QUEUEING MODELS IN NON-MARKOVIAN ENVIRONMENT

BY

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SYNOPSIS

In Chapter I, notations, definitions and some preliminary results which are used in this dissertation are given. The earlier work done by many researchers in connection with our work is also pointed out.

The main theme of Chapter II is to analyse a queueing system with 'C' servers where each server adheres to the general bulk service rule : each server has a finite or infinite capacity 'b' representing the number of customers he can serve concurrently; he takes up to that number into service if, on becoming idle, he finds a quorum of atleast "a" customers awaiting service. It is assumed that additional customers cannot join a batch once it has entered service and that there is infinite waiting space available. Arrivals occur singly such that the interarrival times are independent identically distributed random variables with distribution function A(t).

This GI/M(a, b)/c model is examined in the steady state by Madill and Chaudry [8]. We give a detailed analysis of this paper in section 2 of this chapter.

To analyse the above model in detail we first approach GI/M/c model in section (1) of this chapter to get a clear view.

In Chapter III we anlayse a single server queueing system with time and operation dependent server failures and arrival rate depending on the up and down state of the server. This model is studied by Shantikumar [12] in his paper "Analysis of a single server queue with time and operation dependent server failures".

CONTENTS

PAGE NO.

CHAPTER : I

Time is -

too slow for those who wait

too swift for those who fear

too long for those who grieve

too short for those who rejoice

But for those who love -

time is eternity.

INTRODUCTION

CHAPTER : 1

INTRODUCTION

A queueing system can be described by the flow of units for service, forming or joining the queue, if service is not immediately available and leaving the system after being served.

The basic characteristics of the queueing system are

i) Input Process
ii) Service Mechanism
iii) Queue - Discipline
iv) Number of Servers
v) System Capacity

i) Input Process

It describes the manner in which units arrive (either singly or in a group) and join the system. The interval between two consecutive arrivals is called the interarrival time.

ii) Service Mechanism

The service mechanism describes the manner in which service is rendered. A unit may be served singly or in a batch. The time required for servicing a unit is called the service time.

iii) Queue Discipline

It is the principle by which the units form a queue and served. They are

a) First Come First Served (FCFS)
b) Last Come Last Served (LCFS)
c) Service in Random Order (SIRO)

iv) Number of Servers

A queueing system may contain a number of service channels (either parallel or series or a combination of both) to provide service. When the system has a number of parallel servers it is known as multiserver model.

v) System Capacity

The system capacity me be finite or infinite.

Kendall's Notation

Kendall (1951) formulated the convenient symbolic form A/B/C/X/Y to specify any queueing model completely. A and B denote the type of distributions of inter-arrival time and of service times respectively. C specifies the number of servers, X, the capacity of the system. Y denotes the queue discipline. But only the first three symbols are used commonly. Unless otherwise stated the queue discipline can be considered as FIFO and the system capacity can be considered infinite.

Some of the familiar notations are

M : Exponential (Markovian) distribution
GI : General input distribution
G : General service time distribution

Thus GI/M/C defines a 'C' server queueing system with general input and exponential service time distribution.

GI/M(a, b)/C defines a 'C' server queueing system providing exponential service where each server adheres to the bulk-service rule.

M/G/1 defines a single server queueing system where the inter arrival time is exponential the service time do not possess the memoryless property.

Definition and Preliminaries

1. Markovian queueing models

Queueing models with inter-arrival time of customers and service time exponentially distributed are called Markovian queueing models.

2. Non-Markovian queueing models

In practice, there are models that do not rely on strict Markov assumptions. Queueing models having the inter-arrival times and / or service times which are not exponentially distributed are called Non-Markovian queueing models.

3. Imbedded Markov Chain technique

By suitable choice of regeneration points, the process of extracting Markov Chains in discrete time from $\{ N(t) \}$ (where $N(t)$ is the number in the system at time t) is known as Imbedded Markov chain technique.

4. Parallel Channels

The number of servers or service channels in a queueing model may be finite or infinite. Depending on the model, if the number of servers is more than one, the customers may form separate queues in front of each server. This pattern is called a parallel channel.

Results

1. Rouche's Theorem

If $f(z)$ and $g(z)$ are functions analytic inside and on a closed contour C and if $|g(z)| < |f(z)|$ on C then $f(z) + g(z)$ have the same number of zeros inside C.

2. If $G(z)$ is the probability generating function of P_i's then $R(z) = Z - G(z)$ has unique root inside $|z| = 1$ if $G'(1) > 1$.

3. Let $\tilde{a}(s) = \int_0^\infty e^{-st} \, d\, A(t)$ denote the Laplace-stieltjes transform of $d\, A(t)$. Moreover $L(A(t)) = \frac{\tilde{A}(S)}{S}$

4. Let $D = \frac{d}{d\mu}$ denote the differential operator then $e^{-\mu \omega^b D} \cdot f(\mu) = f(\mu(1 - \omega^b))$. This property can be proved by a Taylor series expansion of $f(\mu(1 - \omega^b))$ about μ.

5. If the random variable A_{a-j-1} represents the time for a-j-1 arrivals, then

$$(\tilde{a}(\mu))^{a-j-1} = \int_0^\infty e^{-\mu t} \, d\, A_{a-j-1}(t)$$

6. Let $W_q(t)$ denote the waiting time distribution, then the expected waiting time in the queue is

$$E(W_q(t)) = \int_0^\infty t \, d\, W_q(t)$$

$$E(A_{a-j-1}(t) = \frac{a-j}{\lambda}, \text{ where } \frac{1}{\lambda} \text{ is the mean of } A(t).$$

Relavant Literature Survey

There are queueing models where the distributions of the interarrival time or the service times do not possess the memoryless property i.e., are not exponential. In these cases the process N(t) giving the state of the system or system size at time t will not be Markovian. However the analysis of the process can be based on an associated process which is Markovian. Two techniques are generally used for this purpose.

1. Imbedded Markov Chain Technique
2. Supplementary Variable Technique

The G/M/1 queue (with general arrivals, exponential service times and single server) and its dual M/G/1 (both with infinite waiting room) have received extensive attention in the literature over the years by many authors. An almost complete bibliography on these systems can be found in Bagohi and Templeton [2] and Bhat [3].

The multiserver queue with general interarrival time has been studied by several authors. Shyu, K.H. [13] has studied the steady state solution and the waiting time distribution for G/M/C queue.

Bulk service models of this type are motivated by their applicability to situations which arise in delivery and shuttle systems.

Neuts, M.F. and Nadarajan, R. [10] and Sim, S.H. and Templeton, J.G.C. [14] have discussed the bulk service model M/M(a, b)/C in the context of transportation system.

Madill, B.R. and Chaudhry, M.L. [8] have considered the generalized form of the interarrival time distribution discussed by Neuts, M.F. and Nadarajan, R. [10] and Sim, S.H. and Templeton, J.G.C. [14]. It also extends the work of Shyu, K.H. [13] and Love, R.F. [7].

Buzacott and Hanifin [4] reported that the machine failures in flow lines and automatic transfer lines can be classified into operation failure and time-dependent failure. Under operation dependent failures, it is assumed that the failure mechanism operates only when the machine is in operation, and under time-dependent failures, it is assumed that the failure mechanism is in operation at all time, independent of whether the machine is in operation or not.

For an M/G/1 queue with only time-dependent or only operation dependent failures, Avi-Itzhak and Naor [1] and Gaver [5] obtained the steady state results for the number in the system and the waiting times. In their analysis, they assumed that the time to failures are exponentially distributed and that the repair time has a general distribution. Shantikumar [11] analyzed a similar model with operation dependent failures and studied the effect of carrying out opportune maintenance during the idle period of the machine.

In the above three studies it is assumed however that the customer arrival rate to the M/G/1 queue, or, equivalently the first stage production rate, is unaffected by the up and down state of the second stage machine.

Recently Shantikumar [12] considered a general M/G/1 queue with the server subjected to time and operation dependent failures and the arrival rate depending on the up and down-state of the server. The time to failures are exponentially distributed and the repair times have general distribution. The failure rates depend on whether the server is idle or busy, and the repair times depend on whether the failure occured during the idle or busy period of the server.

In this dissertation we study about the steady-state Analytic and Algorithmic Results on the Queueing System GI/M(a, b)/C discussed by B.R.Madill and M.L.Chaudhry [8] and about Analysis of a Single Server Queue with Time and Operation-dependent Server Failures by J.G.Shantikumar [12] in detail.

CHAPTER : II

SECTION : 1

STEADY STATE RESULTS IN GI/M/c

In the model GI/M/c it is assumed that there are 'c' service channels each providing exponential distribution with mean $\frac{1}{\mu}$ and that the inter-arrival time is a random variable having an arbitrary general distribution A(t) with mean $\frac{1}{\lambda}$. Let the p.d.f. of A(t) be a(t). Let t_n, n = 1, 2, ... and $t_0 = 0$ be the epochs at which the n^{th} arrival occurs. The process $N(t_n - 0)$, n = 0, 1, ... gives the number in the system immediately before the arrival of the n^{th} unit.

We use the following notations to solve the queueing system at prearrival epoch.

