)$$

$$\text{where } I_S(z) = \left[\left(\frac{1 - V^*(w_X(z))D^*(w_X(z))}{w_X(z)}\right) + D^*(w_X(z))\,\psi(z) + \varphi_S(z)\right] \qquad (3.1.36)$$

Thus the partial PGF of the system size at arbitrary epoch when the server is in different states are obtained by using equations (3.1.16), (3.1.22), (3.1.24), (3.1.34), (3.1.32) and (3.1.26)

$$\left.\begin{aligned}
Q^*(z,0) &= P_1(0)\left(\frac{1 - V^*(w_X(z))}{w_X(z)}\right) \\
R(z) &= P_1(0)\,\psi(z) \\
D^*(z,0) &= P_1(0)\left(\frac{1 - D^*(w_X(z))}{w_X(z)}\right)(V^*(w_X(z)) - w_X(z)\,\psi(z)) \\
U(z) &= P_1(0)\,\phi_S(z) \\
P_1^*(z,0) &= P_1(0)\,\frac{z(S_1^*(w_X(z)) - 1)}{(z - S_{sos}^*(w_X(z)))}\, I_S(z) \\
\text{and } P_2^*(z,0) &= \frac{P_1(0)\, r\, z\, S_1^*(w_X(z))\,(S_2^*(w_X(z)) - 1)}{(z - S_{sos}^*(w_X(z)))}\, I_S(z)
\end{aligned}\right\} \qquad (3.1.37)$$

The partial generating functions corresponding to the busy period ($P_B(z)$) and idle period ($P_I(z)$) are given by

$$P_B(z) = P_1^*(z,0) + P_2^*(z,0)$$

$$= P_1(0)\,\frac{z\,(S^*_{sos}(w_X(z))-1)}{(z-S^*_{sos}(w_X(z)))}\,I_S(z) \qquad (3.1.38)$$

$$P_I(z) = Q^*(z,0) + R(z) + D^*(z,0) + U(z) = P_1(0)\,I_S(z) \qquad (3.1.39)$$

Thus the total PGF is given by

$$\mathbf{P}^{S}_{sos(m,N)}(z) = P_B(z) + P_I(z)$$

$$= P_1(0)\,\frac{(z-1)\,S^*_{sos}(w_X(z))}{(z-S^*_{sos}(w_X(z)))}\,I_S(z) \qquad (3.1.40)$$

where $P_1(0)$ can be calculated by using the normalizing condition $\mathbf{P}^{S}_{sos(m,N)}(1) = 1$.

IV Performance Measures

In this section some useful performance measures of the proposed model are presented. Let P_v, P_{build}, P_{set}, P_{dor} and P_{busy} denote the probability that the server is in vacation, buildup, setup, dormant and in busy state respectively. Then their corresponding probabilities are obtained, by considering the equations in (3.1.37) at z = 1. Thus

(i) $$P_v = \lim_{z\to1} Q^*(z,0) = P_1(0)\,E(V)$$

(ii) $$P_{build} = \lim_{z\to1} R(z) = P_1(0)\,\psi(1)$$

(iii) $$P_{set} = \lim_{z\to1} D^*(z,0) = P_1(0)\,E(D)$$

(iv) $$P_{dor} = \lim_{z\to1} U(z) = P_1(0)\,\phi_S(1)$$

The probability that the server is in idle state (P_I) and in busy state (P_{busy}) are obtained from equations (3.1.39) and (3.1.38) as

(v) $$P_I = P_1(0)\,I_S(1) = P_1(0)\,D_S(m, N)$$

(vi) $P_{busy} = \underset{z\to1}{lt}\, P_B(z) = P_1(0)\,\dfrac{\rho_{sos}}{(1-\rho_{sos})}\,D_S(m, N)$

where $D_S(m, N) = I_S(1) = [E(D) + E(V) + \sum_{n=0}^{m-1}\frac{\psi_n}{\lambda} + \sum_{n=m}^{N-1}\frac{\phi_n^S}{\lambda}]$ (3.1.41)

and $\rho_{sos} = \frac{d}{dz}(S^*_{sos}(w_X(z)))_{z=1} = \lambda\, E(X)\,(E(S_1) + r\,E(S_2))$ (3.1.42)

Thus by using the normalizing condition, $\mathbf{P}^S_{sos(m,N)}(1) = 1$,

$P_1(0)$ is evaluated as

$$P_1(0) = \frac{1-\rho_{sos}}{D_S(m,\ N)} \quad (3.1.43)$$

and hence $P_{busy} = \rho_{sos}$ (3.1.44)

Thus $$\mathbf{P^S_{sos(m,N)}(z)} = \frac{(1-\rho_{sos})(z-1)\,S^*_{sos}(w_X(z))}{(z-S^*_{sos}(w_X(z)))}\,\frac{I_S(z)}{I_S(1)}$$

V Decomposition Property

The system size distribution of the (m,N) policy of single vacation model $M^X_{(m,N)}/G_{sos}/1/SV$ is decomposed into two random variables one of which is the system size of classical SOS model $M^X/G_{sos}/1$ system namely $\frac{(1-\rho_{sos})(z-1)S^*_{sos}(w_X(z))}{(z-S^*_{sos}(w_X(z)))}$ (Choudhury and Paul 2006) and the other is $\frac{I_S(z)}{I_S(1)}$ which gives the PGF of the conditional system size distribution during the server idle period under the steady state condition $\rho_{sos} < 1$.

VI Mean System Size

Let L_v, L_{build}, L_{set}, L_{dor} and L_{busy} denote the expected system size when the server is in vacation, buildup, setup, dormant and in busy state respectively. Then the derivatives of equation (3.1.37) at z = 1 give,

(i) $L_v = P_1(0)\left(\dfrac{\lambda\, E(X)E(V^2)}{2}\right)$

(ii) $L_{build} = P_1(0) \sum_{n=0}^{m-1} \frac{n\,\psi_n}{\pi}$

(iii) $L_{set} = P_1(0)\,\lambda\,E(X)\left(\frac{E(D^2)}{2} + E(D)\left(E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda}\right)\right)$

(iv) $L_{dor} = P_1(0) \sum_{n=m}^{N-1} \frac{n\,\phi_n^S}{\lambda}$

(v) $$L_{busy} = \rho_{sos} + \left[\frac{(\lambda E(X(X-1))\,E(S_{sos}) + (\lambda E(X))^2\,(E(S_{sos}^2)))}{2(1-\rho_{sos})} + \frac{\rho_{sos}\,L_S(m,N)}{D_s(m,N)}\right]$$

where $L_S(m, N) = L_0 + \lambda\,E(X)\,E(D)\,\psi(1) + \sum_{n=0}^{m-1} \frac{n\psi_n}{\lambda} + \sum_{n=m}^{N-1} \frac{n\,\phi_n^S}{\lambda}$

with $L_0 = \lambda\,E(X)\left(\frac{E(D^2)}{2} + \frac{E(V^2)}{2} + E(D)\,E(V)\right)$

$E(S_{sos}) = E(S_1) + r\,E(S_2)$

and $E(S_{sos}^2) = E(S_1^2) + 2\,r\,E(S_1)\,E(S_2) + r\,E(S_2^2)$

Thus the expected system when the server is in idle state is obtained by adding equations (i) to (iv) as,

(vi) $L_I = \frac{(1-\rho_{sos})\,L_S(m,N)}{D_S(m,N)}$

Therefore the expected system size (L_{sos}^S) of the model is given by

(vii) $$\mathbf{L_{sos(m,N)}^S} = \frac{L_S(m,N)}{D_S(m,N)} + L_{SOS} \qquad (3.1.45)$$

where $$L_{SOS} = \left(\rho_{sos} + \frac{\lambda\,E(X(X-1))\,E(S_{sos}) + (\lambda\,E(X))^2\,E(S_{sos}^2)}{2\,(1-\rho_{sos})}\right) \qquad (3.1.46)$$

gives the mean system size of the classical second optional service model ($M^X / G_{sos} / 1$) without N-policy (Choudhury and Paul 2006)

VII Other System Characteristics

Let E(Cycle),E(Busy) , E(I) and $E(W_s)$ denote the expected length of cycle, busy period , idle period and the expected waiting time in the system then IV- (iii) , (vi) and (v) imply ,

(i) $E\,(\text{Cycle}) = \dfrac{1}{P_1(0)} = \dfrac{1-\rho_{sos}}{D_s(m,N)}$

(ii) $E(\text{Busy}) = P_{busy}\,E(\text{cycle}) = \dfrac{\rho_{sos}}{(1-\rho_{sos})} D_S(m, N)$

(iii) $E(\text{idle}) = P_I\,E(\text{cycle}) = D_S(m, N)$

(iv) $E(W_s) = \dfrac{L^S_{sos(m,N)}}{\lambda\, E(x)}$ (using little's formula)

VIII Queue Size Distribution at Department Epoch

By following the argument of PASTA (refer Wolf 1982) a departing customer will see j customers in the system just after the departure if and only if there are (j+1) customers in the system just before the departure.

