Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Background and Motivation

Artificial Neural Networks (ANN) is now widely used as a new method for information processing in the fields of pattern recognition, associative memory and combinatorial optimization etc. Although most neural systems are realized by software simulations, only hardware implementation can fully utilize its advantages of parallel processing and error tolerance. Until now, efforts to construct hardware realizations of artificial neural networks were devoted primarily to the implementation of models, which ignore the dynamical behaviors of neural networks. It is well known that, stability for neural networks is one of the preconditions in the designs, applications and VLSI implementation of neural networks, therefore, the dynamical behaviors of neural networks have received a great deal of interest, for details see Cichoki and Unbehauen (1993), Haykin (1998) and Hagan et al.(1996).

In biological and artificial neural networks, the interactions between neurons are generally asynchronous which inevitably result in time delays. In electronic implementation of analog neural networks, nevertheless, the delays are usually time-varying due to the finite switching speed of amplifiers. It is well known that time delays are often a source of instability of neural networks from Liao et al.(2002 b). So, it is important to study the stability of neural networks with time delays from Cao et al. (2006) and Zhao (2004). Passivity theory is another effective tool to analyze the stability of a nonlinear system. It may deal with nonlinear systems using only the general characteristics of the input-output dynamics, and offers elegant solutions for the proof of absolute stability. Passivity framework is a promising approach to the stability analysis of NNs. This chapter aims to mainly focus on the description of NNs, SDEs, switched NNs, FNNs, Lyapunov stability, passivity analysis and LMI techniques. Further, the occurrence of different types of neural networks in various application fields is also provided.

1.1.2 Artificial Neural Network Model

A human brain consists of approximately 10^{11} computing elements called neurons. They communicate through a connection network of axons and synapses having a density of approximately 10^4 synapses per neuron. The elementary nerve cell, called a neuron, is the fundamental building block of the biological NN. Its schematic diagram is shown in Figure 1.1. A typical cell has three major regions: the cell body, which is also called the soma, the axon, and the dendrites. Dendrites form a dendritic tree, which is a very fine bush of thin fibers around the neurons body. Dendrites receive information from neurons through axons-long fibers that serve as transmission lines. An axon is a long cylindrical connection that carries impulses from the neuron. The end part of an axon splits into a fine arborization. Each branch of it terminates in a small end bulb almost touching the dendrites of neighboring neurons. The axon-dendrite contact organ is called a synapse. The synapse is where the neuron introduces its signal to the neighboring neuron. The signals reaching a synapse and received by dendrites are electrical impulses. The interneuronal transmission is sometimes electrical but is usually effected by the release of chemical transmitters at the synapse. Thus, terminal boutons generate the chemical that affects the receiving neuron. The receiving neuron either generates an impulse to its axon, or produces no response.

The neuron is able to respond to the total of its inputs aggregated within a short time interval called the period of latent summation. The neurons response is generated if the total potential of its membrane reaches a certain level. The membrane can be considered as a shell, which aggregates the magnitude of the incoming signals over some duration. Specifically, the neuron generates a pulse response and sends it to its axon only if the conditions necessary for firing are fulfilled. The hypothesis regarding the modeling of the natural nervous system

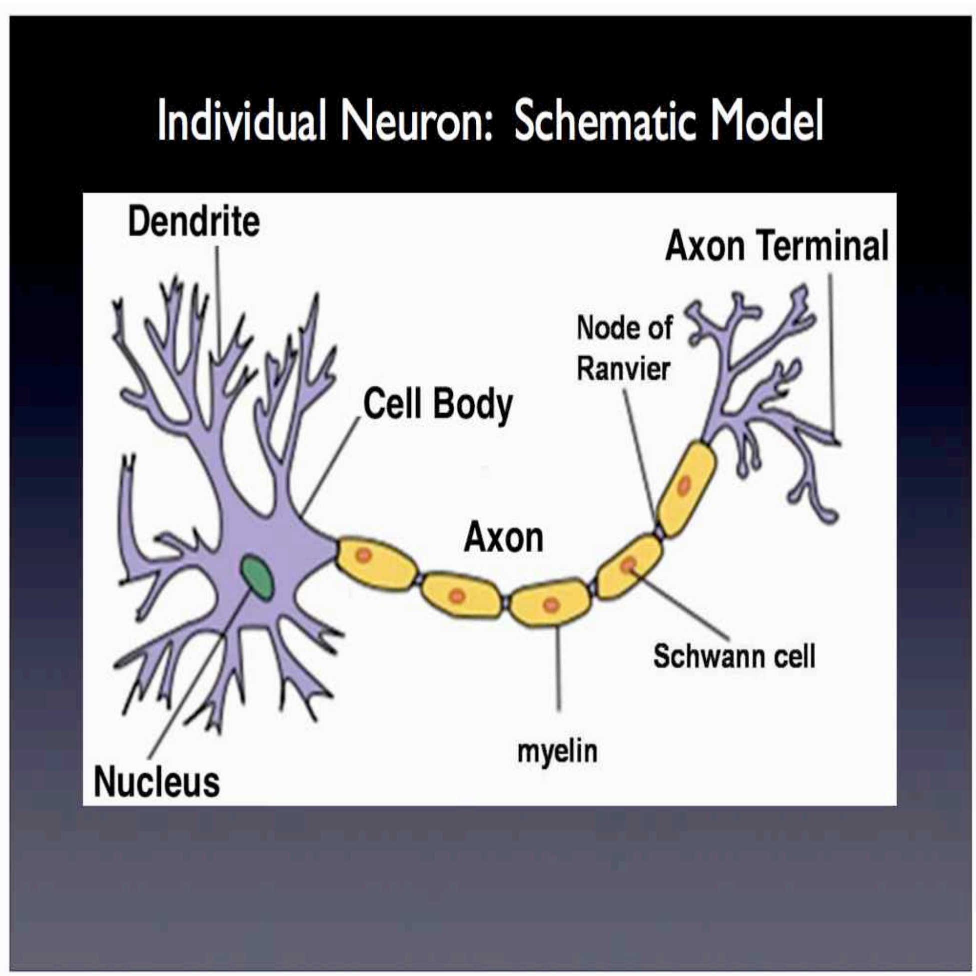

Figure 1.1: Biological Neuron

is that neurons communicate with each other by means of electrical impulses as discussed in Arbib (1987). The neurons operate in a chemical environment that is even more important in terms of actual brain behavior. Thus one can consider the brain to be a densely connected electrical switching network conditioned largely by the biochemical processes. The vast neural networks has an elaborate structure with very complex interconnections. The input to the network is provided by sensory receptors. Receptors deliver stimuli both from within the body, as well as from sense organs when the stimuli originate in the external world. The stimuli are in the form of electrical impulses that convey the information into the network of neurons. As a result of information processing in the central nervous systems, the effectors are controlled and give human responses in the form of diverse actions. Thus, we have a three stage system consisting of receptors, neural networks, and effectors, in control of the organism and its actions.

Neural networks makes an attempt to simulate human brain. The simulation is based on the present knowledge of brain function, and this knowledge is even at its best primitive. So, it is not absolutely wrong to claim that artificial neural networks probably have no close relationship to operation of human brains. The learning of artificial neural networks may be based on two mechanisms: the creation of new connection, and the modification of connections. Each neuron has an activation level which, in contrast to Boolean logic, ranges between some minimum and maximum value.