1. Let (K/c) denote the probability that K customers complete service in an interarrival period when the service rate is c μ. Then

$$(K/c) = \int_0^\infty \frac{e^{-c\mu t} (c\mu t)^k}{K!} \, d\, A(t), \quad K \geq 0$$

If K(z) denotes the p.g.f. of the sequence $\{ (K/c) \}$ with parameter z, then

$$K(z) = \sum_{K=0}^{\infty} (K/c)\, z^{k}$$

$$= \sum_{K=0}^{\infty} \int_{0}^{\infty} \frac{e^{-c\mu t} (c\mu t)^{k}}{K!} dA(t) \,.\, z^{k}$$

$$= \int_{0}^{\infty} e^{-\mu ct} e^{\mu ctz} dA(t)$$

$$= \int_{0}^{\infty} e^{-\mu ct(1-z)} dA(t)$$

$$= \tilde{a}(\mu c(1-z))$$

Where $\tilde{a}(\alpha)$ is the Laplace-stieltjes transform of d A(t) which is defined by

$$\tilde{a}(\alpha) = \int_{0}^{\infty} e^{-\alpha t}\, d\, A(t)$$

2. Let [k/m], $k \leq m \leq c$ denote the probability that K servers complete service during an arbitrary interarrival period, given that 'm' servers were busy at the start of the period. Then [k/m] follows a binomial distribution with $p = 1 - e^{-\mu t}$, that is, probability that a customer is

served. Then,

$$[k/m] = \int_0^\infty \binom{m}{k} (1 - e^{-\mu t})^k e^{-\mu t(m-k)}\, d\, A(t), \quad k \leq m \leq c$$

3. Let $\{k/m\}$, $k \leq c$, $m \geq 0$ denote the probability that k servers are idle at the end of an arbitrary interarrival period, given that 'c' servers were busy at the start of the period and m customers were waiting to enter service. The system starts out with all servers busy and sometime during the interarrival time T, servers start to become idle until finally k servers are idle.

We assure at time V after the arrival comes $(0 < V < T)$ he goes into service all prior customers have left, with H(v) the CDF of V. Thus to find k servers idle at the end of time T we must have k service completions from V to T. Thus

$$\{k/m\} = \int_0^\infty \int_0^t \binom{c}{k} (1 - e^{-\mu(t-v)})^k e^{-\mu(t-v)(c-k)} \cdot$$

$$\frac{(\mu cv)^m e^{-\mu cv}}{v(m-1)!}\, dv\, d\, A(t)$$

Let P_j denote the steady state probability that there are j persons in the system at prearrival epoch. The steady state equations satisfied by the above probabilities are,

For $0 \leq j \leq c-1$

$$P_j = P_{j-1}[o/j] + P_j[1/j+1] + \dots + P_{c-1}[c-j/c] + P_c\{c-j/1\} + P_{c+1}\{c-j/2\} + \dots$$

$$P_j = \sum_{k=j-1}^{c-1} P_k[k-j+1/k+1] + \sum_{k=c}^{\infty} P_k\{c-j/k-(c-1)\} \tag{1}$$

For $j \geq c$,

$$P_j = \sum_{k=j-1}^{\infty} P_k(k-j+1/c) \tag{2}$$

$$P_{j+1} = \sum_{k=j}^{\infty} P_k(k-j/c)$$

$$(E - \sum_{r=0}^{\infty} E^r(r/c))\, P_j = 0, \quad j \geq c-1 \tag{3}$$

It can be proved that $R(z) = z - K(z)$ has a unique root inside the unit circle when $K'(1) > 1$, where $K(z)$ is the probability generating function of (r/c)

$$\text{But} \quad K'(1) = \frac{d}{dz}\left[\sum_{r=0}^{\infty} z^r (r/c)\right]_{z=1}$$

$$= \sum_{r=0}^{\infty} r\ (r/c)$$

$$= \sum_{r=0}^{\infty} \int_0^{\infty} \frac{e^{-\mu ct} (\mu ct)^r}{(r-1)!} d\ A(t)$$

$$= \int_0^{\infty} \mu ct\ d\ A(t)$$

$$= \frac{\mu c}{\lambda}$$

$$= \frac{1}{\rho'}$$

Hence when $\rho' < 1$, let r_0 be the unique root inside the unit circle for $R(z) = 0$. Since $P_j \leq 1$, the solution of (3) is

$$P_j = A_0\ r_0^{\ j}, \text{ for } j \geq c-1$$

Where A_0 is a constant

$$\therefore \quad P_{c-1} = A_0\ r_0^{\ c-1}$$

$$\therefore \quad A_0 = \frac{P_{c-1}}{r_0^{\ c-1}}$$

$$\therefore \quad P_j = P_{c-1}\ r_0^{\ j-c+1}$$

From (2),

$$P_c = \sum_{k=c-1}^{\infty} P_k \, [k-c+1/c]$$

$$P_{c-1} = \left[P_c - A_0 \sum_{k=c}^{\infty} r_0^{\,k} \, [k-c+1/c] \right] / \, [o/c]$$

$$= \frac{A_0}{[o/c]} \left[r_0^{\,c} - \sum_{k=c}^{\infty} r_0^{\,k} \, [k-c+1/c] \right]$$

From (1)

$$P_{j-1} = \frac{1}{[o/j]} \left[P_j - \sum_{k=j}^{c-1} P_k [k+j+1/k+1] \right.$$

$$\left. - A_0 \sum_{k=c}^{\infty} r_0^{\,k} \, \{ c-j/k-(c-1) \} \right] , \quad 1 \leq j \leq c-1 \qquad (4)$$

Hence by using (4), P_0, P_1, ... P_{c-1} can be calculated. The constant A_0 can be determined from the normalising condition,

$$\sum_{j=0}^{\infty} P_j = 1$$

i.e. $$\sum_{j=0}^{c-1} P_j + \sum_{j=c}^{\infty} A_0 r_0^j = 1$$

Let $P_j^1 = \frac{P_j}{A_0}$, then

$$\sum_{j=0}^{c-1} A_0 P_j^1 + A_0 \sum_{j=c}^{\infty} r_0^j = 1$$

$$A_0 \left[\sum_{j=0}^{c-1} P_j^1 + \frac{r_o^c}{1 - r_0} \right] = 1$$

$$\therefore \quad A_0 = \left[r_0^c (1 - r_0)^{-1} + \sum_{j=0}^{c-1} P_j^1 \right]^{-1}$$

Waiting Time Distribution

Let T_q be the random variable denoted "time spent in the queue" and $W_q(t)$ its cummulative probability distribution. Then

$$W_q(0) = P_r \{ T_q \leq 0 \}$$

$$= P_r \{ T_q = 0 \}$$

$$= \sum_{i=0}^{c-1} P_i$$

$$= 1 - \sum_{i=c}^{\infty} P_i$$

$$= A\left[\frac{1}{A} - \sum_{i=c}^{\infty} P_i^{1}\right]$$

$$= A\left[\frac{1}{A} - \frac{r_0^{c}}{1 - r_0}\right]$$

If an arrival finds all the 'c' servers working the mean rate is c μ, that is, if he finds $n \geq c$ customers in the system upon arrival, there are (n-c) waiting and the current arrival does not get into service until n-c+1 customers have been served. Thus his waiting time is the (n-c+1) fold convolution of exponential with mean c μ which is Erlang. Then

$$P_r \{ T_q \leq t \} = W_q(t)$$

$$= \sum_{n=c}^{\infty} P_n \int_0^t \frac{(c\,\mu)^{n-c+1}\, v^{n-c}\, e^{-\mu c v}}{(n-c)!}\, dv + W_q(0)$$

$$= A_0 r_0^{c} \int_0^t \sum_{n=c}^{\infty} \frac{(c\,\mu\, v\, r_0)^{n-c}\, c\,\mu\, e^{-c\,\mu\, v}}{(n-c)!}\, dv + W_q(0)$$

$$= A_0 r_0^{c} \int_0^t \mu c\, e^{-\mu c\, v(1 - r_0)}\, dv + A[\frac{1}{A} - \frac{r_0^{c}}{1 - r_0}]$$

$$= 1 - \frac{A_0\, r_0^{c}\, e^{-\mu c t(1 - r_0)}}{1 - r_0}, \quad t > 0$$

The probability density function of the waiting time is

$$W_q(t) = A_0\, r_0^{\,c}\, \mu c\, e^{-\mu ct(1 - r_0)}, \quad t > 0$$

The expected waiting time of the server in the queue is

$$E(W_q(t)) = \int_0^\infty t\, A_0\, r_0^{\,c}\, \mu c\, e^{-\mu ct(1 - r_0)}\, dt$$

$$= A_0\, r_0^{\,c}\, \mu c \left[\int_0^\infty \frac{e^{-\mu ct(1 - r_0)}}{\mu c(1 - r_0)}\, dt \right]$$

$$= \frac{A_0\, r_0^{\,c}}{(1 - r_0)} \left[\frac{1}{\mu c(1 - r_0)} \right]$$

$$= \frac{A_0\, r_0^{\,c}}{\mu c(1 - r_0)^2}$$

SECTION : 2

STEADY - STATE ANALYTIC AND ALGORITHMIC RESULTS ON THE QUEUEING SYSTEM GI/M(a, b)/c

In this section we consider GI/M(a, b)/c queueing system with 'c' service channels providing exponential service where each server adheres to the following quorum bulk service rule.