Thus the probability that there are j customers in the system at department epoch is given by

$$\pi_j^+ = D((1 - r)\, P_{j+1,1}(0) + P_{j+1,2}(0)), \qquad j \geq 0$$

where D is the normalizing constant

Let $\pi^+(z)$ be the PGF of $\{\pi_j^+ \,;\, j \geq 0\}$

$$\text{then } \pi^+(z) = \sum_{j=0}^{\infty} \pi_j\, z^j = \frac{D}{z}\left((1-r)\sum_{j=1}^{\infty} P_{j,1}(0)\, z^j + \sum_{j=1}^{\infty} P_{j,2}(0)\, z^j \right)$$

$$= \frac{D}{z}[(1 - r)\, P_1(z, 0) + P_2(z, 0)]$$

Substituting for $P_i(z, 0)$ i=1,2 and evaluating D using the normalizing condition, $\pi^+(z) = \dfrac{(1-X(z))}{E(X)\ (1-z)}\, \mathbf{P}^S_{sos(m,N)}(z)$

3.1.2 Optimal Management Policy

In this section the main objective is to find the optimal values of m^* and $N^*(m)$ which minimize the linear cost function of $M^X_{(m,N)}/G_{sos}/1/SV$ queueing system by considering the cost structure as in previous chapter (section2.1).

Let C_y(startup cost per cycle), C_h(holding cost per customer), C_{set}(setup cost), C_{dor}(standby cost), C_{build}(buildup cost), C_{busy} (operating cost) and C_v(reward cost) per unit time.

Then the average cost per unit time of the system is given by

$$T_C(m, N) = \frac{C_y}{E(cycle)} + C_h L^S_{sos(m,N)} + C_{set} P_{set} + C_{dor} P_{dor} + C_{build} P_{build} + C_{busy} P_{busy} - C_v P_v$$

By substituting for various measures, $T_C(m, N)$ can be re-written as,

$$T_C(m,N) = \frac{1}{D_S(m,N)}\left[A^S_{sos} + z^S_{sos}(m) + C_h \sum_{n=m}^{N-1} \frac{n\,\varphi^S_n}{\lambda} + C_{dor} \sum_{n=m}^{N-1} \varphi^S_n (1-\rho_{sos})\right] + A'_{sos}$$

where $A'_{sos} = C_{busy}\,\rho_{sos} + C_h L_{SOS}$

$$A^S_{sos} = (1-\rho_{sos})(C_y + C_{set}E(D) - C_v E(V)) + C_h \frac{\lambda E(X)}{2}(E(D)^2 + 2E(D)E(V) + E(V^2))$$

$$z^S_{sos}(m) = C_h \sum_{n=0}^{m-1} \frac{n\,\psi_n}{\lambda} + [C_h \lambda E(X) E(D) + C_{build}(1-\rho_{sos})] \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda}$$

and $$D_S(m,N) = \left[E(D) + E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda} + \sum_{n=m}^{N-1} \frac{\phi^S_n}{\lambda}\right]$$

By calculation, $$T_C(m, k+1) - T_C(m, k) = \frac{\varphi^S_k}{\lambda\, D_S(m,k+1)\, D_S(m,k)} H^S_{sos}(m,k)$$

where $$H^S_{sos}(m,k) = C_h\left[k\,\ell^S_m + \sum_{n=m}^{k} \frac{(k-n)}{\lambda}\varphi^S_n\right] + C_{dor}(1-\rho_{sos}) - (A^S_{sos} + Z^S_{sos}(m))$$

with $$\ell^S_m = E(D) + E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda}$$

Thus by proceeding as in section (2.1.2) it is found that for a given m the condition under which $T_C(m, k+1) - T_C(m,k) > 0$ for the first value of k is given by ,

$$N^*(m) = \text{Min}\{k \geq 1 \,/\, H^S_{sos}(m, k) > 0\}$$

And this gives the conditional optimal policy for each m.

Thus the optimal value (m^*, N^*) and the corresponding $T_C(m^*, N^*)$ can be obtained by following the algorithm given in section (2.1.2).

SECTION : 3.2

MULTIPLE VACATION QUEUEING SYSTEM

3.2.1 Mathematical Analysis of the System Size Distribution

I Model Description

The model of this section differs from the model of section 3.1 only in vacation policy. The vacation policy considered for the model of this section is repeated (or) multiple vacation policy. According to this vacation policy, the server leaves the system for vacation of random length V_1 as soon as the system empties. After returning from the vacation if the server finds **m** (or) more customers in the system, he immediately starts a setup operation of random length D. Otherwise, he takes another vacation V_2 and so on until he finally finds at least **m**-customers. The random variable V_1, V_2 .....are assumed to be independently and identically distributed with generic representation V. The behaviour after the setup is exactly the same as in the model of section (3.1). For the further development of the model, a grand vacation and the grand vacation process introduced by Lee *et al.,*(1994b) are considered.

Grand Vacation

The first grand vacation (G_1) starts from the time point the system becomes empty and the server leaves for the first vacation (V_1) and lasts until the server finds one (or) more customers after returning from a vacation. At the end of the first grand vacation, if the number of customers is $< m$, the server has to leave for another vacation and it becomes the initiation point of second grand vacation (G_2). This second grand vacation continues until a different system state is observed after a vacation. Thus the grand vacations $\{G_1\ G_2, \ldots\}$ continue in this manner until the number of customers observed after a grand vacation is greater than or equal to **m.**

The Grand Vacation Process

The grand vacation process is a process imbedded in the idle period process in which the imbedded states are the number of customers in the queue just after leaving for grand vacations.

II System Size Distribution at Random Epoch

Further for the multiple vacation model the buildup period is zero, and the random variable Y(t) which denotes the state of the system at time t takes values 0, 2, 3, 4 and 5 according as the system is in vacation, setup, dormant and busy with FES and SOS state respectively.

Let Z(t) = j denote the server is on j^{th} vacation at time t, counting from the idle period initiation point (j = 1, 2, . . .) then,

$$Q_{n,j}(x, t)\, dt = \Pr\left(N(t) = n, x \leq V^{\circ}(t) \leq x + dt, y(t) = 0, z(t) = j\right) \ (n \geq 0), (j \geq 1)$$

The other notations and assumptions are as in section (3.1) . Thus, by assuming the steady state probability $\lim_{t\to\infty} Q_{n,j}(x, t) = Q_{n,j}(x)$, its LST $Q^*_{n,j}(\theta)$, and following the arguments of Cox (1955) the Kolmogrov forward equations under steady state conditions are obtained. The equations corresponding to vacation states and setup states are listed below and the other equations are similar to the corresponding equations of section (3.1).

$$-\frac{d}{dx}(Q_{0,1}(x)) = -\lambda\, Q_{0,1}(x) + P_1(0)\, v(x)$$

$$-\frac{d}{dx}(Q_{0,j}(x)) = -\lambda\, Q_{0,j}(x) + Q_{0,j-1}(0)\, v(x) \qquad j \geq 2$$

$$-\frac{d}{dx}(Q_{n,1}(x)) = -\lambda\, Q_{n,1}(x) + \lambda \sum_{k=1}^{n} Q_{n-k,1}(x)\, g_k \qquad n \geq 1$$

$$-\frac{d}{dx}(Q_{n,j}(x)) = -\lambda\, Q_{n,j}(x) + \lambda \sum_{k=1}^{n} Q_{n-k,j}(x)\, g_k + Q_{n,j-1}(0)\, v(x)$$

$$j \geq 2, \quad 1 \leq n \leq m-1$$

$$-\frac{d}{dx}(Q_{n,j}(x)) = -\lambda\, Q_{n,j}(x) + \lambda \sum_{k=1}^{n} Q_{n-k,j}(x)\, g_k \qquad n \geq m, \; j \geq 2$$

$$-\frac{d}{dx}(D_m(x)) = -\lambda\, D_m(x) + \sum_{j=1}^{\infty} Q_{m,j}(0)\; d(x)$$

$$-\frac{d}{dx}(D_n(x)) = -\lambda\, D_n(x) + \lambda \sum_{k=1}^{n-m} D_{n-k}(x)\, g_k + \sum_{j=1}^{\infty} Q_{n,j}(0)\; d(x) \qquad n \geq m+1.$$

Thus, the Laplace -Stieltjes transform (LST) of the steady state equations of the multiple vacation model are :

$$\theta\, Q^*_{0,1}(\theta) - Q_{0,1}(0) = \lambda\, Q^*_{0,1}(\theta) - P_1(0)\; V^*(\theta) \qquad (3.2.1)$$

$$\theta\, Q^*_{n,1}(\theta) - Q_{n,1}(0) = \lambda\, Q^*_{n,1}(\theta) - \lambda \sum_{k=1}^{n} Q^*_{n-k,1}(\theta)\, g_k \qquad n \geq 1 \qquad (3.2.2)$$

$$\theta\, Q^*_{0,j}(\theta) - Q_{0,j}(0) = \lambda\, Q^*_{0,j}(\theta) - Q_{0,j-1}(0)\; V^*(\theta) \qquad j \geq 2 \qquad (3.2.3)$$

$$\theta\, Q^*_{n,j}(\theta) - Q_{n,j}(0) = \lambda\, Q^*_{n,j}(\theta) - \lambda \sum_{k=1}^{n} Q^*_{n-k,j}(\theta)\, g_k - Q_{n,j-1}(0)\; V^*(\theta)$$

$$j \geq 2, \quad 1 \leq n \leq m-1 \qquad (3.2.4)$$

$$\theta\, Q^*_{n,j}(\theta) - Q_{n,j}(0) = \lambda\, Q^*_{n,j}(\theta) - \lambda \sum_{k=1}^{n} Q^*_{n-k,j}(\theta)\, g_k \qquad j \geq 2,\; n \geq m \qquad (3.2.5)$$