In Artificial Neural Networks, the inputs of the neuron are combined in a linear way with different weights. The result of this combination is then fed into a non-linear activation unit (activation function), which can in its simplest form be a threshold unit. It is illustrated in Figure 1.2. Neural networks offer nonlinearity, input-output mapping, adaptivity and fault tolerance. Nonlinearity is a desired property if the generator of input signal is inherently nonlinear. The high connectivity of the network ensures that the influence or errors in a few terms will be minor, which ideally gives a high fault tolerance.

1.1.3 Network Topologies

The pattern of connections between the units and the propagation of data is classified into two types. As for as this pattern of connections the two main distinctions are:

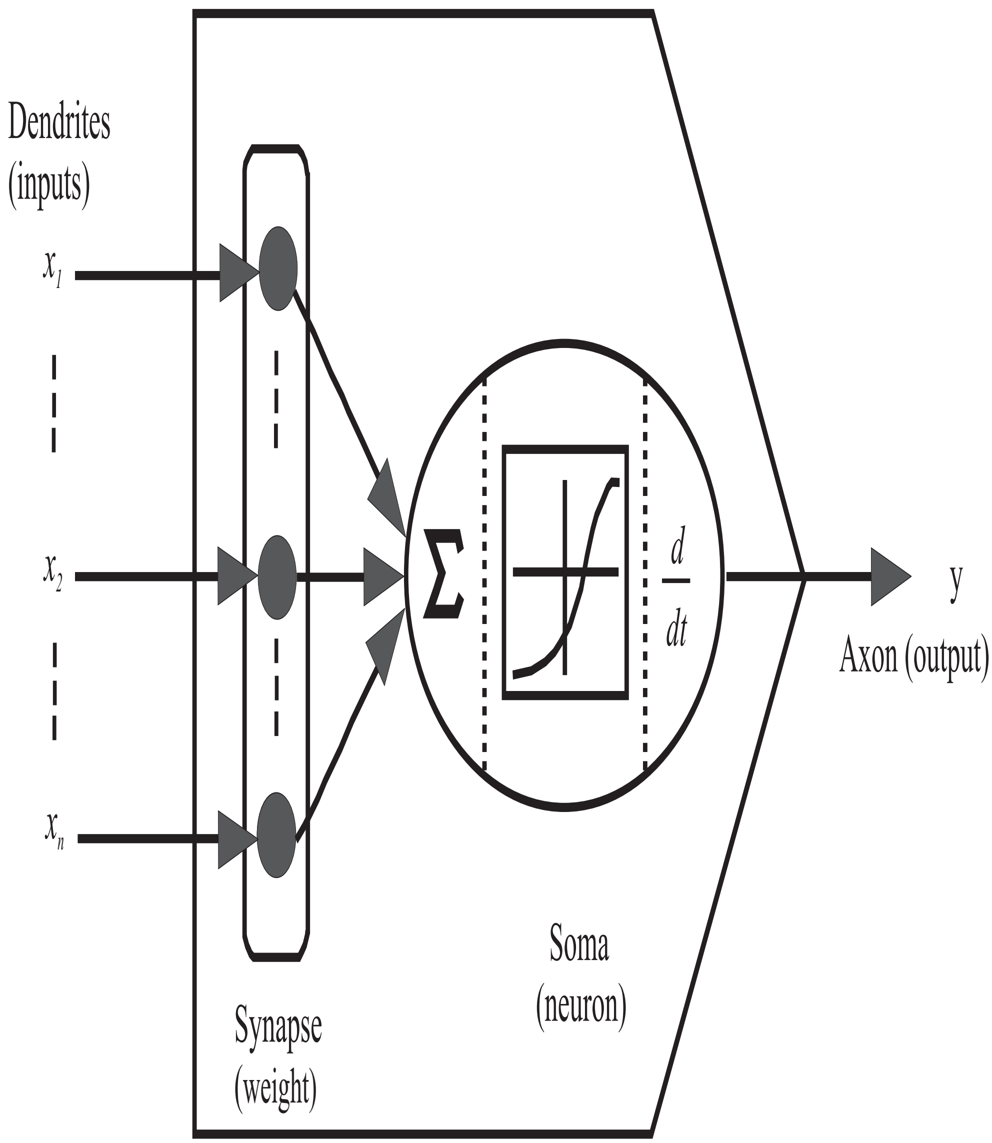

Figure 1.2: An Artificial Neuron

(i) **Feed-Forward Networks**, where the data flow from input to output units is strictly feed-forward. The data processing can extend over multiple (layers of) units, but no feedback connections are present, that is, connections extending from outputs of units to inputs of units in the same layer or previous layers.
(ii) **Recurrent Networks** that contain feedback connections. Contrary to feed-forward networks, the dynamical properties of the network are important. In some cases, the activation values of the units undergo a relaxation process such that the network will evolve to a stable state in which these activations do not change any more. In other applications, the change of the activation values of the output neurons are significant, such that the dynamical behavior constitutes the output of the network. Classical examples of feed-forward networks are the Perceptron and Adaline. Examples of RNNs have been presented by Anderson and Vongpanitlerd (1973), Hopfield (1982) and Kohonen (1977). The details are given in the following section.

1.1.4 Hopfield Neural Networks

One of the earliest Recurrent Neural Networks (RNN) reported in literature was the auto-associator independently described by Anderson (1977) and Kohonen (1977). It consists of a pool of neurons with connections between each unit i and j, $i \neq j$. All connections are weighted. Hopfeild Neural Networks (HNN) are recurrent networks introduced by Hopfield (1982). The earliest Hopfield network, which employs two-state (on/off) neurons, is used for the design of neural content addressable memories. Hopfield later introduced a modified version of his earlier model that employed a continuous nonlinear function to describe the output behavior of the neurons. The Hopfield network (model) consists of a set of neurons and a corresponding set of unit delays, forming a multiple-loop feedback system, as illustrated in Figure 1.3. The number of feedback loops is equal to the number of neurons. Basically, the output of each neuron is fed back via a unit delay element, to each of the other neurons in the network. In other words, there is no self-feedback in the network. The Hopfield network is useful as a content addressable memory or an analog computer for solving combinatorial-type optimization problems.

Hopfield (1982) brings together several earlier ideas concerning Hopfield networks and presents a complete mathematical analysis, see Amit et al.(1986). It is therefore, that this network described in this chapter is generally referred to as the Hopfield network.