If a server immediately after completing his service finds

- less than 'a' units present he does not start his service
- 'a' or more but atmost 'b' he takes them all in batch for service
- more than 'b' units, he takes 'b' units in the batch of service, while others wait.

Additional customers are not permitted to join a batch once it has entered service. Arrivals occur singly such that the interarrival times are identically and independently distributed random vaiables with distribution function A(t) and mean arrival time, $\frac{1}{\lambda}$. The service time for each server is exponentially distributed with mean $\frac{1}{\mu}$ and distribution function B(t) and is independent of the size of the group in service. Further all servers are not permitted to be idle if there is a batch with completed quorum in the queue. All transitions in the system are considered to be instantaneous.

The points at which the system will be examined in the steady state are the pre-arrival epochs (p.a.e.) immediately preceeding the arrival of a customer and the random epochs (r.e) occuring at the end of a random period since the most recent arrival.

Here we define the two dimensional state space of the system as $\{(s, n) \,/\, 0 \leq s \leq c,\ n \geq 0\}$, where 's' denotes the number of busy servers and 'n' denotes the number of customers in the queue. In accordance with the general bulk service rule the possible states for the system are

$$(s, n) \qquad n = hb + j \qquad \left[\begin{array}{l} 0 \leq s \leq c-1 \ ;\ 0 \leq j \leq a-1 \ ;\ h = 0 \\ s = c;\ 0 \leq j \leq b-1 \ ;\ h = 0,1,2,\ldots \end{array}\right.$$

If an arrival finds the system in the state (s, n) he will be the $(j+1)^{st}$ member of his batch and that there will be 'h+s' batches ahead of his batch in the system where 'h' batches of size 'b' in the queue and 's' batches of unknown size in service.

To solve the queueing system at prearrival epoch we recall the following notations.

1. $$(k/c) = \int_0^\infty \frac{e^{-\mu ct} (\mu ct)^k}{k!} \, d\,A(t) \; ; \; k = 0, 1, 2, \ldots$$

If k(z) denotes the p.g.f. of the sequence $\{(k/c)\}$ with parameter z, then

$$k(z) = \tilde{a}(\mu c(1-z))$$

We define $\alpha_y = \tilde{a}(\mu y)$ where y is some real number.

2. $$[k/m] = \int_0^\infty \binom{m}{k} (1 - e^{-\mu t})^k e^{-\mu t(m-k)} \, d\,A(t) \; ; \quad k \leq m \leq c$$

3. $$\{k/m\} = \int_0^\infty \int_0^t \binom{c}{k} (1 - e^{-\mu(t-v)})^k e^{-\mu(t-v)(c-k)} \frac{(\mu c v)^m e^{-\mu cv}}{v(m-1)!} \, dv \, d\,A(t)$$

Let $P^a_{s,n}$ denote the steady state probability that there are 's' servers busy, $0 \leq s \leq c$ and 'n' members in the queue. The steady state equations are

$$P^{a}_{s,0} = P^{a}_{s-1,a-1}\,[o/s] + \sum_{h=1}^{\infty}\sum_{j=a-1}^{b-1} P^{a}_{c,(h-1)b+j}\,\{c-s/h\} + \sum_{v=s+1}^{c}[v-s/v]\,P^{a}_{v-1,a-1} \; ; \; 0 \leq s \leq c \qquad (1)$$

$$P^{a}_{s,j+1} = P^{a}_{s,j}\,[o/s] + \sum_{h=1}^{\infty} P^{a}_{c,hb+j}\,\{c-s/h\} + \sum_{v=s+1}^{c} P^{a}_{v,j}\,[v-s/v] \; ; \; 0 \leq j \leq a-2 \; ; \; 0 \leq s \leq c \text{ and } 0 \leq j \text{ if } s = c \qquad (2)$$

Then from (1) and (2) we get

$$P^{a}_{s-1,a-1} = \left[P^{a}_{s,o} - \sum_{h=1}^{\infty}\sum_{j=a-1}^{b-1} P^{a}_{c,(h-1)b+j}\,\{c-s/h\} - \sum_{v=s+1}^{c}[v-s/v]\,P^{a}_{v-1,a-1} \right] / \,[o/s] \; ; \; 0 \leq s \leq c \qquad (3)$$

$$P^{a}_{s,j} = \left[P^{a}_{s,j+1} - \sum_{h=1}^{\infty} P^{a}_{c,hb+j}\,\{c-s/h\} - \sum_{v=s+1}^{c} P^{a}_{v,j}\,[v-s/v] \right] / \,[o/s] \; ; \; 0 \leq j \leq a-2 \quad (4)$$

$$0 \leq s \leq c$$

$$\text{and} \quad 0 \leq j \quad \text{if } s=c$$

$$P^a_{c,j} = \left[P^a_{c,j+1} - \sum_{h=1}^{\infty} P^a_{c,hb+j} \{ o/h \} \right] / \; [o/c] \; ; \; s = c \qquad (5)$$

To solve the above equations we first find the relations between $\{ o/m \}$, (m/c), $[o/c]$ and (o/c)

$$[o/c] = \int_0^{\infty} e^{-c\mu t} \, d\, A(t)$$

$$= (o/c)$$

$$= \alpha_c$$

$$\{o/m\} = \int_0^{\infty} \int_0^{t} \frac{e^{-\mu ct} (\mu cv)^m}{v(m-1)!} \, dv \, d\, A(t)$$

$$= \int_0^{\infty} e^{-\mu ct} \frac{(\mu c)^m}{(m-1)!} \int_0^{t} v^{m-1} \, dv \, d\, A(t)$$

$$= \int_0^{\infty} e^{-\mu ct} \frac{(\mu ct)^m}{m!} \, d\, A(t)$$

$$= (m/c)$$

When $m = 0$, $\{ o/m \} = (o/c) = \alpha_c$

$$\therefore \; P_{c,j+1} - \sum_{h=0}^{\infty} P_{c,hb+j} \{o/h\} = 0 \quad \text{(from (5))}$$

Let E be the forward shifting operator. Then

$$\left[E - \sum_{h=0}^{\infty} E^{hb} \{ o/h \} \right] P_{c,j} = 0 \qquad (6)$$

$$\text{Let } \phi(z) = \sum_{h=0}^{\infty} z^{hb} \{ o/h \} \text{, then}$$

$$\phi'(1) = \frac{b \mu c}{\lambda}$$

$$= \frac{1}{\rho}$$

If $\phi'(1) > 1$ i.e. if $0 < \rho < 1$ then ω is the unique root of $\phi(z) = z$ which lies within the unit circle $|z| = 1$. The solution of (6) is given by

$$P_{c,j} = A \omega^{j} \text{ if } \rho < 1,\ 0 \leq j \qquad (7)$$

Where A is a constant

Substituting (7) in (3) we get

$$P^{a}_{c-1,a-1} = \frac{P^{a}_{c,0} - \sum_{h=1}^{\infty} \{ o/h \} \sum_{j=a-1}^{b-1} P^{a}_{c,(h-1)b+j}}{[o/c]}$$

$$= \frac{A - \sum_{h=1}^{\infty} \{ o/h \} \sum_{j=a-1}^{b-1} A\, \omega^{(h-1)b+j}}{[o/c]}$$

$$= \frac{A}{[o/c]} \left\{ 1 - \sum_{h=1}^{\infty} \frac{\{o/h\}\ \omega^{hb}}{\omega^{b}} \ \frac{\omega^{a-1} - \omega^{b}}{1 - \omega} \right\}$$

$$= \frac{A}{[o/c]} \left\{ \frac{(1 - \omega) - (\omega^{a-b-1} - 1) \displaystyle\sum_{h=1}^{\infty} \{o/h\}\ \omega^{hb}}{1 - \omega} \right\}$$

$$= \frac{A}{[o/c]} \left\{ \frac{(1 - \omega) - (\omega^{a-b-1} - 1)\ (\omega - \alpha_{c})}{1 - \omega} \right\}$$

$$= \frac{A}{1 - \omega} \left\{ \frac{1 - \omega^{a-b}}{\alpha_{c}} + \omega^{a-b-1} - 1 \right\}$$

The remaining "ca-1" elements of P can be evaluated numerically in a recursive manner from (3) and (4).

Random Epoch

A random epoch in an international period occurs at the end of a random period of time R, since the last pre-arrival epoch.