$$\theta\, D_m^*(\theta) - D_m(0) \quad = \quad \lambda\, D_m^*(\theta) - \sum_{j=1}^{\infty} Q_{m,j}(0)\, D^*(\theta) \qquad (3.2.6)$$

$$\theta\, D_n^*(\theta) - D_n(0) \quad = \quad \lambda\, D_n^*(\theta) - \sum_{k=1}^{n-m} D_{n-k}^*(\theta)\, g_k - \sum_{j=1}^{\infty} Q_{n,j}(0)\, D^*(\theta)$$

$$n \geq m+1 \qquad (3.2.7)$$

$$\lambda\, U_m \quad = \quad D_m(0) \qquad (3.2.8)$$

$$\lambda\, U_n \quad = \quad D_n(0) + \lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k \qquad m+1 \leq n \leq N-1 \qquad (3.2.9)$$

$$\theta\, P_{1,1}^*(\theta) - P_{1,1}(0) \quad = \quad \lambda\, P_{1,1}^*(\theta) - P_{2,1}(0)\, S_1^*(\theta)\, (1-r) - P_{2,2}(0)\, S_1^*(\theta) \quad (3.2.10)$$

$$\theta\, P_{n,1}^*(\theta) - P_{n,1}(0) \quad = \quad \lambda\, P_{n,1}^*(\theta) - (1-r)\, P_{n+1,1}(0)\, S_1^*(\theta) - \lambda \left(\sum_{k=1}^{n-1} P_{n-k,1}^*(\theta)\, g_k \right)$$

$$- P_{n+1,2}^*(\theta)\, S_1^*(\theta) \qquad 2 \leq n \leq N\text{-}1 \qquad (3.2.11)$$

$$\theta\, P_{n,1}^*(\theta) - P_{n,1}(0) = \lambda P_{n,1}^*(\theta) - D_n(0)\, S_1^*(\theta) - (1-r)\, P_{n+1,1}(0)\, S_1^*(\theta) - P_{n+1,2}(0)\, S_1^*(\theta)$$

$$-\lambda \sum_{k=1}^{n-1} P_{n-k,1}^*(\theta)\, g_k - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\, S_1^*(\theta) \qquad n \geq N \quad (3.2.12)$$

$$\theta\, P_{1,2}^*(\theta) - P_{1,2}(0) \quad = \quad \lambda\, P_{1,2}^*(\theta) - r\, P_{1,1}(0)\, S_2^*(\theta) \qquad (3.2.13)$$

$$\theta\, P_{n,2}^*(\theta) - P_{n,2}(0) \quad = \quad \lambda\, P_{n,2}^*(\theta) - r\, P_{n,1}(0)\, S_2^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,2}^*(\theta)\, g_k \qquad n \geq 2$$

$$(3.2.14)$$

III Probability Generating Functions

Since equations (3.2.8) to (3.2.14) are similar to the equations (3.1.7) to (3.1.13) of single vacation model of section (3.1), the partial probability generating functions of the system size, when the server is in busy states with FES, SOS and in dormant state are obtained, using equations (3.2.8) to (3.2.14) (i.e), $$P_1(z,0) = \frac{z\, S_1^*(w_X(z))\,[D(z,0) - P_1(0) - U(z)\, w_X(z)]}{(z - S_{sos}^*(w_X(z)))} \tag{3.2.15}$$

$$P_1^*(z,\theta) = \frac{z\,(S_1^*(w_X(z)) - S_1^*(\theta))\,[D(z,0) - P_1(0) - U(z)\, w_X(z)]}{(\theta - w_X(z))\,(z - S_{sos}^*(w_X(z)))} \tag{3.2.16}$$

$$P_2(z,0) = r\, S_2^*\,(w_X(z))\, P_1(z,0) \tag{3.2.17}$$

$$P_2^*(z,\theta) = r\left(\frac{S_2^*(w_X(z)) - S_2^*(\theta)}{(\theta - w_X(z))}\right) P_1(z,0) \tag{3.2.18}$$

where $P_1(0) = (1-r)\, P_{1,1}(0) + P_{1,2}(0)$,

$$S_{sos}^*(\theta) = S_1^*(\theta)\,(\,(1-r) + r\, S_2^*(\theta)\,) \tag{3.2.19}$$

and $$\lambda\, U_n = \sum_{k=m}^{n} D_k(0)\, \pi_{n-k} \qquad (\, m \le n \le N\text{-}1) \tag{3.2.20}$$

with $\pi_0 = 1,\; \pi_n = \sum_{k=1}^{n} \pi_{n-k}\, g_k \qquad (0 \le n \le m-1)$

The equations (3.2.1) to (3.2.5) are used to obtain $Q_j(z,0)$ and $Q_j^*(z,0)$ where,

$$Q_j(z,0) = \sum_{n=0}^{\infty} Q_{n,j}(0)\, z^n \;;\; Q_j^*(z,\theta) = \sum_{n=0}^{\infty} Q_{n,j}^*(\theta)\, z^n \qquad j \ge 1$$

The equations (3.2.1) and (3.2.2) imply

$$Q_1(z,0) = P_1(0)\;\; V^*(w_X(z)) \tag{3.2.21}$$

and $$Q_1^*(z,\theta) = P_1(0)\, \frac{[V^*(w_X(z)) - V^*(\theta)]}{(\theta - w_X(z))} \tag{3.2.22}$$

Equations (3.2.3) to (3.2.5) lead to

$$Q_j(z,0) = V^*(w_X(z)) \sum_{n=0}^{m-1} Q_{n,\,j-1}(0)\, z^n \qquad j \ge 2 \tag{3.2.23}$$

and $$Q_j^*(z,\theta) = \frac{[V^*(w_X(z)) - V^*(\theta)]}{(\theta - w_X(z))} \sum_{n=0}^{m-1} Q_{n,\,j-1}(0)\, z^n \qquad j \ge 2 \tag{3.2.24}$$

The sum of equations (3.2.21) and (3.2.23) over j = 1 to ∞ yields

$$Q(z, 0) = \sum_{j=1}^{\infty} Q_j(z,0) = V^*(w_X(z))\, y(z) \qquad (3.2.25)$$

where $$y(z) = P_1(0) + \sum_{j=1}^{\infty}\sum_{n=0}^{m-1} Q_{n,j}(0)\, z^n \qquad (3.2.26)$$

Similarly the addition of (3.2.22) and (3.2.24) imply

$$\sum_{j=1}^{\infty} Q_j^*(z,\theta) = \frac{[V^*(w_X(z)) - V^*(\theta)]}{(\theta - w_X(z))}\,(y(z)) \qquad (3.2.27)$$

Next to calculate the PGF corresponding to setup state, equations (3.2.6) and (3.2.7) are used. And these equations give,

$$(\theta - w_X(z))\, D^*(z,\theta) = D(z, 0) - D^*(\theta)\left[\sum_{j=1}^{\infty}\left(Q_j(z,0) - \sum_{n=0}^{m-1} Q_{n,j}(0)\, z^n\right)\right]$$

Then at $\theta = w_X(z)$, and using equation (3.2.25),

$$D(z, 0) = D^*(w_X(z))\left[(V^*(w_X(z)) - 1)\, y(z) + P_1(0)\right] \qquad (3.2.28)$$

and $$D^*(z,\theta) = \frac{\left(D^*(w_X(z)) - D^*(\theta)\right)}{(\theta - w_X(z))}\left[(V^*(w_X(z)) - 1)\, y(z) + P_1(0)\right] \qquad (3.2.29)$$

Next to evaluate y(z), the following arguments are used. By conditioning on the arrival size during previous vacation

$$Q_{n,1}(0) = \alpha_n P_1(0) \qquad n \geq 0 \qquad (3.2.30)$$

and $$Q_{n,j}(0) = \sum_{k=0}^{n} \alpha_k Q_{n-k,j-1}(0) \qquad j \geq 2,\ 0 \leq n \leq m-1 \qquad (3.2.31)$$

where α_n denotes the probability that n customers arrive during a vacation V.