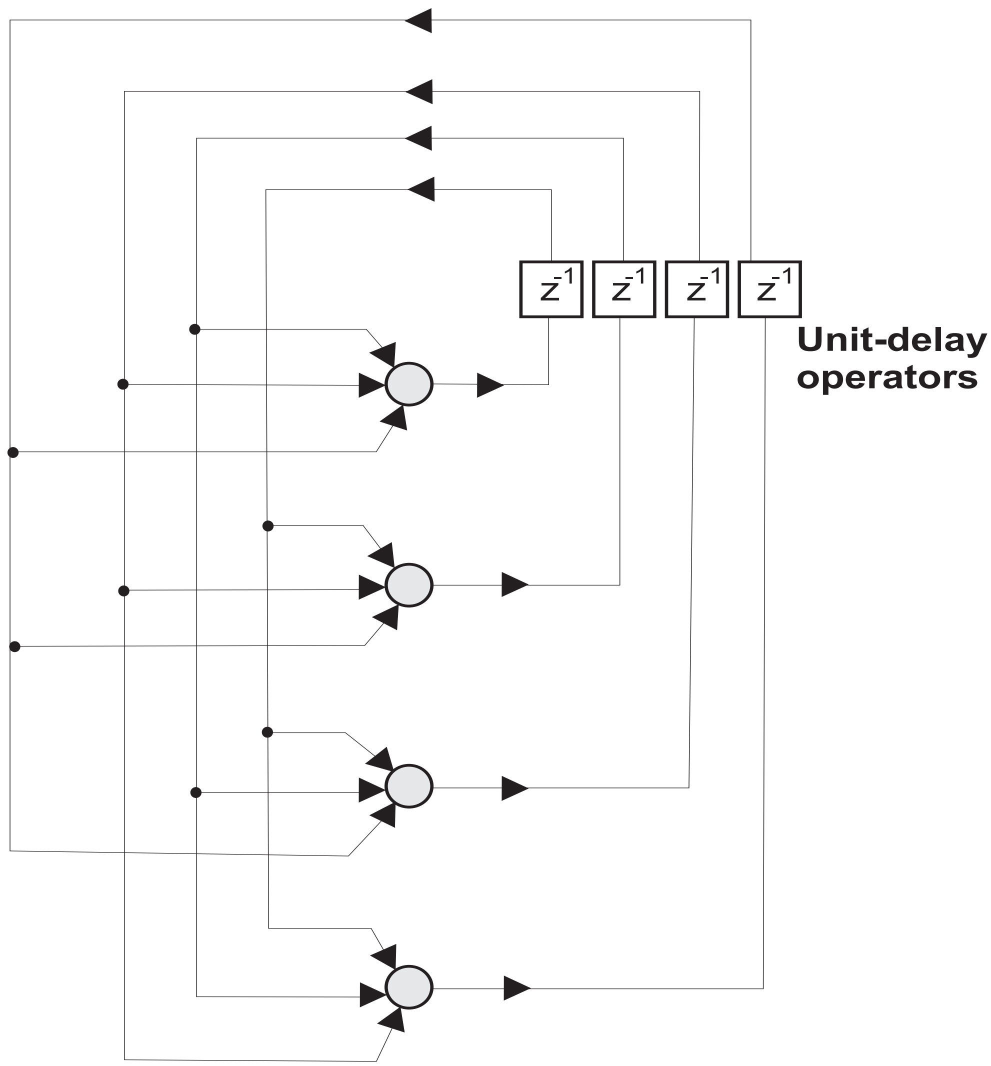

Figure 1.3: Architectural graph of a Hopfield network consisting four neurons

1.1.5 Implementation of Hopfield Neural Networks

A well-known model of dynamic RNNs with some useful collective computational properties is due to Hopfield. A continuous-time model of an analog NN can be described by the following system of nonlinear DEs

$$\begin{aligned} \text{State equation: } C_i \frac{dx_i(t)}{dt} &= -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij} y_j(t) + s_i, \quad i = 1, 2, \cdots, n \\ \text{Output equation: } y_i(t) &= \sigma_i(x_i(t)), \quad i = 1, 2, \cdots, n \end{aligned} \tag{1.1.1}$$

where x_i represents the state of the ith neuron, y_i is the output of the ith neuron, w_{ij} is the synaptic connection weight from the ith neuron to the jth neuron, s_i is a constant external input, $\sigma_i(\cdot)$ is the activation function. This nonlinear system can be implemented by an analog RC (resistance-capacitance) network circuit which is shown in Figure 1.4. A circuit contains a RC network at the input of each amplifier. The capacitance $C_i > 0$ and the resistance $R_i > 0$ represent the total shunt capacitance and shunt resistance at the input of the ith amplifier. Since the intrinsic delay exhibited by any physical amplifier is modelled by an input resistance $R_i > 0$ and $C_i > 0$, which are drawn as external components, an actual operational amplifier can, therefore, be assumed as an ideal amplifier without delay. Furthermore, let R_{ij} be the resistor connecting the output of the jth amplifier to the input of the ith amplifier and s_i the fixed external input current.

$$\begin{aligned} u_i &= x_i : \text{input voltage of the } i\text{th amplifier} \\ V_i &= \sigma_i(u_i) : \text{output of the } i\text{th amplifier, where each operational amplifier has two} \\ &\quad\ \text{output terminals each providing } V_i \text{ and } -V_i. \end{aligned}$$

An electronic circuit consisting of operational amplifiers, capacitors and resistors should be able to operate as a Hopfield network. This circuit can be designed by reconstructing the stable states that have been designed using the proper value of w_{ij} and as long as w_{ij} is symmetric; that is $w_{ij} = w_{ji}$ and the amplifiers are quick compared with the characteristic of the neural time constant R_iC_i. In this case, the neural system converges to stable states and will not oscillate or display chaotic behavior. The innovative concepts and implementations of a single-chip electronic NN along the lines just discussed have been reported by several groups using very-large-scale integration.