Let $(h/c)_r$, $[k/m]_r$ and $\{k/m\}_r$ denote the corresponding probabilities at random epochs. Then using the renewal theory result $d\,R(t) = \lambda(1 - A(t))dt$

We have

$$(h/c)_r = \lambda \int_0^\infty \frac{e^{-\mu ct} (\mu ct)^h (1 - A(t))\, dt}{h!} \qquad (8)$$

$$= \rho\, b\, (1 - \sum_{k=0}^{h} (k/c))$$

$$[k/m]_r = \int_0^\infty \binom{m}{k} (1 - e^{-\mu t})^k e^{-\mu t (m-k)}\, d\, R(t)\ ; \qquad k \leq m \leq c$$

$$\{ k/m \}_r = \int_0^\infty \int_0^t \binom{c}{k} (1 - e^{-\mu(t-v)})^k e^{-\mu(t-v)(c-k)}$$

$$= \frac{(\mu cv)^m e^{-\mu cv}}{v(m-1)!}\, dv\, d\, R(t)$$

$$\therefore \{o/h\}_r = \int_0^\infty \int_0^t e^{-\mu(t-v)c} \frac{(\mu cv)^h\ e^{-\mu cv}}{v(h-1)!}\, dv\, dR(t)$$

$$= \int_0^\infty \frac{e^{-\mu ct} (\mu c)^h}{(h-1)!} \int_0^t \frac{v^h}{v}\, dv\, d\, R(t)$$

$$= \int_0^\infty \frac{e^{-\mu ct} (\mu c)^h}{(h-1)!} \frac{t^h}{h}\, d\, R(t)$$

$$= \int_0^\infty e^{-\mu ct} \frac{(\mu ct)^h}{h!} dR(t)$$

$$= (h/c)_r \quad \text{(from (8))}$$

$$[o/c]_r = \int_0^\infty e^{-\mu ct} dR(t)$$

$$= \int_0^\infty e^{-\mu ct} \lambda (1 - A(t))dt$$

$$= \rho b(1 - (o/c))$$

The steady state probabilities $P^r_{s,j}$ for random epochs satisfy the following equations.

$$P^r_{s,0} = \sum_{v=s}^{c} [v-s/v]_r P^a_{v-1,a-1}$$

$$+ \sum_{h=1}^{\infty} \{c-s/h\}_r \sum_{j=a-1}^{b-1} P_{c,(h-1)b+j} \; ; \quad 0 \leq s \leq c$$

$$P^r_{s,j} = \sum_{v=s}^{c} [v-s/v]_r P^a_{v,j-1}$$

$$+ \sum_{h=1}^{\infty} \{c-s/h\}_r P^a_{c,hb+j-1} \; ; \quad 0 \leq j \leq a-1 \quad (9)$$

$$0 \leq s \leq c-1$$

$$\text{and} \quad 0 \leq j \text{ if } s = c$$

When $s = c$

$$P^{r}_{c,0} = [o/c]_r \, P^{a}_{c-1,a-1} + \sum_{h=1}^{\infty} \{ o/h \}_r \sum_{j=a-1}^{b-1} P^{a}_{c,(h-1)b+j} \tag{10}$$

∴ Equation (10) becomes

$$P^{r}_{c,0} = A \rho \, b(1 - (o/c)) \left[\frac{1 - \omega^{a-b}}{\alpha_c} + \omega^{a-b-1} - 1 \right] / (1 - \omega)$$

$$+ A \sum_{h=1}^{\infty} \rho \, b(1 - \sum_{k=0}^{h} (k/c)) \sum_{j=a-1}^{b-1} \omega^{(h-1)b+j}$$

$$= \frac{A \rho \, b(1 - \alpha_c)}{1 - \omega} \; \frac{1 - \omega^{a-b} + \alpha_c \, \omega^{a-b-1} - \alpha_c}{\alpha_c}$$

$$+ A \rho \, b \sum_{h=1}^{\infty} \sum_{j=a-1}^{b-1} \omega^{(h-1)b+j}$$

$$- A \rho \, b \sum_{h=1}^{\infty} \sum_{j=a-1}^{b-1} \sum_{k=0}^{h} (k/c) \, \omega^{(h-1)b+j}$$

$$= \frac{A \rho b (1 - \alpha_c)}{1 - \omega} \left[\frac{1 - \omega^{a-b} + \alpha_c \omega^{a-b-1} - \alpha_c}{\alpha_c} \right]$$

$$+ \frac{A \rho b}{1 - \omega} \left(\frac{\omega^{a-1} - \omega^b}{1 - \omega^b} \right)$$

$$- A \rho b \left(\frac{\omega^{a-1} - \omega^b}{1 - \omega} \right) \left[\frac{(o/c)}{1 - \omega^b} + \frac{(1/c)}{1 - \omega^b} + \frac{(2/c)\omega^b}{1 - \omega^b} + \frac{(3/c)\omega^{2b}}{1 - \omega^b} + \ldots \right]$$

$$= \frac{A \rho b(1 - \alpha_c)}{1 - \omega} \quad \frac{1 - \omega^{a-b} + \alpha_c \omega^{a-b-1} - \alpha_c}{\alpha_c}$$

$$+ \frac{A\rho b}{1 - \omega} \left(\frac{\omega^{a-1} - \omega^b}{1 - \omega^b} \right)$$

$$- A \rho b \left(\frac{\omega^{a-1} - \omega^b}{1 - \omega} \right) \frac{1}{1 - \omega^b} \left[(o/c) + \frac{1}{\omega^b} \sum_{k=1}^{\infty} (k/c)\, \omega^{kb} \right]$$

$$= \frac{A \rho b (1 - \alpha_c)}{1 - \omega} \quad \frac{1 - \omega^{a-b} + \alpha_c \omega^{a-b-1} - \alpha_c}{\alpha_c}$$

$$+ \frac{A \rho b}{1 - \omega} \quad \frac{\omega^{a-1} - \omega^b}{1 - \omega^b}$$

$$- A \rho b \quad \frac{\omega^{a-1} - \omega^b}{1 - \omega} \quad \frac{1}{1 - \omega^b} \left[(o/c) \left(1 - \frac{1}{\omega^b}\right) + \frac{1}{\omega^b} \cdot \omega \right]$$

(Since ω is the unique root of

$$\phi(z) = z = \sum_{h=0}^{\infty} z^{hb} \{ o/h \}$$

$$= \frac{A \rho b (1 - \alpha_c)}{1 - \omega} \quad \frac{1 - \omega^{a-b} + \alpha_c \omega^{a-b-1} - \alpha_c}{\alpha_c}$$

$$+ \frac{A \rho b}{1 - \omega} \quad \frac{\omega^{a-1} - \omega b}{1 - \omega^b}$$

$$- A \rho b \frac{\omega^{a-1} - \omega^b}{1 - \omega} \left[\frac{-(o/c)}{\omega^b} + \frac{\omega}{\omega^b (1 - \omega^b)} \right]$$

Simplifying further we get

$$= \frac{A \rho b}{1 - \omega^b} \quad \frac{1}{\alpha_c (1 - \omega)} [(1 + \omega^a - \omega^b) - \alpha_c (1 + \omega^a - \omega^b) - \omega^{a-b} + \alpha_c (\omega^{a-b-1}) - \alpha_c (1 - \omega)]$$

$$= \frac{A \rho b}{1 - \omega^b} \left[\frac{1}{\alpha_c (1 - \omega)} [(1 - \alpha_c)(1 + \omega^a - \omega^b) + \alpha_c \omega^{a-b-1} - \omega^{a-b}] - 1 \right]$$

$$= \frac{A \rho b}{1 - \omega^b} \left[\frac{H}{\alpha_c (1 - \omega)} - 1 \right] \quad \text{where}$$

$$H = (1 - \alpha_c)(1 + \omega^a - \omega^b) - (\omega - \alpha_c)\,\omega^{a-b-1}$$

From equation (9),

$$P^r_{c,j} = [o/c]_r\, A\omega^{j-1} + \sum_{h=1}^{\infty} \{ o/h \}\, A\omega^{hb+j-1}$$

$$= A\rho b\,(1 - (o/c))\,\omega^{j-1} + A\omega^{j-1} \sum_{h=1}^{\infty} \rho b\,\left(1 - \sum_{k=0}^{h} (k/c)\right) \omega^{hb}$$

(from (8))

$$= A\omega^{j-1} \rho b \left[1 - \alpha_c + \frac{\omega^b}{1 - \omega^b} - \frac{1}{1 - \omega^b} \left(\alpha_c \omega_b + \sum_{k=1}^{\infty} (k/c)^{hb}\right) \right]$$

$$= A\omega^{j-1} \rho b \left[1 - \alpha_c + \frac{\omega^b}{1 - \omega^b} - \frac{1}{1 - \omega^b} \left[\alpha_c \omega^b + \left[\sum_{k=0}^{\infty} (k/c)\, \omega^{hb} - (o/c) \right] \right] \right]$$

$$= A\omega^{j-1} \rho b \left[1 - \alpha_c + \frac{\omega^b}{1 - \omega^b} - \frac{1}{1 - \omega^b} [\alpha_c \omega^b + \omega - \alpha_c] \right]$$

$$= A^{j-1} \rho b \left[1 + \frac{\omega^b}{1 - \omega^b} - \frac{\omega}{1 - \omega^b} \right]$$

$$= A^{j-1} \rho b \left[\frac{1 - \omega^b + \omega^b - \omega}{1 - \omega^b} \right]$$

$$= \frac{A\omega^{j-1} \rho b(1 - \omega)}{1 - \omega^b}$$

The remaining "ca-1" elements of P can be evaluated numerically in a recursive manner from (9).

Waiting Time Moment

Next we shall calculate the waiting time of the customer in queue. i.e. time between the arrival of a customer at a queueing system and his entry into the service.