Further adding equations (3.2.30) and (3.2.31) over $j \geq 1$, for $0 \leq n \leq m-1$,

$$\sum_{j=1}^{\infty} Q_{n,j}(0) = \alpha_n P_1(0) + \sum_{j=1}^{\infty}\sum_{k=0}^{n} \alpha_k Q_{n-k,j}(0) \qquad (3.2.32)$$

By letting $\delta_0 = P_1(0) + \sum_{j=1}^{\infty} Q_{0,j}(0)$ and

$$\delta_n = \sum_{j=1}^{\infty} Q_{n,j}(0) \qquad 1 \le n \le m-1$$

the equation (3.2.32) leads to

$$\delta_0 = \frac{P_1(0)}{(1-\alpha_0)} \quad (\text{at } n = 0)$$

and for $1 \le n \le m-1$

$$\delta_n = \alpha_n P_1(0) + \sum_{k=0}^{n-1} \alpha_k \delta_{n-k} + \alpha_n \sum_{j=1}^{\infty} Q_{0,j}(0)$$

$$= \alpha_n \delta_0 + \sum_{k=0}^{n-1} \alpha_k \delta_{n-k} = \sum_{k=0}^{n} \alpha_k \delta_{n-k}$$

For further simplification β_n's are introduced (as in section (2.2)) as , $\beta_0 = 1;\ \beta_n = \sum_{j=1}^{n} \frac{\alpha_j \beta_{n-j}}{(1-\alpha_0)}$ $(1 \le n \le m)$. Also by following the argument of theorem (2.2.1) of section (2.2), δ_n's can be expressed in terms of β_n's by induction, as

$$\delta_n = \frac{\beta_n}{(1-\alpha_0)} P_1(0) \qquad (0 \le n \le m-1)$$

Thus $y(z) = P_1(0) + \sum_{j=1}^{\infty} \sum_{n=0}^{m-1} Q_{n,j}(0) z^n$ implies

$$y(z) = \sum_{n=0}^{m-1} \delta_n z^n = P_1(0) \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)}$$

Substituting for y(z) in equations (3.2.27), (3.2.29) and (3.2.28),

$$\sum_{j=1}^{\infty} Q_j^*(z,\theta) = \frac{[V^*(w_X(z)) - V^*(\theta)]}{(\theta - w_X(z))} \left(\sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)} \right) P_1(0) \qquad (3.2.33)$$

$$D^*(z,\theta) = \frac{\left(D^*(w_X(z)) - D^*(\theta)\right)}{(\theta - w_X(z))} \left[(V^*(w_X(z)) - 1) \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)} + 1 \right] P_1(0)$$

(3.2.34)

and $$D(z,0) = D^*(w_X(z))\left((V^*(w_X(z))-1)\sum_{n=0}^{m-1}\frac{\beta_n z^n}{(1-\alpha_0)}+1\right)P_1(0) \qquad (3.2.35)$$

For further simplification of U_n of equation (3.2.20) , theorem (2.2.2) is used.

$$D_k(0) = \text{co-eff. of } z^k \text{ in } D(z, 0)$$

$$= P_1(0) \text{ co-eff. of } z^k \left[D^*(w_X(z))\left((V^*(w_X(z))-1)\sum_{n=0}^{m-1}\frac{\beta_n z^n}{(1-\alpha_0)}+1\right)\right]$$

$D_k(0) = P_1(0)\sum_{i=m}^{k} S_i h_{k-i}$ where $S_i = \sum_{j=0}^{m-1}\frac{\alpha_{i-j}\beta_j}{(1-\alpha_0)}$, follows from theorem(2.2.2)(ii).

Substituting for $D_k(0)$ in (3.2.20)

$$\lambda U_n = P_1(0)\sum_{k=m}^{n}\pi_{n-k}\left(\sum_{i=m}^{k} S_i h_{k-i}\right)$$

$$= P_1(0)\sum_{k=m}^{n} S_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i}$$

$$= P_1(0)\,\phi_n^R$$

(i.e.) $$U(z) = P_1(0)\sum_{n=m}^{N-1}\frac{\phi_n^R z^n}{\lambda}$$

$$= P_1(0)\,\phi_R(z) \qquad (3.2.36)$$

where $\phi_R(z) = \sum_{n=m}^{N-1}\frac{\phi_n^R z^n}{\lambda}$; $\phi_n^R = \sum_{k=m}^{n} S_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i}$

By using equations (3.2.35) and (3.2.36) and by defining ,

$$I_R(z) = \left[D^*(w_X(z))\left(\frac{1-V^*(w_X(z))}{w_X(z)}\right)\sum_{n=0}^{m-1}\frac{\beta_n z^n}{(1-\alpha_0)} + \frac{(1-D^*(w_X(z)))}{w_X(z)} + \sum_{n=m}^{N-1}\frac{\phi_n^R z^n}{\lambda}\right]$$

(3.2.37)

We have, $D(z, 0) - P_1(0) - U(z)\, w_X(z) = -P_1(0)\, w_X(z)\, I_R(z)$, (3.2.38)

Thus equations (3.2.16) and (3.2.18) imply,

$$P_1^*(z,\theta) = \frac{z(S_1^*(w_X(z)) - S_1^*(\theta))}{(\theta - w_X(z))} \frac{(-P_1(0)(w_X(z))\, I_R(z))}{(z - S_{sos}^*(w_X(z)))} \tag{3.2.39}$$

and $$P_2^*(z,\theta) = \frac{z\, r\, S_1^*(w_X(z))(S_2^*(w_X(z)) - S_2^*(\theta))}{(\theta - w_X(z))} \frac{(-P_1(0)(w_X(z))\, I_R(z))}{(z - S_{sos}^*(w_X(z)))} \tag{3.2.40}$$

Hence the partial PGFs of the system size for the proposed multiple vacation model are obtained from equations (3.2.33, 3.2.34, 3.2.36, 3.2.39, and 3.2.40) as,

$$\left.\begin{aligned}
\sum_{j=1}^{\infty} Q_j^*(z,0) &= P_1(0) \frac{(1 - V^*(w_X(z)))}{w_X(z)} \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)} \\
D^*(z,0) &= P_1(0) \frac{(1 - D^*(w_X(z)))}{w_X(z)} \left[(V^*(w_X(z)) - 1) \sum_{n=0}^{m-1} \frac{\beta_n z^n}{(1-\alpha_0)} + 1 \right] \\
U(z) &= P_1(0) \sum_{n=m}^{N-1} \phi_n^R z^n (1/\lambda) \qquad = P_1(0)\, \phi_R(z) \\
P_1^*(z,0) &= P_1(0) \frac{z(S_1^*(w_X(z)) - 1) I_R(z)}{(z - S_{sos}^*(w_X(z)))} \\
P_2^*(z,0) &= P_1(0) \frac{z\, r\, S_1^*(w_X(z))(S_2^*(w_X(z)) - 1)}{(z - S_{sos}^*(w_X(z)))} I_R(z)
\end{aligned}\right\} \tag{3.2.41}$$

Hence the PGF of the system size when the server is in busy state ($P_B(z)$) and in idle state ($P_I(z)$) are given by ,

$$P_B(z) = P_1^*(z,0) + P_2^*(z,0) = \frac{z(S_{sos}^*(w_X(z)) - 1)}{(z - S_{sos}^*(w_X(z)))} P_1(0)\, I_R(z)$$

and $$P_I(z) = \sum_{j=1}^{\infty} Q_j^*(z,0) + D^*(z,0) + U(z) = P_1(0)\, I_R(z)$$

Thus the total PGF $\mathbf{P}_{sos(m,N)}^{R}(z) = P_B(z) + P_I(z)$ gives,

$$\mathbf{P}_{sos(m,N)}^{R}(z) = P_1(0) \frac{(z-1)\, S_{sos}^*(w_X(z))}{(z - S_{sos}^*(w_X(z)))} I_R(z) \tag{3.2.42}$$

IV Performance Measures

Let P_v, P_{set}, P_{dor} P_I and P_{busy} denote the probability that the server is in vacation state, setup state, dormant state, idle state and in busy state respectively. Then the equation (3.2.41) gives,

(i) $$P_v = P_1(0)\, E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)}$$

(ii) $$P_{set} = P_1(0)\, E(D)$$

(iii) $$P_{dor} = P_1(0) \sum_{n=m}^{N-1} \phi_n^R\, (1/\lambda) \text{ and}$$

(iv) $$P_{busy} = P_1(0) \left(\frac{\rho_{sos}}{1-\rho_{sos}} \right) I_R(1)$$

(v) $$P_I = P_1(0)\, I_R(1)$$

where $\rho_{sos} = \lambda\, E(X)\, E(S) = \lambda\, E(X)\, (E(S_1) + r\, E(S_2))$

Also, equation (3.2.37) implies,

$$I_R(1) = E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)} + E(D) + \sum_{n=m}^{N-1} \phi_n^R\, (1/\lambda) = D_R(m, N) \tag{3.2.43}$$

Thus the normalizing condition gives,

$$P_1(0) = \frac{(1-\rho_{sos})}{D_R(m,N)} \tag{3.2.44}$$

and hence $P_{busy} = \rho_{sos}$

By substituting $P_1(0)$ in the equation (3.2.42) the total PGF of the model can be written as

$$\mathbf{P^R_{sos(m,N)}(z)} = \left(\frac{(1-\rho_{sos})(z-1)\, S^*_{sos}(w_X(z))}{(z - S^*_{sos}(w_X(z)))} \right) \left(\frac{I_R(z)}{I_R(1)} \right) \tag{3.2.45}$$

V Decomposition Property

The system size distribution of the (m,N) policy of multiple vacation model $M^X_{(m,N)}/G_{sos}/1/MV$ is decomposed into two random variables one of which is the system size of classical SOS model $M^X/G_{sos}/1$ system namely $\frac{(1-\rho_{sos})(z-1)S^*_{sos}(w_X(z))}{(z-S^*_{sos}(w_X(z)))}$ (Choudhuryand Paul 2006) and the other is $\frac{I_R(z)}{I_R(1)}$ which gives the PGF of the conditional system size distribution during the server idle period under the steady state condition $\rho_{sos} < 1$.