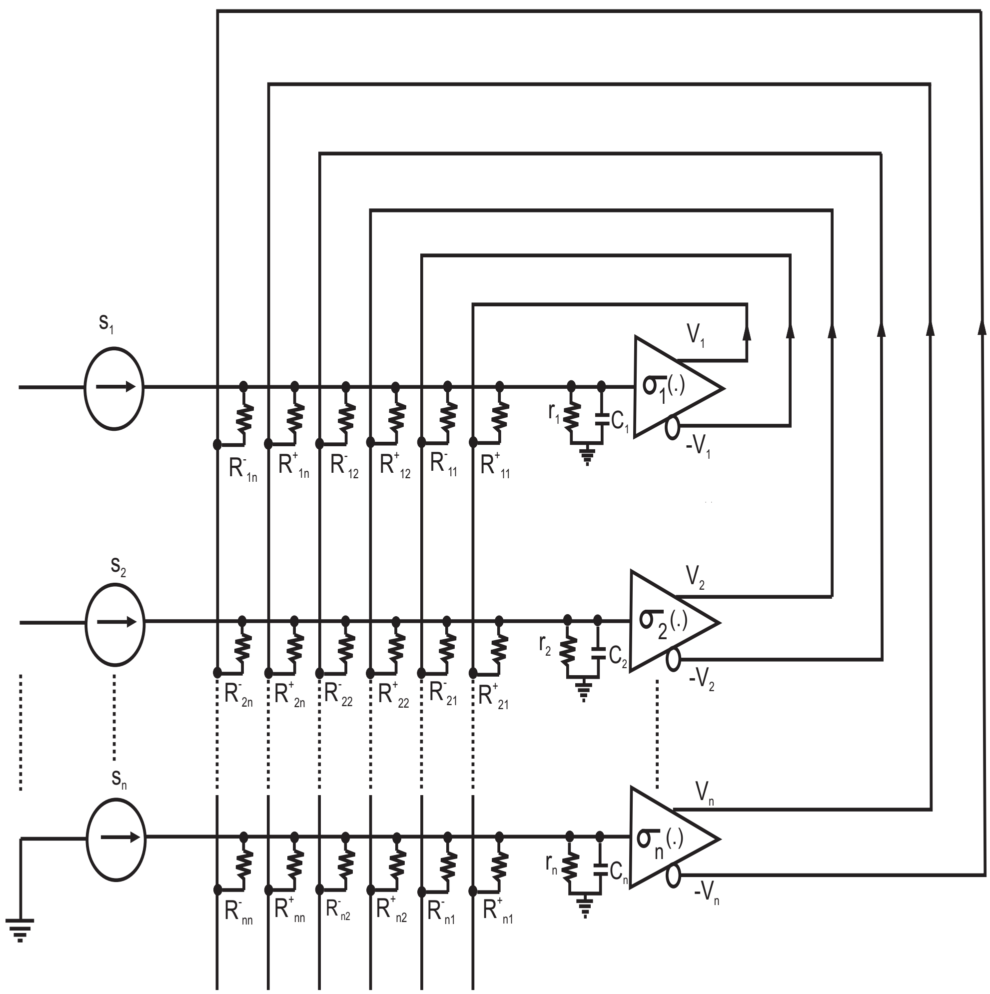

Figure 1.4: Dynamic NN Circuit Structure

1.1.6 Cohen-Grossberg Neural Networks

A large class of neural networks which can function as stable content addressable memories was proposed by Cohen and Grossberg (1983). In this model, the feedback terms consist of amplification and stabilizing functions which are generally nonlinear. These terms provide the model with a special kind of generalization where in many neural networks models that are capable of content addressable memory such as additive neural networks, Cohen-Grossberg Neural Networks(CGNN) and Bidirectional Associative Memory Neural Networks (BAMNN) and also biological models such as Lotka-Volterra models of population dynamics are included as special cases. CGNNs were designed to include additive neural networks followed by Hopfield (1982, 1984). In the original analysis, Cohen and Grossberg assumed that the weight matrix was symmetric. Meanwhile, the activation functions are assumed to be continuous, differentiable, monotonically increasing, and bounded such as sigmoid-type function. CGNNs and their various generalizations with or without transmission delays have been investigated by several authors Gao and Cui (2009), Li (2009 a, 2009 b).

1.1.7 Stochastic Process

The word “stochastic” means “pertaining to chance” (Greek roots), and is thus used to describe subjects that contain some element of random or stochastic behavior. For a system to be stochastic, one or more parts of the system has randomness associated with it. Unlike a deterministic system, for example, a stochastic system does not always produce the same output for a given input. A few components of systems that can be stochastic in nature include stochastic inputs, random time-delays, noisy (modelled as random) disturbances, and even stochastic dynamic processes.

A stochastic process is one whose behavior is non-deterministic, in which a system’s subsequent state is determined both by the process’s predictable actions and by a random element.

An example of a stochastic process in the natural world is pressure in a gas as modelled by the Wiener process. Even though each molecule is moving in a deterministic path, the motion of a collection of them is computationally and practically unpredictable. A large enough set of molecules will exhibit stochastic characteristics, such as filling the container, exerting equal pressure, diffusing along concentration gradients, etc.

Markov Property

In probability theory and statistics, the term Markov property refers to a property of a stochastic process which is initiated by the Russian Mathematician Andrey Markov (1954). Moreover, stochastic process has the Markov property if the conditional probability distribution of future states of the process depend only upon the present state; that is, given the present, the future does not depend on the past. A process with this property is called Markov process. The term strong Markov property is similar to this, except that the meaning of "present" is defined in terms of a certain type of random variable, which might be specified in terms of the outcomes of the stochastic process itself, known as stopping time.

An extension of the Markov property to other circumstances is encompassed by the idea of a Markov random field, which extends the formulation of the property to apply to two or more dimensions, or to random variables defined for an interconnected network of items.

Markov Jump Process

One of the main issues in control systems is their capability of maintaining an acceptable behavior and meeting some performance requirements even in the presence of abrupt changes in the system dynamics. These changes can be due to abrupt environmental disturbances, component failures or repairs, changes in subsystems interconnections, abrupt changes in the operation point for a non-linear plant, etc. Examples of these situations can be found, in economic systems, aircraft control systems, control of solar thermal central receivers, robotic manipulator systems, large flexible structures for space stations, etc. In some cases these systems can be modelled by a set of discrete-time linear systems with model transition given by a Markov chain. This family is known in the specialized literature as Markov jump linear systems.

Markov Jump Process (MJP) plays an important role in a large number of application domains. However, realistic systems are analytically intractable and they traditionally have been analyzed using simulation based techniques, which do not provide a framework for statistical inference. A mean field approximation is proposed to perform posterior inference and parameter estimation. The approximation allows a practical solution to the inference problem, while still retaining a good degree of accuracy. The author illustrates the approach on two biologically motivated systems. MJPs provide a rigorous probabilistic framework to model the joint dynamics of groups (species) of interacting individuals, with applications ranging from information packets in a telecommunications network to epidemiology and population levels in the environment. These processes are usually non-linear and highly coupled, giving rise to non-trivial steady states (often referred to as emerging properties). Unfortunately, this also means that exact statistical inference is unfeasible and approximations must be made in the analysis of these systems. A traditional approach, which has been very successful throughout the past century, is to ignore the discrete nature of the processes and to approximate the stochastic process with a deterministic process whose behavior is described by a system of non-linear and coupled ODEs.

Brownian Motion

Brownian motion named after the Scottish Botanist Robert Brown (1828) is the seemingly random movement of particles suspended in a fluid (i.e. a liquid such as water or air) or the mathematical model used to describe such random movements, often called a particle theory. Brown in $1826 - 1827$ observed the irregular motion of pollen particles suspended in water and noted that

✠ the path of a given particle is very irregular, having a tangent at no point, and

✠ the motions of two distinct particles appear to be independent.

The mathematical model of Brownian motion has several real-world applications including stock market fluctuations. However, movements in share prices may arise due to unforeseen events which do not repeat themselves, further physical and economic phenomena are not comparable. Brownian motion is among the simplest of the continuous-time stochastic processes, and it is a limit of both simpler and more complicated stochastic processes. This universality is closely related to that of the normal distribution. In both cases, it is often mathematical convenience rather than the accuracy of the models that motivates their use.

First time Bachelier attempted to describe fluctuations in stock prices mathematically and essentially discovered certain results later rederived and extended by Einstein.(for which he received the Nobel prize in 1921). The motion was later explained by random collisions with the molecules of water. To describe it mathematically, the concept of a stochastic process $B_t(\omega)$ is used and interpreted as the position of the pollen grain ω at time t.

For a dynamic system the simplest continuous stochastic perturbation is naturally considered to be a Brownian motion, since it is a Normal process (or say, a

Guassian process) with independent increments which are also normally distributed. In general, a continuous stochastic perturbation will be modelled as some stochastic integral with respect to the Brownian motion.