An arriving customer may find the system in any one of the following states

i) (k, j) ; $0 \leq k < c$; $0 \leq j \leq a-2$

ii) $(c, hb+j)$; $a-1 \leq j \leq b-1$; $h = 0, 1, 2, \ldots$

iii) $(c, hb+j)$; $0 \leq j \leq a-2$; $h = 0, 1, 2, \ldots.$

The arriving customer has to wait

- in case (i) for $(a-1-j)$ customers to arrive
- in case (ii) for $(h+1)$ service completion
- in case (iii) till either the services of $h+1$ batches are completed or $a-1-j$ units arrive whichever occurs later

Let $W_q(t)$, $w_q(t)$ denote the waiting time distribution and probability density function of the waiting time respectively.

Let A_{a-j-1} represent the time for $a-j-1$ arrivals and B_{h+1} represent the time for $h+1$ service completions. Then

$$w_q(t) = q_{a-1} \; \delta(t) \; dt + \sum_{j=0}^{a-2} q_j . dA_{a-j-1}(t)$$

$$+ \sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} P_{c,\, hb+j} \; d \, B_{h+1}(t)$$

$$+ \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} P_{c,\, hb+j} \; d \, W_q^{(c,hb+j)}(t)$$

Where

1. $d\,A_{a-j-1}$ is the p.d.f. of A_{a-j-1}

2. $$B_{h+1}(t) = 1 - \sum_{i=0}^{h} \frac{(\mu ct)^{i} e^{-\mu ct}}{i!}$$

3. $$d\,B_{h+1}(t) = \frac{(\mu c)^{h+1} t^{h} e^{-\mu ct}}{h!}\,dt, \quad t > 0$$

4. $$W_{q}^{c,hb+j}(t) = \text{maximum of } (A_{a-j-1}(t), B_{h+1}(t))$$

5. $$d\,W_{q}^{c,hb+j}(t) = A_{a-j-1}(t)\,dB_{h+1}(t) + dA_{a-j-1}(t)\,B_{h+1}(t)$$

6. q_j = The probability that the number of customers in the queue is j given that atleast one of the servers is idle at pre arrival epoch.

$$\therefore\ q_j = \sum_{s=0}^{c-1} P^{a}_{s,j}$$

7. $\delta(t)$ denotes the Dirac delta function

The expected waiting time of the customer in the queue is

$$E(w_q(t)) = \int_0^{\infty} t \sum_{j=0}^{a-2} q_j\, d\,A_{a-j-1}(t)$$

$$+ \int_0^\infty t \sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} P_{c,\ hb+j}\ d\ b_{h+1}\ (t)$$

$$+ \int_0^\infty t \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} P_{c,\ hb+j}\ d\ W_q^{c,\ hb+j}\ (t) \qquad (11)$$

Consider

$$\int_0^\infty t \sum_{j=0}^{a-2} q_j \cdot d\ A_{a-j-1}\ (t)$$

$$= \sum_{j=0}^{a-2} q_j \int_0^\infty t\ d\ A_{a-j-1}\ (t)$$

$$= \sum_{j=0}^{a-2} q_j\ E(A_{a-j-1}) \qquad (12)$$

The second term of (11) becomes

$$\sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} A\,\omega^{hb+j} \int_0^\infty \frac{t(\mu c)^{n+1}\ t^h\ e^{-\mu c t}}{h!}\ dt$$

$$= \sum_{h=0}^{\infty} \sum_{j=a-1}^{b-1} A\,\omega^{hb+j}\ \frac{(\mu c)^{h+i}}{h!}\ \frac{(h+1)!}{(\mu c)^{h+2}}$$

$$= A \sum_{h=0}^{\infty} \frac{h+1}{\mu c}\ \omega^{hb} \left[\frac{\omega^{a-1} - \omega^{b}}{1 - \omega} \right]$$

$$= A\ \frac{\omega^{a-1} - \omega^{b}}{1 - \omega}\ \frac{1}{\mu c}\ (1 - \omega^{b})^{-2} \qquad (13)$$

The last term of (11) becomes

$$\sum_{h=0}^{\infty} \sum_{j=0}^{a-2} A\,\omega^{hb+j} \int_0^{\infty} t \left[A_{a-j-1}(t)\; d\, B_{h+1}(t) + d\, A_{a-j-1}(t)\; B_{h+1}(t) \right] \tag{14}$$

Substituting for $d\, B_{h+1}(t)$ and for $B_{h+1}(t)$ the expression

$$\int_0^{\infty} t \left[A_{a-j-1}(t)\; d\, B_{h+1}(t) + d\, A_{a-j-1}(t)\; B_{h+1}(t) \right]$$

$$= \int_0^{\infty} t\, A_{a-j-1}(t)\, \frac{(\mu c)^{h+1}\, t^h\, e^{-\mu ct}}{h!}\, dt$$

$$+ \int_0^{\infty} t \left(1 - \sum_{i=0}^{h} \frac{(\mu ct)^i\, e^{-\mu ct}}{i!}\right) d\, A_{a-j-1}(t)$$

$$= \frac{(\mu c)^{h+1}}{h!} \int_0^{\infty} t^{h+1}\, e^{-\mu ct}\, A_{a-j-1}(t)\, dt + \int_0^{\infty} t\, d\, A_{a-j-1}(t)$$

$$- \int_0^{\infty} t \sum_{i=0}^{h} \frac{(\mu c\, t)^i}{i!}\, e^{-\mu ct}\, d\, A_{a-j-1}(t)$$

$$= \frac{(\mu c)^{h+1}}{h!} (-D)^{h+1}_{\mu c} \frac{(\tilde{a}(\mu))^{a-j-1}}{\mu c} + \int_0^{\infty} t\, dA_{a-j-1}(t)$$

$$- \sum_{i=0}^{h} \frac{(\mu c)^i}{i!} (-D)^{i+1}_{\mu c} (\tilde{a}(\mu))^{a-j-1}$$

$\therefore$ The expression (14) simplifies as follows

$$\sum_{h=0}^{\infty}\sum_{j=0}^{a-2} A\,\omega^{hb+j}\left[\frac{(\mu c)^{h+1}}{h!}(-D)^{h+1}_{\mu c}\frac{(\tilde{a}(\mu))^{a-j-1}}{\mu c} + \int_{0}^{\infty} t\; d\,A_{a-j-1}(t) - \sum_{i=0}^{h}\frac{(\mu c)^{i}}{i!}(-D)^{i+1}_{\mu c}(\tilde{a}(\mu))^{a-j-1}\right]$$

$$= \sum_{h=0}^{\infty}\sum_{j=0}^{a-2} A\,\omega^{hb+j}\frac{(\mu c)^{h+1}}{h!}(-D)^{h+1}_{\mu c}\frac{(\tilde{a}(\mu))^{a-j-1}}{\mu c}$$

$$+ \sum_{h=0}^{\infty}\sum_{j=0}^{a-2} A\,\omega^{hb+j}\int_{0}^{\infty} t\; d\,A_{a-j-1}(t)$$

$$- \sum_{h=0}^{\infty}\sum_{j=0}^{a-2} A\,\omega^{hb+j}\sum_{i=0}^{h}\frac{(\mu c)^{i}}{i!}(-D)^{i+1}_{\mu c}(\tilde{a}(\mu))^{a-j-1}$$

$$= \sum_{j=0}^{a-2}{}^{j}\sum_{h=0}^{\infty} A\,\omega^{hb}\frac{(\mu c)^{h+1}}{h!}(-D)^{h+1}_{\mu c}\frac{(\tilde{a}(\mu))^{a-j-1}}{\mu c}$$

$$+ A\sum_{h=0}^{\infty}\sum_{j=0}^{a-2}\omega^{hb+j}\,E(A_{a-j-1})$$

$$- A\sum_{j=0}^{a-2}\omega^{j}\sum_{h=0}^{\infty}\omega^{hb}\sum_{i=0}^{h}\frac{(\mu c)^{i}}{i!}(-D)^{i+1}_{\mu c}(\tilde{a}(\mu))^{a-j-1}$$

$$= - A \sum_{j=0}^{a-2} \omega^{j} \mu c \, e^{-\mu c D_{\mu c} \omega^{b}} D_{\mu c} \frac{(\tilde{a}(\mu))^{a-j-1}}{\mu c}$$

$$+ A \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} \omega^{hb+j} E(A_{a-j-1})$$

$$+ A \sum_{j=0}^{a-2} \frac{\omega^{j}}{1 - \omega^{b}} e^{-\mu c D_{\mu c} \omega^{b}} D_{\mu c} (\tilde{a}(\mu))^{a-j-1}$$

$$= - A \sum_{j=0}^{a-2} \omega^{j} \mu c \, e^{-\mu c D_{\mu c} \omega^{b}} \left(\frac{\mu c D_{\mu c}(\tilde{a}(\mu))^{a-j-1} - (\tilde{a}(\mu))^{a-j-1}}{(\mu c)^{2}} \right)$$

$$+ A \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} \omega^{hb+j} E(A_{a-j-1})$$

$$+ A \sum_{j=0}^{a-2} \frac{\omega^{j}}{1 - \omega^{b}} e^{-\mu c D_{\mu c} \omega^{b}} D_{\mu c} (\tilde{a}(\mu))^{a-j-1} \qquad (15)$$