VI Expected System Size:

Let L_v, L_{set}, L_{dor} and L_{busy} denote the mean number of customers in the system when the server is in vacation, setup, dormant and in busy state respectively. Then the differentiation of equation (3.2.41) at z = 1 imply,

(i) $$L_v = \left(\lambda E(X)\frac{E(V^2)}{2}\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)} + E(V)\sum_{n=0}^{m-1}\frac{n\beta_n}{(1-\alpha_0)}\right)P_1(0)$$

(ii) $$L_{set} = \lambda E(X)\left(\frac{E(D^2)}{2} + E(D)E(V)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)}\right)P_1(0)$$

(iii) $$L_{dor} = \sum_{n=m}^{N-1}\left(\frac{n\phi_n^R}{\lambda}\right)P_1(0)$$

(iv) $$L_{busy} = P_{busy} + P_1(0)\left[\frac{(\lambda E(X(X-1))E(S_{sos}) + (\lambda E(X))^2 E(S^2_{sos}))}{2(1-\rho_{sos})^2}D_R(m, N) + \frac{\rho_{sos}}{(1-\rho_{sos})}L_R(m,N)\right]$$

where $$L_R(m, N) = I'_R(1) = \lambda E(X)\left[\left(\frac{E(V^2)}{2} + E(D)E(V)\right)\sum_{n=0}^{m-1}\frac{\beta_n}{(1-\alpha_0)} + \frac{E(D^2)}{2}\right] + E(V)\sum_{n=0}^{m-1}\frac{n\beta_n}{(1-\alpha_0)} + \sum_{n=m}^{N-1}\frac{n\phi_n^R}{\lambda} \qquad (3.2.46)$$

Let $\mathbf{L^R_{sos(m,N)}}$ denote the expected system size of the model then,

(v) $$\mathbf{L^R_{sos(m,N)}} = L_{sos} + \frac{L_R(m,N)}{D_R(m,N)}$$

where L_{sos} denotes the expected system size of the classical $M^X/G_{sos}/1$ second optional service queueing model given in (3.1.46)

VII Other System Characteristics

Let E(Cycle), E(Busy), E(I) and $E(W_s)$ denote the expected length of cycle, busy period, idle period and the expected waiting time in the system then IV- (ii), (iv) and (v) imply

(i) $E\,(\text{Cycle}) = \dfrac{1}{P_1(0)} = \dfrac{1-\rho_{sos}}{D_R(m,N)}$

(ii) $E(\text{Busy}) = P_{busy}\, E(\text{cycle}) = \dfrac{\rho_{sos}}{(1-\rho_{sos})} D_R(m, N)$

(iii) $E(\text{idle}) = P_I\, E(\text{cycle}) = D_R(m, N)$

(iv) $E(W_s) = \dfrac{L^R_{sos(m,N)}}{\lambda\, E(x)}$ (using little's formula)

VIII Queue Size Distribution at Departure Epoch:

If π_j^+ denotes the probability that there are j - customers in the system at departure epoch, then its PGF $\pi_j^+(z)$ for the multiple vacation model is given by $\pi^+(z) = \dfrac{(1-X(z))}{E(X)(1-z)} \mathbf{P}^R_{sos(m,N)}(z)$

3.2.2 Optimal Management Policy

By following the cost structure used in the multiple vacation model of section (2.2) the average cost per unit time for the $M^X_{(m,N)}/G_{sos}/1/MV$ model is given by

$$T_C(m, N) = \frac{C_y}{E(\text{cycle})} + C_h\, L^R_{sos(m,N)} + C_{set}\, P_{set} + C_{dor}\, P_{dor} + C_{build}\, P_{build} + C_{busy}\, P_{busy} - C_v\, P_v$$

By substituting various performance measures,

$$T_C(m, N) = \frac{1}{D_R(m,N)}[A^R_{sos} + Z^R_{sos}(m) + C_h \sum_{n=m}^{N-1} \frac{n\,\phi^R_n}{\lambda} + (1-\rho_{sos}) \sum_{n=m}^{N-1} \frac{\phi^R_n}{\lambda}] + A'_{sos}$$

where $A^R_{sos} = C_y (1-\rho_{sos}) + C_{set}\, E(D)\,(1-\rho_{sos}) + C_h \frac{\lambda\, E(X)}{2} E(D^2)$

$$z^R_{sos}(m) = C_h [\lambda\, E(X)\,(E(D)\,E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)} + \frac{E(V^2)}{2} \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)})$$

$$+ E(V) \sum_{n=0}^{m-1} \frac{n\,\beta_n}{(1-\alpha_0)}] - C_v\, E(V)\,(1-\rho_{sos}) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)}$$

$$A'_{sos} = C_{busy}\, \rho_{sos} + C_h\, L_{sos}$$

and $D_R(m,N) = E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)} + E(D) \sum_{n=m}^{N-1} \phi^R_n\, (1/\lambda)$

By calculation

$$T^R_C(m, k+1) - T^R_C(m, k) = \frac{\phi^R_k}{\lambda\, D_R(m,k)\, D_R(m,k+1)} H_R(m, k)$$

where $H_R(m, k) = (1-\rho_{sos})\, C_{dor}\, \ell^R_m + C_h[k\, \ell^R_m + \sum_{n=m}^{k-1} (k-n) \frac{\varphi^R_n}{\lambda}]$

$$- (A^R_{sos} + z^R_{sos}(m)) > 0$$

$$\ell^R_m = E(D) + E(V) \sum_{n=0}^{m-1} \frac{\beta_n}{(1-\alpha_0)}$$

Then the value of first k, for which $H_R(m, k) > 0$ decides the conditional optimal policy of the model.

(i.e.) $\mathbf{N^*_R(m) = Min\, \{ k \geq 1 \,/\, H_R(m, k) > 0\}}$.

3.3 Particular Cases

(1) Let $M^X_{(m,N)}/G_1+G_2/1/SV$ denote the (m,N) policy single vacation queueing model in which each customer undergoes both types of services one after the other. Then when r = 1,

$$S^*_{G_1+G_2}(w_X(z)) = S^*_1(w_X(z)))\, S^*_2(w_X(z)))$$

and $\rho = E(S) = \lambda E(X)(E(S_1) + E(S_2))$

follow from equations (3.1.31) and (3.1.42) respectively.

Thus the PGF of $M^X_{(m,N)}/G_1+G_2/1/SV$ is given by

$$P^S(z) = \frac{(1-\rho)(z-1)(S^*_{G_1+G_2}(w_X(z)))}{(z-S^*_{G_1+G_2}(w_X(z)))}\frac{I_S(z)}{I_S(1)}$$

(2) Let $M^X_{(m,N)}/G/1/SV$ I denote the single service (m,N) policy queueing model under single vacation .

When r = 0, equations (3.1.31) and (3.1.42)

imply $S^*(w_X(z))) = S^*_1(w_X(z)))$ and $\rho = \lambda E(X)E(S)$.

The PGF is given by

$$P^S(z) = \frac{(1-\rho)(z-1)(S^*(w_X(z)))}{(z-S^*(w_X(z)))}\frac{I_S(z)}{I_S(1)}$$ (Lee *et al.*, 2003)

(3) Let $M^X_{(m,N)}/G_1+G_2/1/MV$ denote the (m,N) policy queueing model with multiple vacation, in which each customer undergoes both type of services one after the other. Then the PGF of the model is obtained by letting r = 1, in equations (3.2.19) and (3.2.45), it is found that

$$P^R(z) = \left(\frac{(1-\rho_{sos})(z-1)\ S^*_{G_1+G_2}(w_X(z))}{(z-S^*_{G_1+G_2}(w_X(z)))}\right)\left(\frac{I_R(z)}{I_R(1)}\right)$$

where $S^*_{G_1+G_2}(w_X(z)) = S^*_1(w_X(z)))\,S^*_2(w_X(z)))$

and $\rho = \lambda E(X)(E(S_1) + E(S_2))$

(4) The $M^X_{(m,N)}/G/1/MV$ queueing model

When r = 0, then the PGF is given by,

$$P^R((z)) = \frac{(1-\rho)(z-1)(S^*(w_X(z)))}{(z-S^*(w_X(z)))}\frac{I_R(z)}{I_R(1)}$$ (Lee *et al.*,2003)

(5) When m= n, the result s of corresponding N-policy queueing models (Lee *et al.*, 1994a) can be obtained.

3.4 NUMERICAL ANALYSIS

In this section some numerical results are presented to study the effect of various parameters namely the batch arrival rate (λ), mean service time of FES ($E(S_1)$), SOS($E(S_2)$) , mean setup time E(D) mean vacation time E(V) and the SOS probability r on the mean system size for single vacation ($L^{S}_{sos(m,N)}$) and multiple vacation ($L^{R}_{sos(m,N)}$) queueing models. The optimal threshold values (m*,N*) of (m,N) policy and the minimum average cost Tc(m*,N*) per unit time are also obtained for both the models. The bi-level policy is compared with the corresponding N policy through the total expected cost per unit time .The values of the parameters are chosen such that they satisfy the stability condition of the model.

For the computation purpose the following distributions are assumed for different random variables.