1.1.8 Stochastic Differential Equations

Stochastic Differential Equation (SDE) is a combination of differential equations, probability theory and stochastic process. SDEs constitute an ideal mathematical model for a multitude of phenomena and processes encountered in areas such as filtering, optimal stopping, stochastic control, signal processes and mathematical finance, most notably in option pricing. In some real physical engineering problems such as wind excitation or seismic impact, it is very difficult to describe the dynamic behavior of the system by a mathematical model. The possible way to model these excitations is by the use of probabilistic Mathematics instead of deterministic Mathematics. Unlike their deterministic counterparts SDEs do not have explicit solutions, apart from a few exceptional cases, hence the necessity for a sound theory of their numerical approximations. There are two classes of their numerical methods for approximating SDEs. The objective of the first is to produce a pathwise approximations of the solution. The second method involves approximating the distribution of the solution at a particular instance in time.

In sciences, SDEs are used to model systems that are inherently random or subject to random external perturbations. Furthermore, systems in continuum mechanics or in financial economics have governing equations, which involve integral terms representing the effect of the past. All kinds of dynamics with stochastic influence in nature or complex system created by mankind are modelled by SDEs.

Example 1.1.1

Considering the estimation of the horizontal distance that certain seeds traverse when falling from a given height under the influence of a randomly varying wind. It is assumed that the seeds experience a frictional force proportional to the square of the speed of the air on the seeds.

First, consider a deterministic model where the wind speed is constant and not varying randomly. Let $V_\omega(t) = v_\omega(t)i$ be the wind velocity, $V(t) = v_x(t)i + v_y(t)j$ be the seed velocity, and $V_a(t) = V_\omega(t) - V(t)$ be the air velocity on the seed at time t. Let$F_f(t) = k|V_a(t)|^2 \frac{V_a(t)}{|V_a(t)|} = k|V_a(t)|V_a(t)$ be the frictional force on the seed at time t, where k is a constant of proportionality. Let $F_g(t) = -mgj$ be the force of gravity on the seed of mass m. Finally, let $s(t) = x(t)i + y(t)j$ be the position of the seed at time t. It is assumed that $V(0) = 0 = 0i + 0j$ and $s(0) = hj$ where h is the initial height. It is straightforward to check that the velocity and the position of the seed at any time t satisfy the following initial-value system

$$\begin{cases} \frac{ds(t)}{dt} = V(t), \\ \frac{dV(t)}{dt} = \frac{1}{m}\Big(F_f(t) + F_g(t)\Big) = \frac{k}{m}|V_a(t)|V_a - gj, \\ V_a(t) = V_m(t) - V(t) \text{ with } V_\omega(t) = \upsilon_\omega i, \\ s(0) = hj, \quad V(0) = 0. \end{cases} \tag{1.1.2}$$

Now suppose that the wind speed randomly varies about a mean speed υ_e. In particular, the wind speed can experience a change of $\pm\alpha$ during a small time interval Δt with the probabilities listed in Table 1.1 . Here it is assumed that Δt is sufficiently small so that $p_1, p_2 > 0$ (p_1 and p_2 can be defined as $p_l = max\Big\{0, (\lambda + (-1)^l\beta(\upsilon_e - \upsilon_\omega(t)))\Delta t\Big\}$ for $l = 1, 2$ to guarantee nonnegativity of the probabilities). The value $\lambda\Delta t$ represents the probability associated with random diffusion of the wind speed and does not depend on $\upsilon_\omega(t)$. The term $\pm\beta(\upsilon_e - \upsilon_\omega(t))$ represents the probability associated with drift towards the mean wind speed of υ_e. When $\upsilon_\omega(t) \neq \upsilon_e$, the probability that the wind speed moves closer to υ_e is greater than the probability that the wind speed moves further from υ_e. Thus, υ_e can be regarded as a mean wind speed.

The next step is to find the mean wind speed change and the variance in the change. It is straightforward to show that the required expectations to order $(\Delta t)^2$ are

$$E(\Delta\upsilon_\omega) = 2\alpha\beta(\upsilon_e - \upsilon_\omega(t))\Delta t$$

and

$$E((\Delta\upsilon_\omega)^2) = 2\alpha^2\lambda\Delta t.$$

Based on the above arguments, a reasonable SDE model for the randomly wind speed is

$$d\upsilon_\omega(t) = 2\alpha\beta(\upsilon_e - \upsilon_\omega(t))dt + \sqrt{2\alpha^2\lambda}dW(t).$$

Table 1.1: **Possible change in the wind speed with the corresponding probabilities**

Change $\varDelta\upsilon_\omega$	Probability
$(\varDelta\upsilon_\omega)_1 = -\alpha$	$p_1 = (\lambda - \beta(\upsilon_e - \upsilon_\omega(t)))\varDelta t$
$(\varDelta\upsilon_\omega)_2 = \alpha$	$p_2 = (\lambda + \beta(\upsilon_e - \upsilon_\omega(t)))\varDelta t$
$(\varDelta\upsilon_\omega)_3 = 0$	$p_3 = 1 - p_1 - p_{2=1-2\lambda\varDelta t}$

Indeed, this SDE can be solved exactly to yield

$$\upsilon_\omega(t) = \upsilon_e + exp(-2\alpha\beta t)\Big(- \upsilon_e + \upsilon_\omega(0) + \int_0^t \sqrt{2\alpha^2\lambda}\ exp(2\alpha\beta s)dW(s)\Big).$$

This solution implies, for large time t, that the wind speed $\upsilon_\omega(t)$ is approximately normally distributed with mean υ_e and variance $2\alpha^2\lambda/(4\alpha\beta) = \alpha\lambda/(2\beta)$.

The complete model for the seed dispersal dynamics in a randomly varying wind is given as

$$\begin{cases} \frac{ds(t)}{dt} = V(t), \\ \frac{dV(t)}{dt} = \frac{1}{m}\Big(F_f(t) + F_g(t)\Big) = \frac{k}{m}|V_a(t)|V_a - g_j, \\ \frac{d\upsilon_\omega}{dt} = 2\alpha\beta(\upsilon_e - \upsilon_\omega(t)) + \sqrt{2\alpha^2\lambda}\ \frac{dW(t)}{dt}, \\ V_a(t) = V_\omega(t) - V(t), \\ s(0) = hj, \quad V(0) = 0, \quad V_\omega(0) = \upsilon_w i. \end{cases}$$

This problem illustrates the ease with which a deterministic model can be transformed into a SDE model for certain physical problems after agreeing upon a discrete stochastic model. One can notice that there are other interesting possibilities for modeling the wind speed besides the model described here. For example, in constructing the probabilities for the wind speed changes, consider $p_l = (\lambda + (-1)^l\beta(\upsilon_e - \upsilon_\omega(t))^{\frac{k_1}{k_2}})\varDelta t$ for $l = 1, 2$ for some positive odd integers k_1 and k_2.

Selecting the parameter values and the discrete stochastic process that represents a given physical situation may involve many computational comparisons with physical data.