Substituting (12), (13) and (15) in (11) we get

$$E(w_q(t)) = \sum_{j=0}^{a-2} q_j E(A_{a-j-1}) + A \left(\frac{\omega^{a-1} - \omega^{b}}{1 - \omega} \right) \frac{1}{\mu c (1 - \omega^{b})^{2}}$$

$$+ A \sum_{h=0}^{\infty} \sum_{j=0}^{a-2} \omega^{hb+j} E(A_{a-j-1})$$

$$- A \sum_{j=0}^{a-2} \omega^{j} \left\{ \frac{\mu c \, e^{-\mu c \, D_{\mu c} \, \omega^{b}}}{(1 - \omega^{b}) \, \mu c} \, D_{\mu c} \, (\tilde{a}(\mu))^{a-j-1} \right.$$

$$\left. - \frac{1}{(\mu c)^{2}} \, \frac{e^{-\mu c \, D_{\mu c}}}{(1 - \omega^{b})^{2}} \, b \, (\tilde{a}(\mu))^{a-j-1} \right\}$$

$$+ \frac{1}{1 - \omega^{b}} \, e^{-\mu c \omega^{b} D_{\mu c}} \, D_{\mu c} \, (\tilde{a}(\mu))^{a-j-1}$$

$$= \sum_{j=0}^{a-2} q_{j} \frac{(a-j-1)}{\lambda} + A \, \frac{\omega^{a-1} - \omega^{b}}{1 - \omega} \, \frac{1}{\mu c(1 - \omega^{b})^{2}}$$

$$+ A \sum_{h=0}^{\infty} \omega^{hb+j} \sum_{j=1}^{a-1} \frac{a-j}{\lambda}$$

$$+ A \sum_{j=0}^{a-2} \frac{\omega^{j}}{\mu c(1 - \omega^{b})^{2}} \left[e^{-\mu c \, \omega^{b} \, D_{\mu c}} \right] (\tilde{a}(\mu))^{a-j-1}$$

(Since $E(A_{a-j-1}) = \frac{a-j}{\lambda}$)

$$= \sum_{j=0}^{a-2} q_{j} \frac{(a-j-1)}{\lambda} + A \left(\frac{\omega^{a-1} - \omega^{b}}{1 - \omega} \right) \frac{1}{\mu c(1 - \omega^{b})^{2}}$$

$$+ A \sum_{h=0}^{\infty} \sum_{j=1}^{a-1} \omega^{hb+j} \frac{(a-j)}{\lambda}$$

$$+ A \sum_{j=0}^{a-2} \omega^{j} \left[\frac{1}{\mu c(1-\omega^{b})^{2}} \right] (\tilde{a}\,\mu(1-\omega^{b}))^{a-j-1}$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + A \left(\frac{\omega^{a-1} - \omega^{b}}{1-\omega} \right) \frac{1}{\mu c\,(1-\omega^{b})^{2}}$$

$$+ \frac{A}{\mu c(1-\omega^{b})^{2}} \sum_{j=0}^{a-2} \omega^{j}\,\omega^{a-j-1} + A \sum_{h=0}^{\infty} \sum_{j=1}^{a-1} \omega^{hb+j} \frac{(a-j)}{\lambda}$$

$$(\text{Since } \tilde{a}\,\mu(1-\omega^{b}) = \omega)$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{\rho b}{\lambda} A \left(\frac{\omega^{a-1} - \omega^{b}}{1-\omega} \right) \frac{1}{(1-\omega^{b})^{2}}$$

$$+ A \sum_{h=0}^{\infty} \omega^{hb} \left[\sum_{j=1}^{a-1} \omega^{j-1} \frac{a-j}{\lambda} \right] + \frac{\rho bA}{\lambda(1-\omega^{b})^{2}} (a-1)\,\omega^{a-1}$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{A\rho b}{\lambda(1-\omega^{b})^{2}} \left[a\,\omega^{a-1} + \frac{\omega^{a} - \omega^{b}}{1-\omega} \right]$$

$$+ \frac{A}{\lambda} \frac{1}{1-\omega^{b}} \left[a \sum_{j=0}^{a-2} \omega^{j} - \sum_{j=1}^{a-1} j\,\omega^{j-1} \right]$$

$$+ A \sum_{j=0}^{a-2} \omega^{j} \left[\frac{1}{\mu c(1 - \omega^{b})^{2}} \right] (\tilde{a} \mu (1 - \omega^{b}))^{a-j-1}$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + A \left(\frac{\omega^{a-1} - \omega^{b}}{1 - \omega} \right) \frac{1}{\mu c (1 - \omega^{b})^{2}}$$

$$+ \frac{A}{\mu c(1 - \omega^{b})^{2}} \sum_{j=0}^{a-2} \omega^{j} \omega^{a-j-1} + A \sum_{h=0}^{\infty} \sum_{j=1}^{a-1} \omega^{hb+j} \frac{(a-j)}{\lambda}$$

(Since $\tilde{a} \mu (1 - \omega^{b}) = \omega$)

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{\rho b}{\lambda} A \left(\frac{\omega^{a-1} - \omega^{b}}{1 - \omega}\right) \frac{1}{(1 - \omega^{b})^{2}}$$

$$+ A \sum_{h=0}^{\infty} \omega^{hb} \left[\sum_{j=1}^{a-1} \omega^{j-1} \frac{a-j}{\lambda} \right] + \frac{\rho b A}{\lambda (1 - \omega^{b})^{2}} (a-1) \omega^{a-1}$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{A \rho b}{\lambda(1 - \omega^{b})^{2}} \left[a \omega^{a-1} + \frac{\omega^{a} - \omega^{b}}{1 - \omega} \right]$$

$$+ \frac{A}{\lambda} \frac{1}{1 - \omega^{b}} \left[a \sum_{j=0}^{a-2} \omega^{j} - \sum_{j=1}^{a-1} j \omega^{j-1} \right]$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{A\rho b}{\lambda(1-\omega^b)^2}\left[a\,\omega^{a-1} + \frac{\omega^a - \omega^b}{1-\omega}\right]$$

$$+ \frac{A}{\lambda(1-\omega^b)}\left[\frac{a}{1-\omega} - \frac{a\,\omega^{a-1}}{1-\omega} - \sum_{k=1}^{a-1} k\,\omega^{k-1}\right]$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{A\rho b}{\lambda(1-\omega^b)^2}\left[a\,\omega^{a-1} + \frac{\omega^a - \omega^b}{1-\omega}\right]$$

$$+ \frac{A}{\lambda(1-\omega^b)}\left[\frac{a}{1-\omega} - \frac{a\,\omega^{a-1}}{1-\omega} - \frac{d}{d\omega}\left(\sum_{k=1}^{a-1} \omega^k\right)\right]$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{A\rho b}{\lambda(1-\omega^b)^2}\left[a\,\omega^{a-1} + \frac{\omega^a - \omega^b}{1-\omega}\right]$$

$$+ \frac{A}{\lambda(1-\omega^b)}\left[\frac{a}{1-\omega} - \frac{a\,\omega^{a-1}}{1-\omega}\frac{d}{d\omega}\left(\frac{\omega-\omega^a}{1-\omega}\right)\right]$$

$$= \sum_{j=0}^{a-2} q_j \frac{(a-j-1)}{\lambda} + \frac{A\rho b}{\lambda(1-\omega^b)^2}\left[a\,\omega^{a-1} + \frac{\omega^a - \omega^b}{1-\omega}\right]$$

$$+ \frac{A}{\lambda(1 - \omega^{b})} \left[\frac{a}{1 - \omega} - \frac{a\,\omega^{a-1}}{1 - \omega} - \left(\frac{(1 - \omega)\,(1 - a\omega^{a-1}) + \omega - \omega^{a}}{(1 - \omega)^{2}} \right) \right]$$

Simplifying further we get

$$\left[\sum_{j=0}^{a-2} (a-j-1)q_{j} + \frac{A\,\rho\, b}{(1 - \omega^{b})^{2}} \left(a\,\omega^{a-1} + \frac{\omega^{a} - \omega^{b}}{1 - \omega} \right) \right.$$

$$\left. + A \frac{1}{(1 - \omega)\,(1 - \omega^{b})} \left(a + \frac{\omega^{a} - 1}{1 - \omega} \right) \right] / \lambda$$

CHAPTER : III

ANALYSIS OF A SINGLE SERVER QUEUE WITH TIME AND OPERATION DEPENDENT SERVER FAILURES

In this section we consider a general M/G/1 queue where the server is subjected to time and operation dependent failures and arrival rate depending on the up and down state of the system. The time to failures are exponentially distributed and the repair times have general distribution. The failure rates depend on whether the server is idle or busy and the repair times depend on whether the failure occurred during the idle or busy period of the server.

Here we discuss this model in detail and we derive the steady state distribution of the number of customers in the system just after a service completion.

We list the characteristic of this model as

1. γ_0 - Failure rate of the server when it is idling
   γ - Failure rate of the server when it is busy
   The time to failure in both the cases are exponential
2. $G_0(.)$ - General distribution of the server repair times if failure occurred during idle period

G(.) - General distribution of the server repair times if failure occurred during busy period

3. B(.) - Cumulative distribution function (c.d.f) of the service time of the customers

The customers arrive to the system according to the Poisson distribution with rate $\lambda (\lambda_0)$ during the upstate (downstate) of the server.

Let Q_n be the number of customers in the system just after the n^{th} customer service completion.