Random variables	Distribution	Mean	Second order moments
FES S_1	Two stage hyper exponential	$E(S_1)=\frac{a_1}{\mu_{11}}+\frac{1-a_1}{\mu_{12}}$, $0 \le a_1 \le 1$	$E(S_1^2)=2\left(\frac{a_1}{\mu_{11}^2}+\frac{1-a_1}{\mu_{12}^2}\right)$
SOS S_2	Two stage hyper exponential	$E(S_2)=\frac{b_1}{\mu_{21}}+\frac{1-b_1}{\mu_{22}}$, $0 \le b_1 \le 1$	$E(S_2^2)=2\left(\frac{b_1}{\mu_{21}^2}+\frac{1-b_1}{\mu_{22}^2}\right)$
Setup time D	Erlang 3 type distribution	$E(D)=\frac{1}{\nu}$	$E(D^2)=\frac{4}{3\nu^2}$
Vacation V	Erlang 3 type distribution	$E(V)=\frac{1}{\eta}$	$E(V^2)=\frac{4}{3\eta^2}$
Batch size X	Geometric distribution	$E(X)=\frac{1}{(1-p)}$	$E(X(X-1))=\frac{2p}{(1-p)^2}$
	Binomial distribution	$E(X)= n\,p$	$E(x^2)=n\,p\,(q+n\,p)$

Applying the algorithms of sections (3.1.2) and (3.2.2) the conditional optimal values (m,N*(m)) and Tc(m,N*(m)) are calculated and presented in table (3.1) and in figures (3.1a) and (3.1b) for both the models, for a given set of parameters and cost elements. The batch size follows Geometric distribution G(p) in tables (3.1) to (3.7). All the parametric values and the cost elements except C_h are chosen the same for both the models in Table (3.1) C_h is taken as 300 for single vacation and 60 for multiple vacation model.

Table 3.1 : Optimum values (m ,N*(m)) and Tc(m ,N*(m)) for each m

$(C_{build}, C_{set}, C_{dor}, C_{busy}, C_y\ C_v)$ =(8, 50, 500, 1500 ,10000 , 8)

$(p, \lambda, v, \eta, r_1\ E(S))$ = (.75, .2, 1, .1, .2, .5).

(m,N*)	(1,8)	(2,8)	(3,8)	(4,8)	(5,8)	(6,8)
Tc(m,N*)	3485.54	3477.31	3469.25	3461.97	3456.18	3452.78
(m,N*)	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
Tc(m,N*)	1526.28	1522.99	1520.13	1517.96	1516.71	1516.59
(m,N*)	(7,8)	(8,8)	(9.9)	(10,10)	(11,11)	(12,12)
Tc(m,N*)	3462.68	3487.68	3526.39	3577.69	3640.26	3712.86
(m,N*)	(7,8)	(8,8)	(9,9)	(10,10)	(11,11)	(12,12)
Tc(m,N*)	1517.77	1520.38	1525.13	1531.78	1540.03	1549.69

Single vacation   Multiple vacation

Fig. (3.1a) convex curve for single vacation

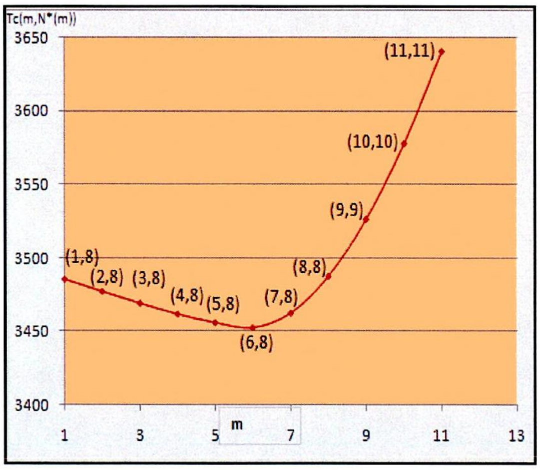
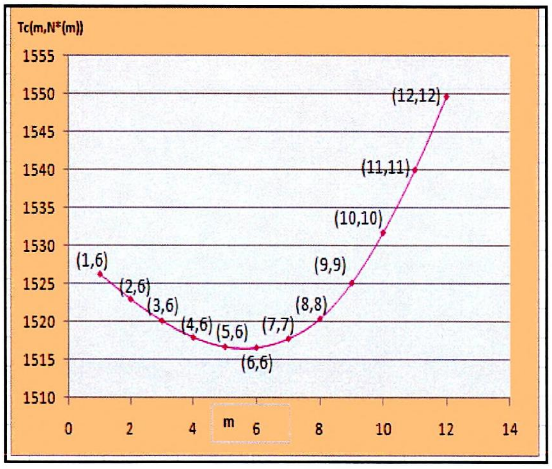


Fig. (3.1b) convex curve for multiple vacation



The expected cost Tc(m,N) is represented in figures (3.2 a) and (3.2b) for single and multiple vacation models respectively , for different values of m and N . The minimum cost **1951.694** per unit time is achieved at m*=11 and N*=14 for single vacation model (fig 3.2a) and **1725.167** at m*=5 and N*=8 for multiple vacation model (fig 3.2b). This justifies the solution procedure given in sections (3.1.2) and (3.2.2) .The numeric values corresponding to the figures (3.2a) and (3.2b) for a given set of parameters and cost elements are listed in the table (3.2).

Table 3.2 : The expected cost Tc(m,N) for single and multiple vacation model

$(C_{build}, C_{set}, C_{dor}, C_{busy}, C_y\ C_v)$ =(8, 1000, 500, 1000 ,10000 , 8) $(p, \lambda, v, \eta, r, E(S_{sos}))$ = (.75, .4, 1, .5, .2, 5)

m \ N	5	6	7	8	9	10	11	12	13	14	15
5	2013.07	2002.45	1991.87	1981.90	1972.97	1965.51	1959.68	1955.69	1953.63	1953.58	1955.54
	1738.09	1731.06	1726.41	**1725.16**	1727.92	1734.92	1746.15	1761.42	1780.44	1802.86	1828.34
6		1999.35	1989.65	1980.29	1971.81	1964.58	1958.91	1955.01	1952.99	1952.95	1954.92
		1730.97	1727.30	1726.24	1728.59	1734.82	1745.07	1759.29	1777.28	1798.74	1823.36
7			1987.05	1978.49	1970.53	1963.65	1958.18	1954.39	1952.43	1952.40	1954.35
			1731.69	1730.74	1732.58	1737.86	1746.92	1759.85	1776.54	1796.79	1820.29
8				1976.47	1969.19	1962.74	1957.54	1953.89	1951.89	1951.97	1953.89
				1738.74	1740.01	1744.23	1751.92	1763.34	1778.47	1796.79	1820.29
9					1967.79	1961.89	1957.03	1953.55	1951.73	1951.81	1953.60
					1750.95	1754.07	1760.26	1769.97	1783.34	1800.35	1820.85
10						1961.16	1956.71	1953.44	1951.71	1951.71	1953.53
						1767.45	1772.10	1779.97	1791.36	1806.39	1824.99
11							1956.64	1953.66	1952.02	**1951.69**	1953.77
							1787.52	1793.48	1802.76	1815.59	1832.04
12								1954.27	1952.77	1952.73	1954.40
								1810.62	1817.71	1828.19	1842.26
13									1954.04	1953.99	1955.54
									1836.29	1844.35	1855.88
14										1955.93	1957.32
										1864.19	1873.08
15											1951.88
											1893.99

C_h = 300 for Single vacation (blue) C_h = 60 for Multiple vacation (red)