1.1.9 Stability

Lyapunov (1892) introduced the concept of stability of a dynamic system. Roughly speaking, the stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are "close" to each other at a specific instant should therefore remain close to each other at all subsequent instants. Many parts of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions and the trajectories. The simplest kind of behavior is exhibited by equilibrium points, or fixed points, and by periodic orbits. If a particular orbit is well understood, it is natural to ask whether a small change in the initial condition will lead to similar behavior. Stability means that the trajectories do not change too much under small perturbations. The opposite situation, where a nearby orbit is getting repelled from the given orbit is also of interest. In general, perturbing the initial state in some directions results in the trajectory asymptotically approaching the given one and in other directions to the trajectory getting away from it. There may also be directions for which the behavior of the perturbed orbit is more complicated (neither converging nor escaping completely), and then stability theory does not give sufficient information about the dynamics. One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit. In particular, at each equilibrium of a smooth dynamical system with an n-dimensional phase space, there is a certain $n \times n$ matrix A whose eigenvalues characterize the behavior of the nearby points. More precisely, if all eigenvalues are negative real numbers or complex numbers with negative real parts then the point is a stable attracting fixed point and the nearby points converge to it at an exponential rate. If none of the eigenvalues is purely imaginary (or zero) then the attracting and repelling directions are related to the eigen spaces of the matrix A with eigenvalues whose real part is negative and positive respectively. Analogous statements are known for perturbations of more complicated orbits.

Concept of stability in common and Engineering sense reflects necessity to keep response of a disturbed system within accepted limits. If deviations describing response of the system from a given regime (e.g. state of equilibrium) lie within prescribed limits, the system is called stable. Otherwise, the system is called unstable. Disturbances, response and prescribed limits can be specified in each case in different ways.

1.1.10 Lyapunov Stability Theory

In Mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. Lyapunov developed the stability theory of dynamical systems determined by nonlinear time-varying ordinary differential equations. In the concepts of stability and instability, he has developed two general methods for the stability analysis of an equilibrium: Lyapunov's direct method, also called the second method of Lyapunov, and the indirect method of Lyapunov, also called the first method. The former involves the existence of scalar-valued auxiliary functions of the state space (called Lyapunov functions) to ascertain the stability properties of an equilibrium, whereas the latter seeks to deduce the stability properties of an equilibrium of a system described by a nonlinear differential equation from the stability properties of its linearization. In the process of discovering the first method, Lyapunov established some important stability results for linear systems (involving the Lyapunov matrix equation).

An orbit is called Lyapunov stable if the forward orbit of any point in a small enough neighborhood of it stays in a small (but perhaps, larger) neighborhood. Various criteria have been developed to prove stability or instability of an orbit. Under favourable circumstances, the problem of stability may be reduced to a well-studied problem involving eigenvalues of matrices. A more general method involves Lyapunov functions where as Lyapunov stability theorems give only sufficient condition.

Lyapunov method is a powerful method for determining the stability or instability of fixed points of nonlinear autonomous systems. In Mathematics, Lyapunov functions are functions which can be used to prove the stability of a certain fixed point in a dynamical system or autonomous differential equation. One must be aware that the basic Lyapunov theorems for autonomous systems are sufficient, but not necessary tool to prove the stability of an equilibrium. Finding a Lyapunov function for a certain equilibrium might be a matter of luck. Trial and error is the method to apply, when testing Lyapunov-candidate functions on some equilibrium.

1.1.11 Basic Theorems of Lyapunov

Lyapunov theorem states as follows: Let $V(x,t)$ be a non-negative function with derivative $\dot{V}(x,t)$ along the trajectories of the system.

- ✠ If $V(x,t)$ is locally positive definite and $\dot{V}(x,t) \leq 0$ locally in x and for all t, then the origin of the system is locally stable (in the sense of Lyapunov).
- ✠ If $V(x,t)$ is locally positive definite and decrescent, further $\dot{V}(x,t) \leq 0$ locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).
- ✠ If $V(x,t)$ is locally positive definite and decrescent, further $-\dot{V}(x,t)$ is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.
- ✠ If $V(x,t)$ is positive definite and decrescent, also $-\dot{V}(x,t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.

The above theorem gives sufficient conditions for the stability of the origin of a system. It does not, however, give a prescription for determining the Lyapunov function $V(x,t)$. Since the theorem only gives sufficient conditions, the search for a Lyapunov function establishing stability of an equilibrium point could be arduous. However, it is a remarkable fact that the converse of theorem also exists: if an equilibrium point is stable, then there exists a function $V(x,t)$ satisfying the conditions of the theorem. However, the utility of this and other converse theorems is limited by the lack of a computable technique for generating Lyapunov functions. Theorem also stops short of giving explicit rates of convergence of solutions to the equilibrium. It may be modified to do so in the case of exponentially stable equilibria.

1.1.12 Basic Ideas for Lyapunov's Framework

In the Lyapunov framework, there are two basic ideas

- ✠ To construct some simple quadratic Lyapunov candidates (Lyapunov–Krasovskii functional, or Lyapunov-Razumikhin functions) leading to some sufficient stability conditions, more or less conservative.

✠ To construct more "complicated" quadratic Lyapunov candidates, which lead to necessary and sufficient asymptotic stability conditions, but which are difficult to check for practical problem. One of the ideas to handle such situation is to use discretization techniques.

In both cases, the corresponding stability condition will be expressed in terms of LMIs, as feasibility (delay-independent stability) or optimization problems (computing the maximal allowable delay, or maximal allowable ellipsoids in the delay-parameter space, etc).

1.1.13 A Brief History of LMIs

The history of LMIs in the analysis of dynamical systems goes back more than 100 years. The story begins in about 1890, when Lyapunov published his seminal work introducing what we now call Lyapunov theory. He showed that the DE

$$\frac{d}{dt}x(t) = Ax(t) \tag{1.1.3}$$

is stable (i.e., all trajectories converge to zero) if and only if there exists a positive-definite matrix P such that

$$A^T P + PA < 0. \tag{1.1.4}$$

The requirement $P > 0,\ A^T P + PA < 0$ is what we now call a Lyapunov inequality on P which is a special form of an LMI. Lyapunov also showed that this first LMI could be explicitly solved. Indeed, one can pick any $Q = Q^T > 0$ and then solve the linear equation $A^T P + PA = -Q$ for the matrix P, which is guaranteed to be positive-definite if the system (1.1.3) is stable. In summary, the first LMI used to analyze the stability of a dynamical system was the Lyapunov inequality (1.1.4) , which can be solved analytically (by solving a set of linear equations).

Lyapunov stability method has more theoretical importance than practical value and can be used to derive and prove other stability results. Its final statement for linear time invariant system is elegant and easily tested using MATLAB. However, it is computationally more involved than the other methods for examining the stability of linear systems. Its importance lies in its generality since it can be applied to all nonlinear and linear systems without taking into account whether or not these systems are time invariant or time varying.

Linear Matrix Inequalities

LMIs and LMI techniques have emerged as powerful design tools in areas ranging from control Engineering to system identification and structural design. Three factors make LMI techniques appealing:

- ✠ A variety of design specifications and constraints can be expressed as LMIs.
- ✠ Once formulated in terms of LMIs, a problem can be solved exactly by efficient convex optimization algorithms (the "LMI solvers").
- ✠ While most problems with multiple constraints or objectives lack analytical solutions in terms of matrix equations, they often remain tractable in the LMI framework. This makes LMI-based design a valuable alternative to classical "analytical" methods.

LMI has of the form

$$F(x) \;=\; F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{1.1.5}$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n},\ i = 0, \ldots, m$ are given. The inequality symbol in (1.1.5) means that $F(x)$ is positive-definite, that is, $u^T F(x) u > 0$ for all nonzero $u \in \mathbb{R}^n$. Of course, the LMI (1.1.5) is equivalent to a set of n polynomial inequalities in x, that is, the leading principal minors of $F(x)$ must be positive.