Let r_i be the probability that i customers will arrive during an interval starting with the initiation of a service of a customer and ending with the completion of its service.

Let r_i^0 be the probability that i customers will arrive during an interval starting with the arrival epoch of the first customer to an empty system and ending with the service completion of the first customer.

$\{ Q_n, n = 1, 2, \ldots \}$ forms a Markov Chain with the transition probability matrix

$$P = \begin{bmatrix} r_0^{\,0} & r_1^{\,0} & r_2^{\,0} & \cdot & \cdot & \cdot & \cdot \\ r_0 & r_1 & r_2 & \cdot & \cdot & \cdot & \cdot \\ 0 & r_0 & r_1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & r_0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Then conditioning on the actual service length of the customer and the number of server breakdowns during the service we get

$$r_i = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{i} \frac{e^{-\lambda x} (\lambda x)^m}{m!} \frac{e^{-\lambda_0 y} (\lambda_0 y)^{i-m}}{(i-m)!} \frac{e^{-\gamma x} (\gamma x)^j}{j!} \, d\,G^j(y) \, d\,B(x); \; i \geq 0$$

Where $G^j(.)$ is the j fold convolution of G(.). Let

$$\tilde{F}(s) = \int_0^{\infty} e^{-st} \, d\,F(t)$$

The probability generating function of $\{r_i\}$ is given by

$$r(z) = \sum_{i=0}^{\infty} r_i z^i$$

$$= \sum_{i=0}^{\infty} \left[\int_{x=0}^{\infty} \int_{y=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{i} \frac{e^{-\lambda x} (\lambda x)^m}{m!} \frac{e^{-\lambda_0 y} (\lambda_0 y)^{i-m}}{(1-m)!} \frac{e^{-\gamma x} (\gamma x)^j}{j!} \, d\, G^j(y) \, dB(x) \right] z^i$$

$$= \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-\lambda x}_{\;x} \, e^{-\lambda_0 y} \sum_{j=0}^{\infty} e^{\lambda_0 yz} e^{\lambda xz} e^{-\gamma x} \frac{(\gamma x)^j}{j!} \, d\, G^j(y) \, dB(x)$$

$$= \int_{x=0}^{\infty} e^{-\lambda x(1-z)} e^{-\gamma x} \left[\int_{y=0}^{\infty} e^{-\lambda_0 y(1-z)} \sum_{j=0}^{\infty} \frac{(\gamma x)^j}{j!} \, d\, G^j(y) \right] d\, B(x)$$

$$= \int_{x=0}^{\infty} e^{-x(\gamma + \lambda(1-z))} e^{\gamma x \tilde{G}(\lambda_0(1-z))} \, dB(x)$$

$$= \tilde{B}(\gamma + \lambda(1-z) - \gamma \tilde{G}(\lambda_0(1-z)))$$

$$\lim_{z \to 1} r(z) = \tilde{B}(\gamma - \gamma\tilde{G}(0))$$
$$= 1 \qquad \text{(Since } \tilde{G}(0) = 1\text{)}$$

The arrivals during the time interval starting from the arrival epoch of the first customer to an empty system and ending with the service completion of this first customer can be separated into two parts.

i) The arrivals during the time interval starting from the arrival epoch of the first customer to an empty system and ending with the initiation of its service.

ii) The arrivals during the service period of the first customer.

Let u_i be the probability of i arrivals in the first part.

Then $$r_i^0 = \sum_{m=0}^{i} u_m r_{i-m}, \quad i \geq 0$$

Let u(z) and $r^0(z)$ be the p.g.f. of $\{u_i\}$ and $\{r_i^0\}$ respectively.

Then $$r^0(z) = \sum_{i=0}^{\infty} r_i^0 z^i$$
$$= \sum_{i=0}^{\infty} z^i \sum_{m=0}^{i} u_m r_{i-m}$$

$$= \quad u_0 r_0 + z(u_0 r_1 + u_1 r_0) +$$

$$z^2(u_0 r_2 + u_1 r_1 + u_2 r_0) + \dots$$

$$= \quad u_0 r(z) + u_1 z\, r(z) + u_2 z^2 r(z) + \dots$$

$$= \quad r(z)\, u(z)$$

Next we calculate u(z)

The first arrival to an empty system may find an up or down state of the server.

When the arrival finds up state then $u_0 = 1$ and $u_i = 0$, for all $i > 0$.

When the arrival finds down state then u_i will depend on the c.d.f. of remaining repair time. Let H(.) be its c.d.f. Then

$$u_0 \quad = \quad w.1 + v\,\tilde{H}(\lambda_0) \text{ and}$$

$$u_i \quad = \quad w.0 + v \int_0^\infty e^{-\lambda_0 t} \frac{(\lambda_0 t)^i}{i!}\, d\,H(t), \quad i > 0 \qquad (1)$$

where

v = probability that a customer finds a down state

w = probability that a customer finds an up-state

$$u(z) = u_0 + \sum_{i=1}^{\infty} u_i z^i$$

$$= v \tilde{H}(\lambda_0) + w + \sum_{i=1}^{\infty} \left[v \int_0^{\infty} e^{-\lambda_0 t} \frac{(\lambda_0 t)^i}{i!} \, d\,H(t) \right] z^i \quad \text{(from (1))}$$

$$= w + v \tilde{H}(\lambda_0) + v \int_0^{\infty} e^{-\lambda_0 t} (e^{\lambda_0 tz} - 1) \, d\,H(t)$$

$$= w + v \tilde{H}(\lambda_0) + v \int_0^{\infty} e^{-\lambda_0 t(1-z)} \, dH(t)$$

$$- v \int_0^{\infty} e^{-\lambda_0 t} \, d\,H(t)$$

$$= w + v \tilde{H}(\lambda_0(1-z)) \qquad (2)$$

It now remains to find v, w and H(.)

Consider the alternating renewal process $\{ X_n, Y_n, n = 1, 2, \ldots \}$ starting from time zero. Let x_n have an exponential c.d.f. X(.) and Y_n have the c.d.f. $G_0(.)$ for all $n \geq 1$

Therefore $\tilde{X}(x) = 1 - e^{-\gamma_0 x}, \quad x \geq 0$

$$\tilde{X}(s) = \int_0^\infty e^{-sx}\, d(1 - e^{-\gamma_0 x})$$

$$= \gamma_0 \int_0^\infty e^{-x(s + \gamma_0)}\, dx$$

$$= \frac{\gamma_0}{s + \gamma_0} , \quad Re(s) > 0$$

The probability that no arrival occur in the first up period

$$= \int_0^\infty e^{-\lambda x}\, dX(x)$$

$$= \tilde{X}(\lambda) \tag{3}$$

The probability that an arrival with occur in the first up period is $1 - \tilde{X}(\lambda) = \frac{\lambda}{\lambda + \gamma_0}$

The probability that no arrival occur in first down period

$$= \int_0^\infty e^{-\lambda_0 x}\, d(G_0(x)) = \tilde{G}_0(\lambda_0)$$

The probability that an arrival will occur during first down period given that it did not occur during the first up period is $1 - \tilde{G}_0(\lambda_0)$

$\therefore$ Probability that first customer to an empty system will see an up state

= Probability that the customer arrives in the first up period or probability that the arrival neither occurs during the first up period nor during the first down period but in some other up period

(i.e.) $$= 1 - \tilde{X}(\lambda) + \tilde{X}(\lambda)\ \tilde{G}_0(\lambda_0) .$$

(i.e.) $$[1 - \tilde{X}(\lambda)\ \tilde{G}_0(\lambda_0)] = 1 - \tilde{X}(\lambda)$$

(i.e.) $$= \frac{1 - \tilde{X}(\lambda)}{1 - \tilde{X}(\lambda)\ \tilde{G}_0(\lambda_0)}$$

$$= \frac{\lambda}{\lambda + \gamma_0\ (1 - \tilde{G}_0(\lambda_0))} \quad \text{(using (3))} \qquad (4)$$

Probability that first customer to an empty system will see down state

= Probability that the arrival did not occur in the first up period but occured during the first down period or probability that the arrival neither occured in the first up period nor during the first down period but occurred during some other down period

$$\text{(i.e.)} \quad v = \tilde{X}(\lambda)\,(1 - \tilde{G}_0(\lambda_0)) + \tilde{X}(\lambda)\,\tilde{G}_0(\lambda_0)\,v$$

$$v(1 - \tilde{X}(\lambda)\,\tilde{G}_0(\lambda_0)) = \tilde{X}(\lambda)\,(1 - \tilde{G}_0(\lambda_0))$$

$$\therefore \quad v = \frac{\tilde{X}(\lambda)\,(1 - \tilde{G}_0(\lambda_0))}{1 - \tilde{X}(\lambda)\,\tilde{G}_0(\lambda_0)}$$

$$= \frac{\gamma_0\,(1 - \tilde{G}_0\,(\lambda_0))}{\lambda + \gamma_0\,(1 - \tilde{G}_0(\lambda_0))}$$

(by using (3)) (5)

Imposing the condition that the arrival did not occur during the first up period, we get.

