Introduction

INTRODUCTION

"What makes society turn is science and the language of science is math, and the structure of math is logic, and the bedrock of logic is Aristotle, and that's what goes out with fuzzy."

- Bart Kosto

The ordered sets have many times been the object of research of many great topologists - Eilenberg, S., Wolk, E.S., Nachbin, L., Lutzer, D.J., Sierpinski, W., Thron, W.J., Zimmerman, S.J., Kaufman, R., and many others. Nachbin, L., [19] initiated the study of topological ordered space.

In 1965, Zadeh, L.A., [24] defined fuzzy sets with a view to describe, study and formulate mathematically those situations which are imprecise and vaguely defined. Since then, the theory of fuzzy sets has evoked tremendous interest among researchers working in different branches of mathematics.

Using the concept of fuzzy sets Chang, C.L., [8] has developed the theory of fuzzy topological spaces.

The notion of a fuzzy topological vector spaces was given for the first time by Katsaras, A.K., in 1977 [11].

The study of the relationship between fuzzy topology and order was initiated by Katsaras, A.K., in 1981 [13].

Bakier, M.Y., and El-Saady, K., [3] applied the concept of a fuzzy set to the elementary theory of order and vector space structures. They consider the case in which, on a vector space structure, a fuzzy topology has been given as well as an order relation.

The first axiomatic definition of proximity was given by Efremovič, V.A., in 1950. Since then the theory of proximity spaces was developed by Smirnov, Y.M., Pervin,W.J., Leader, S., Gal, Thron, W.J., Lodato, M.W., Harris, D. and many others. A detailed account of these developments are found in the book "Proximity Spaces" by Naimpally, S.A. and Warrack, B.D., [20]. In 1976, Singal, M.K., and Sundar Lal [22] introduced the notion of proximity ordered spaces.

In 1999, Alcantud, J.C.R., analysed the topological properties of a relevant class of topologies associated with spaces ordered by preferences (asymmetric negatively transitive binary relations).

This thesis is devoted to the study of some interesting results on Topological ordered spaces, Proximity ordered spaces, Preference ordered spaces, Ordered fuzzy topological spaces and Fuzzy topological ordered vector spaces.

The first chapter deals with Preliminary definitions and results.

Chapter II is devoted to the study of topological ordered spaces. In this chapter some fundamental definitions related to topological ordered spaces are given. I-continuous, D-continuous, B-continuous, I-open, D-open, B-open, I-closed, D-closed and B-closed maps for topological ordered spaces are introduced and studied. Characterizations of x-continuous maps for x = I, D, B are given in theorems 2.12, 2.13, 2.14. Characterizations of x-open maps for x = I, D, B are given in theorems 2.22, 2.23, 2.24. Characterizations of x-closed maps for x = I, D, B are given in theorems 2.32, 2.33, 2.34.

Interrelations between

(1) continuous maps and x-continuous maps (x = I, D, B),

(2) open maps and x-open maps (x = I, D, B),

(3) closed maps and x-closed maps (x = I, D, B),

are analysed with examples.

Regarding compositions of these maps the following results are proved :

"Let f $(X, \tau, \leq_1) \rightarrow (Y, \tau, \leq_2)$ and g : $(Y, \sigma, \leq_2) \rightarrow (Z, \eta, \leq_3)$ be any two mappings. Then

(1) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-continuous if f and g are x-continuous for x = I, D, B.

(2) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-continuous if f is x-continuous and g is continuous for x = I, D, B.

(3) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-continuous if f is x-continuous and g is y-continuous for all x, y∈{I, D, B}.

(4) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-open if f is open and g is x-open for x = I, D, B.

(5) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-open if both f and g are x-open for x = I, D, B.

(6) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-open if f is y-open and g is x-open for all x, y∈{I, D, B}.

(7) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-closed if f is closed and g is x-closed for x = I, D, B.

(8) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-closed if both f and g are x-closed for x = I, D, B.

(9) $g \circ f : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is x-closed if f is y-closed and g is x-closed for all x, y$\in$\{I, D, B\}".

Using the concept of x-continuous maps (x = I, D, B), the concept of x-homeomorphisms (x = I, D, B) is introduced. Here also the interrelations, compositions and characterization theorems are studied.

Chapter III deals with separation axioms in topological ordered spaces. The following separation properties of range spaces under some of the mappings x-continuous, x-open, x-closed and x-homeomorphisms (x = I, D, B) are examined;

1) Strongly T_i-ordered (i = 1, 2, 3, 4),
2) strongly regular ordered and
3) strongly normally ordered.

Chapter IV is devoted to the study of proximity ordered spaces. Using quasi-proximity as a primitive concept, Singal, M.K., and Sundar Lal introduced the notion of proximity ordered spaces (Definition 4.9) and proved that quasi-proximities are related to completely regular ordered spaces of Nachbin [19].

Following are the interesting results proved in this chapter.

1) A topological pre-ordered space $(X, \tau, \leq)$ is said to be completely regular pre-ordered if it satisfies the conditions.
    i) $p \in P \in \tau$ implies the existence of two continuous functions $f, g : X \rightarrow [0, 1]$ such that f is increasing, g is decreasing $f(p) = g(p) = 1$ and $\min\{f(x), g(x)\} = 0$ for $x \in X - P$

ii) $x \not\lesssim y$ implies the existence of a continuous increasing real-valued function f such that f(x) > f(y).

2) Let (X, τ, $\leq$) be a completely regular pre-ordered space. Then there exists a proximity δ^* on X such that (X, τ, δ^*) becomes a proximity pre-ordered space and that $\tau(\delta^*) = \tau$, where $\tau(\delta^*)$ is the topology associated with the proximity δ^*.

Topological properties of spaces ordered by preferences are studied in the fifth chapter. Generalizations of the concept of LOTS (Linearly Ordered Topological Spaces) and GO spaces (Generalized Ordered Spaces) to the preference case namely POTS (Preference Ordered Topological Spaces) and GPO spaces (Generalized Preference Ordered Spaces) are defined and analysed. Interesting characterizations of these spaces are given in theorems 3.1, 3.2 and 3.4.

Chapter VI is devoted to the study of ordered fuzzy topological spaces. In the first section of this chapter, fuzzy ordered sets are defined and some properties of these sets are studied. In the second section of this chapter ordered fuzzy topological spaces are defined and studied. Locally order convex and order-convex preordered fuzzy topological spaces are defined and interesting properties of these spaces are studied (Theorems 5.2.11 and 5.2.13). Following characterization of these spaces is also obtained:

"Let X be a preordered fuzzy topological space. Then X is locally order-convex iff there exist an ordered fuzzy topological space Y whose topology is locally order-convex and an increasing function f from X to Y such that a fuzzy set μ in X is open iff $\mu = f^{-1}(\rho)$ for some open fuzzy set ρ in Y".

Chapter VII deals with separation axioms in ordered fuzzy topological spaces. FT_i-order separation axioms (i = 1, 2, 3, 4) for ordered fuzzy topological spaces are introduced and studied. The relationship between some of the FT_i-order separation axioms are analysed. Regularly ordered spaces and normally ordered spaces are also studied.

Chapter VIII is devoted to the study of fuzzy topological ordered vector spaces. In the first part of this chapter fuzzy sets on an ordered vector space are studied. In the second part of this chapter fuzzy topological ordered vector spaces are defined (Definition 8.26) and some interesting properties of these spaces are obtained.