The LMI (1.1.5) is a convex constraint on x, that is, the set $\{x \mid F(x) > 0\}$ is convex. Although the LMI (1.1.5) may seem to have a specialized form, it can represent a wide variety of convex constraints on x. In particular, linear inequalities, (convex) quadratic inequalities, matrix norm inequalities, and constraint that arise in control theory, such as Lyapunov and convex quadratic matrix inequalities can all be cast in the form of LMI.

Multiple LMIs $F^{(1)}(x) > 0,\ \ldots\ , F^{(p)}(x) > 0$ can be expressed as the single LMI given as $\mathrm{diag}(F^{(1)}(x), \ldots\ , F^{(p)}(x)) > 0$. Therefore, there is no distinction between a set of LMIs and a single LMI.

1.1.14 Stability of Time-Delay Systems

The stability of time-delay systems has been widely investigated in the last two decades. Practical examples of time-delay systems include chemical Engineering, communications and biological systems. Current efforts can be divided into two classes: namely, frequency-domain approach and time-domain approach. In the time-domain approach, the direct Lyapunov method is a powerful tool. There are two different ideas how one can use this method. They are the Lyapunov-Krasovskii approach and the Lyapunov-Razumikhin approach. In the frequency domain approach, two variables polynomial method, matrix pencils, μ-analysis, Integral Quadratic Constraint (IQC) analysis are considered in this case. Both approaches can be used to handle systems with time-varying delay. The former usually requires both the upper bound of the time-varying delay and additional information on the derivative of the time-varying delay, while the latter has no restriction on the derivative of the time-varying delay, which allows a fast time-varying delay. The obtained results using the Lyapunov-Krasovskii approach are usually less conservative than those using the Lyapunov-Razumikhin approach since the former takes advantage of the additional information of the delay. It is well known that there are systems which are stable with some non-zero delay, but are unstable without delay. For such case, if there is a time-varying perturbation on the non-zero delay, it is of great significance to consider the stability of systems with interval time-varying delay. Other typical examples of systems with interval time-varying delay are networked control systems. Time-varying delay is relevant while considering the following situations.

- ✠ Network congestion control - delay depends on the length of queue and network traffic.
- ✠ Real-time control systems with dynamic scheduling - delay depends on the size of the processes.
- ✠ Control of chemical processes.
- ✠ Biological systems.

Two different types of stability criteria can be considered.

- ✠ Delay-independent stability - stability for any length of delay.
- ✠ Delay-dependent stability - assume a priori knowledge on the upper-bounds of the delay.

One of the most widely used tools for investigating the stability of linear systems is the second (direct) method of Lyapunov (1892), presented in his dissertation. The idea of this method is to investigate stability of a given system by measuring the rate of change of the energy of the system. The advantage of this approach is that it allows one to infer the stability of differential (and difference) equations without explicit knowledge of solutions.

1.1.15 Stability of Neural Networks

Complex nonlinear structures of dynamic neural networks used in computing tasks such as information processing, or in associative memory for storing patterns, present a challenge in stability investigations. The notion of the stability of an equilibrium point of a dynamic system is of fundamental importance in dynamic neural networks. The stability of the equilibrium points of a dynamic neural network is one of the most basic and important properties for many Engineering applications. For dynamic neural networks, it refers to stability in the sense of Lyapunov. The dynamic behavior and the notions of stability of continuous time-dynamic neural networks described by a set of nonlinear differential equations have been widely studied since the early 1990s.

As mentioned before, time-delay appears in the electronic implementations of neural networks and can lead to complicated dynamic behaviors such as chaos, oscillation and instability. In fact, the neural networks applied to optimization problems in Xia (2004) must have a unique and stable equilibrium point. That is, stability is one of the major properties of neural networks and is a crucial feature in the design of neural networks. The stability analysis of neural networks with time-delays has thus received a great deal of attention. Various approaches, including the nonsingular M-matrix based approach in Berman and Plemmons (1979), nonlinear measure approach, nonsmooth analysis approach, Clarke (1983) as well as LMI approach, have been developed for the stability analysis of DNNs. In Zhang et al.(2003), the global asymptotic stability of DNNs is discussed, where the activation functions may be non-Lipschitz continuous. Some inequalities are exploited to obtain more general stability criteria, which can be applied to a broad range of activation functions.

Since neural networks usually have a spatial extend due to the presence of a multitude of parallel pathways with a variety of axon sizes and length,there is a distribution of propagation delays over a period of time. It is worth noting that, although the signal propagation is sometimes instantaneous and can be modelled with discrete delays, it may also be distributed during a certain time period so that the distributed delays should be incorporated in the model. In other words, it is often the case that the neural networks model possesses both discrete and distributed delays , Cao (2001). Recently, it is noted that stability analysis of HNNs, CGNNs and BAMNNs with distributed delays have been discussed in Han (2004) and Wang et al. (2006 a) and the references therein. In addition, in hardware implementation of neural networks, stochastic disturbances are nearly inevitable owing to thermal noise in electronic devices. Due to stochastic disturbances, stability of the neural networks may be affected. Recently, some results on stability of SNNs with time-varying delays have been reported in Huang et al. (2005), Huang and Feng (2007), Huang and Cao (2007), Wan and Sun (2005), Wang et al. (2006 c) and Zhang et al. (2007 a) and the references therein.

1.1.16 Robust Stability Analysis

Robustness is an approach to feature persistence in systems for which we do not have the mathematical tools to use the approaches of stability theory. In some cases the problem could be reformulated as one of stability theory, but only in a formal sense that would bring little in the way of new insight or control methodologies. The robust stability problem considers the stability problem of systems that contain some uncertainties.

In Engineering applications, it is now very common that one does not know exactly the system under investigation; that is, the system contains some elements (blocks) that are uncertain. Usually it is known that these uncertain elements belong to some specific admissible domains, which in turn depend on the nature of the elements and also on the information available about the system. In other words, it is known only that the system belongs to the family of systems that arises when the uncertain elements (blocks) range over the admissible domains and therefore, one may treat the family as a new object for analysis. This family is referred to as an uncertain system. When it is possible to show that all systems of the family are stable, the stability of the original system that is a particular member of the family is guaranteed.

As it is well known, it is usually impossible to describe a practical system exactly. First, there are often parameters or parasitic processes that are not completely known. Second, due to the limitation of mathematical tools available, we usually try to use a relatively simple model to approximate a practical system. As a result, some aspects of the system dynamics (known as unmodeled dynamics) are ignored. Third, some control systems are required to operate within a range of different operating condition.To capture these uncertain factors, it is often possible to identify a bounding set such that all the possible uncertainties fall within this set and yet it, is not too difficult to analyze mathematically.