$$v(1 - H(x)) = \tilde{X}(\lambda) \int_0^\infty \lambda_0\, e^{-\lambda_0 t}\,(1 - G_0(t + x)\;dt + \tilde{X}(\lambda)\,\tilde{G}_0(\lambda_0)\;v(1 - H(x))$$

$$v(1 - H(x))\,(1 - \tilde{X}(\lambda)\,\tilde{G}_0(\lambda_0))$$

$$= \tilde{X}(\lambda) \int_0^\infty \lambda_0\, e^{-\lambda_0 t}\,(1 - G_0(t + x))\;dt$$

$$\therefore v(1 - H(x)) = \frac{\tilde{X}(\lambda)}{1 - \tilde{X}(\lambda)\,\tilde{G}_0(\lambda_0)} \int_0^\infty \lambda_0\, e^{-\lambda_0 t}\,(1 - G_0(t + x))\,dt$$

Taking Laplace transform on both sides we get

$$v\left(\frac{1}{s} - \frac{\tilde{H}(s)}{s}\right) = \frac{\tilde{X}(\lambda)}{1 - \tilde{X}(\lambda)\, \tilde{G}_0(\lambda_0)} \int_0^\infty e^{-sx} \int_0^\infty \lambda_0\, e^{-\lambda_0 t} (1 - G_0(t + x))\, dt\, dx$$

Substituting $u = t + x$, we get $du = dt$ and as t varies from 0 to ∞ x varies from u to ∞

$$\therefore \quad v\left(\frac{1}{s} - \frac{\tilde{H}(s)}{s}\right) = \frac{\tilde{X}(\lambda)}{1 - \tilde{X}(\lambda)\, \tilde{G}_0(\lambda_0)} \int_0^\infty \lambda_0\, e^{-\lambda_0 u} (1 - G_0(u))\, x \left[\int_u^\infty e^{-sx}\, e^{\lambda_0 x}\, dx\right] du$$

$$= \frac{\tilde{X}(\lambda)}{1 - \tilde{X}(\lambda)\, \tilde{G}_0(\lambda_0)} \int_0^\infty \lambda_0\, e^{-\lambda_0 u} (1 - G_0(u))\, \frac{e^{-su}\, e^{\lambda_0 u}}{\lambda_0 - s}\, du$$

(Since $\lambda_0 - s$ is a negative quantity

$$\therefore \quad \operatorname{lt}_{x \to \infty} e^{\lambda_0 - s} = 0)$$

$$= \frac{\tilde{X}(\lambda)}{1 - \tilde{X}(\lambda)\, \tilde{G}_0(\lambda_0)} \int_0^\infty \lambda_0 \,(1 - G_0(u))\, e^{-su}\, du$$

$$\frac{v}{s}(1-\tilde{H}(s)) = \frac{\lambda_0\, \tilde{X}(\lambda)\,(1 - \tilde{G}_0(s))}{s(\lambda_0 - s)\,(1 - \tilde{X}(\lambda)\, \tilde{G}_0(\lambda_0))}$$

$$v(1-\tilde{H}(s)) = \frac{\lambda_0}{\lambda_0 - s} \quad \frac{v}{1 - \tilde{G}_0(\lambda_0)} \quad (1 - \tilde{G}_0(s))$$

$$-v\tilde{H}(s) = -v + \frac{\lambda_0}{\lambda_0 - s} \quad \frac{v}{1 - \tilde{G}_0(\lambda_0)} \quad (1 - \tilde{G}_0(s))$$

$$v\, \tilde{H}(s) = \left[v(\lambda_0 - s) - \frac{\lambda_0\, v}{1 - \tilde{G}_0(\lambda_0)} \quad (1 - \tilde{G}_0(s)) \right] \frac{1}{s - \lambda_0}$$

$$= \frac{\gamma_0}{\lambda + \gamma_0(1 - \tilde{G}_0(\lambda_0))}$$

$$[\lambda_0(\tilde{G}_0(s) - \tilde{G}_0(\lambda_0)) - s(1 - \tilde{G}_0(\lambda_0))] \; \frac{1}{\lambda_0 - s}$$

$$\left[\text{Since} \quad \frac{v}{1 - \tilde{G}_0(\lambda_0)} = \frac{\gamma_0}{\lambda + \gamma_0(1 - \tilde{G}_0(\lambda_0))} \right]$$

It is easy to show that $u(1) = 1$

The expected number of arrivals during the time period beginning with the entrance of the first customer to an empty system and ending with the initiation of its service is

$$u'(1) = \underset{z \to 1}{\mathrm{lt}} \; u'(z)$$

$$= \underset{z \to 1}{\mathrm{lt}} \; \frac{\gamma_0}{\lambda + \gamma_0 \, (1 - \tilde{G}_0(\lambda_0))}$$

$$[-\lambda_0 \, z \, \tilde{G}_0^1 \, (\lambda_0 - \lambda_0 \, z) - \tilde{G}_0(\lambda_0 - \lambda_0 \, z) + \tilde{G}_0(\lambda_0)]$$

$$= \frac{\gamma_0}{\lambda + \gamma_0 \, (1 - \tilde{G}_0(\lambda_0))}$$

$$[- \gamma_0 \, \tilde{G}_0^{\,1}(0) - \tilde{G}_0(0) + \tilde{G}_0(\lambda_0)]$$

$$= \frac{- \gamma_0 \, \lambda_0 \, E(G_0) - \gamma_0 \, (1 - \tilde{G}_0(\lambda_0))}{\lambda + \gamma_0 \, (1 - \tilde{G}_0(\lambda_0))}.$$

To derive the steady state probability distribution function π_n, $n = 0, 1, 2, \ldots.$ of the Markov Chain $\{ Q_n \}$. Using the stationary equation $\pi P = \pi$ we get

$$(\pi_0\ \pi_1\ \ldots\ \pi_n\ \ldots)\begin{bmatrix} r_0^0 & r_1^0 & r_2^0 & \cdot & \cdot & \cdot & \cdot \\ r_0 & r_1 & r_2 & \cdot & \cdot & \cdot & \cdot \\ & r_0 & r_1 & \cdot & \cdot & \cdot & \cdot \\ & & r_0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$= (\pi_0\ \pi_1\ \ldots\ \pi_n\ \ldots)$$

$$\therefore\quad \pi_n = \pi_0\, r_n^0 + \sum_{j=1}^{n+1} \pi_j\, r_{n-j+1}, \qquad n = 0, 1, 2, \ldots$$

Taking p.g.f. on both sides,

$$\sum_{n=0}^{\infty} \pi_n z^n = \pi_0 \sum_{n=0}^{\infty} r_n^0 z^n + \sum_{n=0}^{\infty} \sum_{j=1}^{n+1} \pi_j\, r_{n-j+1}\, z^n$$

$$\pi(z) = \pi_0[r_0^0 + r_1^0 z + r_2^0 z^2 + \ldots] + \pi_1 r_0 +$$

$$+ (\pi_1 r_1 + \pi_2 r_0) z + \ldots$$

$$= \pi_0 r^0(z) + \pi_1 r(z) + \pi_2 z\, r(z) + \pi_3 z^2 r(z) + \ldots$$

$$= \pi_0 r^0(z) + \frac{r(z)}{z} (\pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \ldots)$$

$$= \pi_0 r^0(z) + \frac{r(z)}{z} (\pi(z) - \pi_0)$$

$$\therefore \quad \pi(z)\left(1 - \frac{r(z)}{z}\right) = \pi_0\, r^0(z) - \pi_0 \frac{r(z)}{z}$$

$$\pi(z) = \frac{\pi_0(z\, r^0(z) - r(z))}{z - r(z)}, \quad |z| \leq 1$$

π_0 can be calculated using $\operatorname*{lt}_{z \to 1} \pi(z) = 1$

$$\operatorname*{lt}_{z \to 1} \pi(z) = \frac{\pi_0 \left[\operatorname*{lt}_{z \to 1} u(z)\, r(z) - \operatorname*{lt}_{z \to 1} r(z) \right]}{1 - \operatorname*{lt}_{z \to 1} r(z)}$$

Applying L' Hospital's rule and using $u(1) = 1$ and $r(1) = 1$ we get

$$1 = \frac{\pi_0[1 + \rho_0 - \rho]}{1 - \rho}$$

$$\pi_0 = \frac{1 - \rho}{1 + \rho_0 - \rho}$$

where

$$\rho = \operatorname*{lt}_{z \to 1} \frac{d}{dz} r(z)$$

$$= \frac{}{dz} \tilde{B}(\gamma + \lambda(1 - z) - \gamma\, \tilde{G}(\lambda_0(1 - z)))$$

$$= - \lambda\, E(B) + \lambda_0\, \gamma\, E(G)\, E(B)$$

$$\rho_0 = \lim_{z\to 1} \frac{d}{dz} r^0(z)$$

$$= \lim_{z\to 1} \frac{d}{dz} [u(z)\ r(z)]$$

$$= \rho + \frac{\gamma_0\ (\lambda_0\ E(G_0) - (1 - \tilde{G}_0(\lambda_0)))}{\lambda + \gamma_0(1 - \tilde{G}_0(\lambda_0))}$$

Therefore

$$\pi(z) = \left(\frac{1 - \rho}{1 + \rho_0 - \rho}\right) \left(\frac{z\ r^0(z) - r(z)}{z - r(z)}\right)$$

Thus the p.d.f. $\{\pi_n\}$ of the number of customers in the system just after a service completion is derived.

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