Fig. (3.2) The expected cost Tc(m,N) for different values of m,N

Fig. (3.2a) Single vacation model

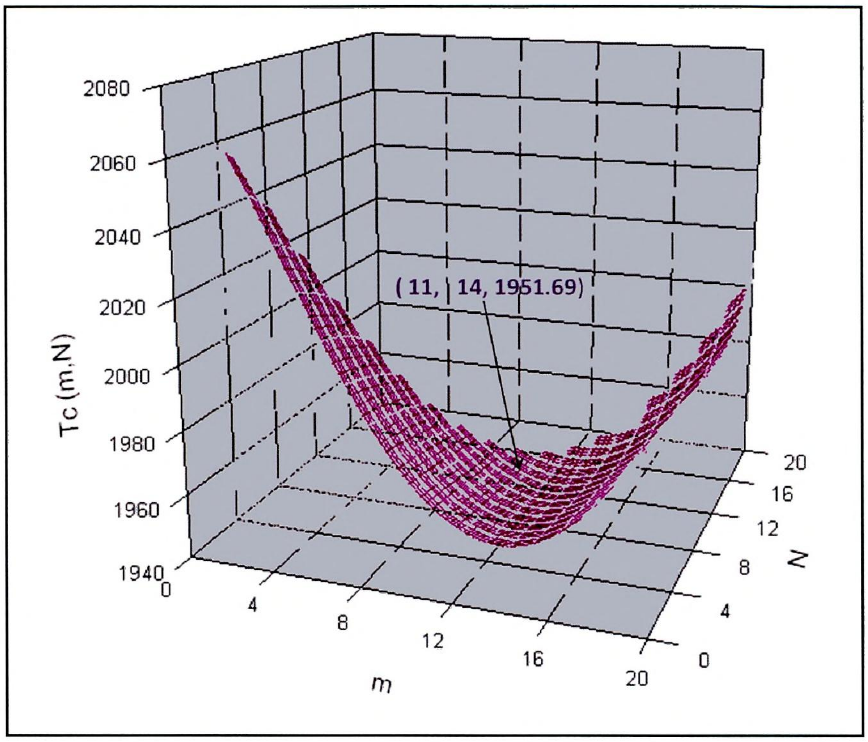


Fig (3.2b) multiple vacation model

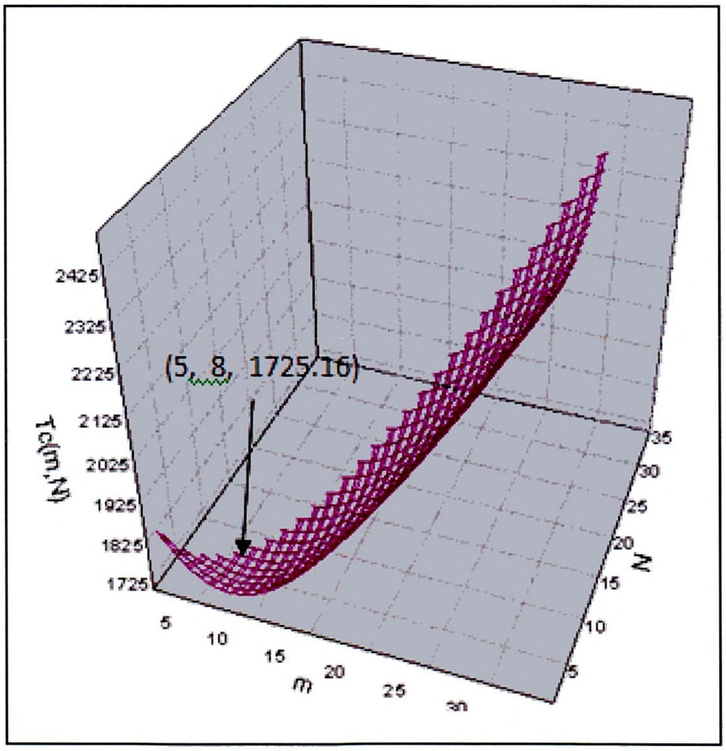


Table (3.3) gives the optimal values as the standby cost (dormant cost) C_{dor} changes. For the queueing system with (N,N) policy, since there is no stand by period, (N* , Tc(N,N)*) of (N, N) policy do not vary with C_{dor} . It is noted that, as C_{dor} increases, the optimal values of (m, N) policy approach the optimal value of (N,N) policy. It is interesting to note that in all the tables, (m, N) policy provides lower average cost than the single threshold (N, N) policy.

Table 3. 3 : Optimal policy as dormant cost changes

(C_{build}, C_{set} , C_h, C_{busy}, C_y C_v) =(8, 50, 300, 1500 ,10000 , 8) and

(p, λ ,v , η ,r, $E(S_{sos})$) = (.75, .2, 1, .1, .2, .5).

C_{dor}	m*,N*	Tc(m*,N*)	$L_{sos(m^*,N^*)}$	E(cycle)	N*	Tc(N*,N*)	$L_{sos(N^*)}$	E(cycle)
100	(1,10)	5784.11	16.19	36.42	8	5833.2	16.38	37.1
	(1,12)	2463.21	15.44	38.67	3	2663.81	17.09	38.01
300	(1,9)	5797.35	16.17	36.43	8	5833.2	16.38	37.1
	(1,11)	2476.58	15.48	38.71	3	2663.81	17.09	38.01
500	(3,9)	5808.63	16.20	36.6	8	5833.2	16.38	37.1
	(1,10)	2489.12	15.54	38.79	3	2663.81	17.09	38.01
1000	(6,9)	5827.09	16.28	37.15	8	5833.2	16.38	37.1
	(1,9)	2514.85	15.58	38.92	3	2663.81	17.09	38.01
1500	(7,8)	5833.36	16.33	37.18	8	5833.2	16.38	37.1
	(1,7)	2533.82	15.59	38.96	3	2663.81	17.09	38.01
2500	(8,8)	5835.28	16.38	37.18	8	5833.2	16.38	37.1
	(3,3)	2547.98	16.24	38.96	3	2663.81	17.09	38.01
5000	(8,8)	5835.28	16.38	37.18	8	5833.2	16.38	37.1
	(3,3)	2547.98	16.24	38.99	3	2663.81	17.09	38.01

■ Single vacation ■ Multiple vacation

Table (3.4) presents optimal values as the holding cost increases. When the startup cost per cycle is very high , the server would not initiate service until a large number of customers accumulate. The values of table (3.4) show that as the holding cost increases, the system starts setup and service earlier than in the case of lower holding cost. Thus the holding cost exceeds the effect of startup cost.

Table 3. 4 : Optimal policy as holding cost changes

The cost values and the parameters are as in Table (3.3) with C_{dor} = 100

C_h	m*,N*	Tc(m*,N*)	$L_{sos(m^*,N^*)}$	N*	Tc(N*,N*)	$L_{sos(N^*)}$
50	(9,15)	1906.72	17.39	13	1908.79	17.48
	(1,15)	1911.53	17.53	3	1962.94	18.27
100	(4,13)	2769.05	17.20	11	2777.98	17.33
	(1,13)	2784.83	17.43	1	2872.87	18.11
500	(1,12)	9633.81	17.16	9	9698.85	17.29
	(1,12)	9542.24	17.42	1	10117.53	18.11
1000	(1,11)	18211.98	17.15	9	18343.9	17.29
	(1,12)	18459.64	17.42	1	19173.35	18.11
5000	(1,11)	86827.41	17.15	9	87504.27	17.29
	(1,12)	88118.81	17.41	1	91619.95	18.11
10000	(1,11)	172596.69	17.15	9	17359.74	17.29
	(1,12)	175192.78	17.41	1	182178.19	18.11

■ Single vacation ■ Multiple vacation

Table (3.5) shows that as the mean setup time increases, an early setup is more beneficial than usual N policy. This is because more customers arrive during the setup time. The effect of the mean vacation time E(V) on the threshold policies , mean system size and the total average cost per unit time is shown in table (3.6).The Tables 3.5 and 3.6 show that the system size increases as the mean setup time E(D) or mean vacation time E(V) increases. The parameters used to construct Table (3.5) are same as in Table (3.1) except, C_{dor} = 100 and η=.1.

Table 3.5 : Optimal policy as mean setup time E(D) changes

v	E(D)	(m*,N*)	Tc(m*,N*)	$L_{sos(m^*,N^*)}$	N*	Tc(N*,N*)	$L_{sos(N^*)}$
.5	2	(6,12)	2720.35	16.8	11	2725.55	16.9
		(1,13)	2737.45	17.03	3	2814.69	17.79
.33	3	(4,13)	2769.05	17.21	11	2779.41	17.34
		(1,14)	2786.13	17.44	2	2862.61	18.09
.2	5	(1,14)	2862.27	17.96	10	2882.52	18.18
		(1,14)	2871.13	18.23	2	2954.56	18.87
.1	10	(1,16)	3102.57	20.03	9	3141.95	20.41
		(1,17)	3122.29	20.35	1	3195.03	20.99

■ Single vacation ■ Multiple vacation

Table 3.6 : Optimal policy as mean vacation time E(V) changes

(C_h , C_{build}, C_{set} , C_h, C_{busy}, C_y C_v) = (300, 8, 500, 300, 1500 ,10000 , 8)

and (p, λ ,v , r, E(S_{sos})) = (.75, .2, 1, .2, .5).

η	E(V)	(m*,N*)	Tc(m*,N*)	$L_{sos(m*,N*)}$	N*	Tc(N*,N*)	$L_{sos(N*)}$
.05	20	**(11,11)**	**4665.01**	**13.37**	**11**	**4655.12**	**13.35**
		(1,11)	**4688.17**	**13.34**	**1**	**4953.04**	**14.27**
.1	10	**(6,6)**	**3272.58**	**8.17**	**6**	**3271.83**	**8.17**
		(1,6)	**3337.05**	**8.29**	**2**	**3553.58**	**9.08**
.5	2	**(4,4)**	**2552.59**	**4.96**	**4**	**2552.59**	**4.96**
		(1,4)	**2550.73**	**4.72**	**3**	**2736.34**	**5.09**
1	1	**(4,4)**	**2528.76**	**4.84**	**4**	**2528.76**	**4.84**
		(2,4)	**2490.03**	**4.54**	**4**	**2696.52**	**4.62**
10	.1	**(3,3)**	**2522.07**	**4.47**	**3**	**2522.07**	**4.47**
		(3,3)	**2423.46**	**4.19**	**4**	**2697.86**	**4.08**

Single vacation   Multiple vacation

Further Table (3.7) gives the optimal threshold policies as λ increases It shows that m decreases as λ increases .The optimal queue length $L_{sos(m*,N*)}$ and the optimal cost Tc (m*,N*) also increase along with λ. The parameters chosen are as in Table (3.6) with η = 1.

Table 3.7: Optimal policy as λ changes

λ	(m*,N*)	Tc(m*,N*)	$L_{sos(m*,N*)}$
.1	(6,6)	942.93	3.91
	(5,6)	899.245	3.21
.2	(5,10)	1596.19	8.31
	(3,9)	1443.09	5.79
.3	(4,12)	2313.63	14.27
	(1,9)	1929.78	8.47
.4	(1,14)	3414.14	24.6
	(1,19)	2598.67	14.47
.5	(1,16)	6543.68	55.59
	(1,7)	4631.17	34.99

Single vacation   Multiple vacation

The effects of batch size E(X) and the probability of opting second optional service (r) on system size $L_{sos(m,N)}$ and optimal cost value Tc(m*,N*) are shown in Table (3.8). It is found that, L and Tc(m*,N*) increase as r or E(X) increases. The numerical values obtained for two different batch size of Geometric distributions G(p) and Binomial distributions B(3,p) are shown in Table (3.8) and in fig (3.3a,3.3b).

Table 3.8 : Optimal policy with respect to r and p for for Geometric G(p) and Binomial distributions B(n,p)

$(C_h, C_{build}, C_{set}, C_{dor}, , C_{busy}, C_y, C_v) = (300, 8, 1000, 500, 1000, 10000, 8)$; $(\lambda, \nu, \eta\, E(S_{sos})) = (.2, 1, .5, .7)$

r	Single vacation						Multiple vacation					
	p=.5			p=.75			p=.5			p=.75		
	(m*,N*)	$L_{sos(m,N)}$	Tc((m*,N*)	(m*,N*)	$L_{sos(m,N)}$	Tc((m*,N*)	(m*,N*)	$L_{sos(m^*,N^*)}$	Tc((m*,N*)	(m*,N*)	$L_{sos(m,N)}$	Tc((m*,N*)
.1	(4,4)	2.05	1462.01	(4,4)	4.34	2371.21	(4,4)	2.35	1483.74	(1,4)	4.07	2368.47
	(4,4)	2.29	1548.43	(4,4)	2.93	1943.74	(3,3)	2.27	1595.2	(1,4)	2.57	1908.53
.3	(4,4)	2.24	1527.49	(3,3)	5.41	2763.46	(3,3)	2.14	1550.79	(1,4)	5.48	2767.69
	(3,3)	2.07	1614.63	(3,3)	3.16	2073.91	(2,3)	2.18	1669.37	(1,4)	3.14	2052.09
.5	(4,4)	2.44	1598.84	(3,3)	7.37	3325.45	(3,3)	2.36	1624.72	(1,3)	7.39	3351.42
	(3,3)	2.28	1679.32	(1,3)	3.68	2263.32	(2,3)	2.43	1744.68	(1,3)	3.89	2277.49
.7	(3,3)	2.54	1672.43	(1,3)	10.3	4233.05	(3,3)	2.61	1705.63	(1,3)	10.67	4302.58
	(3,3)	2.54	1756.77	(1,3)	5.08	2629.89	(1,3)	2.49	1831.11	(1,3)	5.45	2711.9
.9	(3,3)	2.62	1750.82	(1,2)	16.19	5985.63	(3,3)	2.89	1797.43	(1,3)	16.93	6146.33
	(3,3)	2.86	1854.27	(1,2)	8.71	3684.82	(1,3)	2.89	1938.96	(1,3)	9.67	3940.24

■ Single vacation ■ Multiple vacation

The system size Vs r for different values of E(X) for two distributions

Fig. (3.3a) Single vacation model

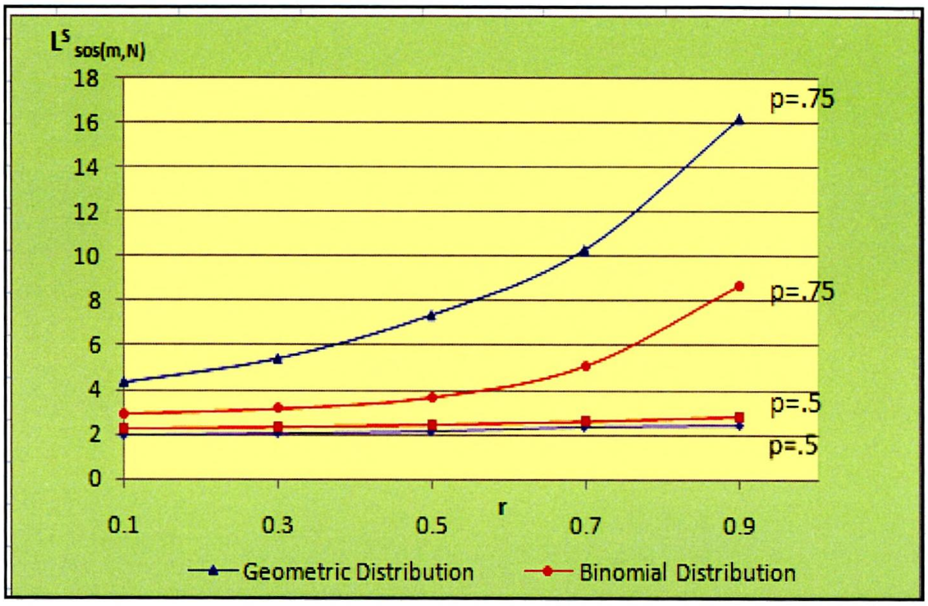


Fig. (3.3b) Multiple vacation model

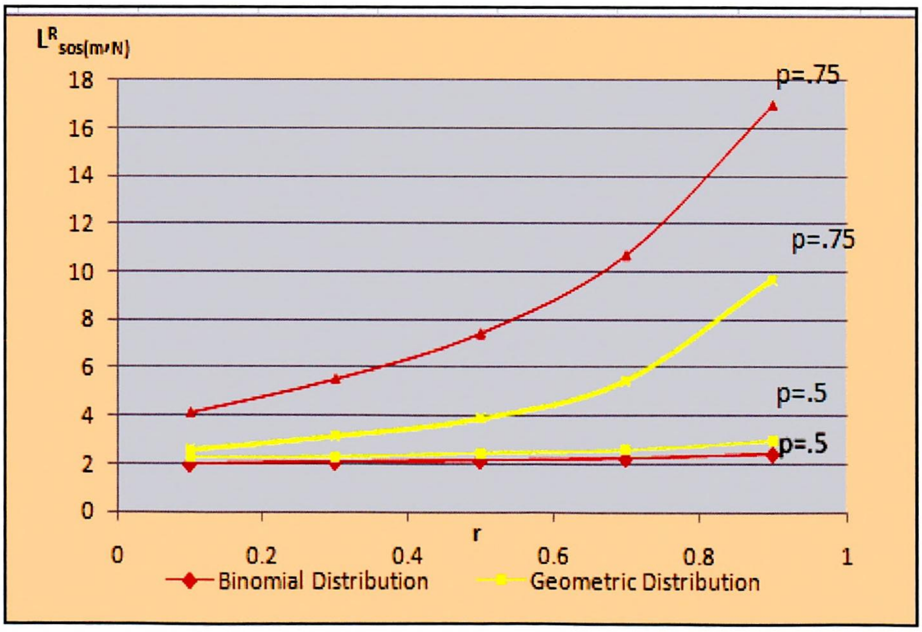


Table (3.9) gives the values of system size probabilities and the expected system size, when the server is in different states, for different values of λ and two different values of SOS probability r. The values are presented for two batch size distributions namely (i) Geometric distributions G(.75) (cases 1 to 4) and (ii) Binomial distribution B(3,.75) (cases5 to 8).

Case 1: r =.1, λ =.2, Geo (.75) ; Case 2: r =.1, λ =.4, Geo(.75) ;
Case 3: r =.35, λ =.2, Geo (.75) ; Case 4: r =.35, λ =.4, Geo(.75) ;
Case 5: r =.1, λ =.2, B (3, .75) ; Case 6: r =.1, λ =.3, B(3, .75) ;
Case 7: r =.35, λ =.2, B (3, .75) ; Case 8: r =.35, λ =.3 B(3, .75) ;

Table 3.9: various performance measures with respect to r , λ and two different distribution of X.

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v)$ = (300, 8,1000, 500,1000,10000,8);

$(p, v, \eta, E(S_{sos}))$ = (.75, 1, .1, .5).

	P_{busy}	P_{build}	P_{set}	P_{dor}	P_v	L_{busy}	L_{build}	L_{set}	L_{dor}	L_v
Case 1	0.346	.185	0.040	0.028	0.401	4.352	0.445	0.49	0.166	2.140
	0.346	-	0.040	0.099	0.515	4.143	-	0.434	0.324	2.747
Case 2	0.692	0.025	0.024	0.024	0.235	20.63	0.067	0.443	0.019	2.512
	0.692	-	0.023	0.040	0.246	17.91	-	0.417	0.226	2.622
Case 3	0.479	0.147	0.032	0.022	0.32	6.79	0.355	0.390	6.792	1.71
	0.479	-	0.032	0.079	0.410	6.704	-	0.345	0.258	2.187
Case 4	0.959	0.001	0.003	0.006	0.031	139.69	0	0.005	0.037	0.339
	0.959	-	0.003	0.005	0.032	125.938	-	0.057	0.030	0.358
Case 5	0.389	0.173	0.037	0.026	0.375	3.23	0.416	0.515	0.155	2.249
	0.389	-	0.036	0.163	0.412	4.597	-	0.33	0.328	2.091
Case 6	0.584	0.063	0.063	0.031	0.259	8.168	0.155	0.507	0.216	2.657
	0.584	-	0.028	0.069	0.320	6.882	-	0.457	0.394	2.877
Case 7	0.539	0.113	0.030	0.02	0.298	4.688	0.211	0.388	0.098	1.787
	0.539	-	0.027	0.099	0.335	4.597	-	0.33	0.328	2.091
Case8	0.809	0.006	0.014	0.035	0.137	15.153	0	0.205	0.174	1.233
	0.809	-	0.013	0.027	0.151	13.407	-	0.210	0.181	1.321

■ Single vacation ■ Multiple vacation