Uncertainty Characterization

Consider the system

$$\dot{x}(t) = A_0(x_t, t)x(t) + A_1(x_t, t)x(t - r), \tag{1.1.6}$$

where $A_0 \in \mathrm{R}^{n\times n}, A_1 \in \mathrm{R}^{n\times n}$ are uncertain coefficient matrices not known completely, except that they are within a compact set $\mho$ which refer the uncertainty set

$$(A_0(x_t, t),\ A_1(x_t, t)) \in \omega \text{ for all } t \geq 0.$$

The uncertainty set characterizes the uncertainties and serves as basic information needed to carry out robust stability analysis. Notice also that the coefficients may depend on the time t as well as the current and previous state variable $x(t+\xi), -r \leq \xi \leq 0$. For the sake of convenience, we will not explicitly show these dependences or only show the dependence on time t when no confusion may arise.

A good choice of uncertainty set is a compromise between minimizing conservatism (and therefore, it is desirable to make the uncertainty set "small") and the mathematical tractibility (and therefore, it is desirable to make the uncertainty set structurally simple).

1.1.17 Concept of Passive System

A dissipative system is one for which the increase in internal energy is not greater than the energy supplied to it in which the storage function quantifies internal energy or stored energy and the supply rate function prescribes the rate of energy supplied to the system. A passive system is then a dissipative system having a particular form of supply rate function, namely one expressed as an inner product of system input and output vectors. The power of these dissipative system quantities lies in their links with system stability results and their ability to analyze physical systems described by nonlinear and linear models. Passive systems are intuitively appealing. Such systems do not generate energy internally and hence are easier to control and to guarantee that the controlled response is stable. Passive systems are a class of processes that dissipate certain types of physical or virtual energy, described by Lyapunov-like functions. Passivity theory has been one of the cornerstones of nonlinear control theory since 1970s. Defined as an input output property of process systems, the concept of passivity is particularly useful in stability analysis for interconnected systems. For example, a strict passive system with a negative feedback is stable. For a given system, its excess or shortage of passivity of a system can be compensated by the excess of passivity of another system to maintain closed-loop stability.

1.2 Review of Literature

The history of neural networks can be traced back to the work of trying to model the neuron. A large class of neural networks which can function as stable content addressible memories was proposed by Cohen and Grossberg (1983, 1988). In this model the feedback terms consist of amplification and stabilizing functions which are generally nonlinear. These terms provide the model with a special kind of generalization where in many neural network models that are capable of content addressable memory such as additive neural networks, Cohen - Grossberg neural networks and Bidirectional Associative Memory neural networks and also biological models such as Lotka-Volterra models of population dynamics are included as special cases. CGNNs were designed to include additive neural networks, followed by Hopfield (1982,1984). In the original analysis, Cohen and Grossberg assumed that the weight matrix was symmetric.

Arik (2000 a, 2004, 2005) and Yu and Li (2007), studied global asymptotic stability of a class of dynamical neural networks, global robust stability, global asymptotic stability of delayed cellular neural network and an analysis of exponential stability of delayed neural networks with time varying delays. Balasubramaniam et al.(2009) and Balasubramaniam and Lakshmanan (2009), studied delay-interval dependent robust stability criteria for stochastic neural networks with linear fractional uncertainties, delay-range dependent stability criteria for neural networks with Markovian jumping parameters and passivity analysis for neural networks of neutral type with Markovian jumping parameters and time delay in the leakage term. Stability of stochastic delay neural networks was studied by Blythe et al.(2001).

Cao (2001), Cao and Li (2005) and Cao et al.(2006) had given global stability conditions for delayed CNNs, stability in delayed Cohen-Grossberg neural networks:

LMI optimization approach and global asymptotical stability of recurrent neural networks with multiple discrete delays and distributed delays.Chen et al.(2009 a, 2009 b), Chen and Wu (2009) and Gao et al. (2009), had given improved delay-dependent stability analysis for uncertain stochastic Hopfield neural networks with time-varying delays, improved results on passivity analysis of uncertain neural networks with time-varying discrete and distributed delays, novel delay-dependent stability criteria of neural networks with time-varying delay and passivity and passification for networked control systems. Mean square exponential stability of impulsive stochastic fuzzy cellular neural networks with distributed delays was studied by Chen et al.(2011).

When working with neural network based control, stability is required for the neural network and the overall system. Stability criteria must be established for both for the controller and the controlled system. Li and Chen (2009), Li and Fu (2010), Li (2009 a, 2009 b, 2010), Li and Cao (2007)Liu et al.(2009) and Mahmoud and Xia (2011) studied stability properties for Hopfield neural networks with delays and impulsive perturbations, existence and global exponential stability of periodic solution for impulsive Cohen - Grossberg-type BAM neural networks with continuously distributed delays, stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks, existence and global exponential stability of periodic solution for delayed neural networks with impulsive and stochastic effects, robust passive filtering for neutral-type neural networks with time-varying discrete and unbounded distributed delays, exponential stability of Cohen - Grossberg type BAM neural networks with time-varying delays via impulsive control and existence and global exponential stability of periodic solution for impulsive Cohen-Grossberg type BAM neural networks with continuously distributed delays, delay-independent exponential stability of stochastic Cohen-Grossberg neural networks with time-varying delays and reaction - diffusion terms and existence and global stability analysis of equilibrium of fuzzy cellular neural networks with time delay in the leakage term under impulsive perturbations and stability and synchronization of discrete-time Markovian jumping neural networks with mixed mode-dependent time delays.

Rong (2005) had given LMI-based criteria for robust stability of Cohen-Grossberg neural networks with delays. R. Rakkiyappan et al.(2008) and Rakkiyappan and Balasubramaniam (2008) studied robust stability results for uncertain stochastic neural networks with discrete interval and distributed time-varying delays and LMI conditions for global asymptotic stability results for neutral-type neural networks with distributed time delays. Singh (2004, 2005), studied by using LMI approach, Global robust stability of delayed neural networks and a generalised LMI-based approach to the global asymptotic stability of delayed cellular neural networks.

Zhang et al. (2007 a, 2010 a, 2010 b , 2010 c),studied delay-dependent global stability results for delayed Hopfield neural networks, Novel weighting-delay-based stability criteria for recurrent neural networks with time-varying delay, new passivity criteria for neural networks with time-varying delay, neural network and stochastic stability analysis of neutral-type impulsive neural networks with mixed time-varying delays and Markovian jumping.

The literature has shown that neural networks have been applied to control non-linear systems. There has been a large number of papers applying neural networks for various problems are addressed. Considerable amount of work has been done with the neural network in the LMI techniques, see the references.

1.3 Basic Definitions and Lemmas

Some useful definitions and lemmas are stated below which are useful to derive the main results in the following chapters.

Definition 1.3.1. *All the possible outcomes, the elementary events are grouped together to form a set Ω with typical element $\omega \in \Omega$. Not every subset of Ω is in general an observable or interesting event. So we only group these observable or interesting events together as a family $\mathcal{F}$ of subsets of Ω. For the purpose of probability theory, such a family $\mathcal{F}$ should have the following properties:*

(i) $\emptyset \in \mathcal{F}$, *where* $\emptyset$ *denotes the empty set,*

(ii) $\alpha \in \mathcal{F} \Rightarrow \alpha^C \in \mathcal{F}$, *where* $\alpha^C = \Omega - \alpha$ *is the complement of* α *in* Ω,

(iii) $\{\alpha_i\}_{i\geq 1} \subset \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} \alpha_i \in \mathcal{F}$.

*A family $\mathcal{F}$ with these three properties is called a σ-**algebra.** The pair $(\Omega, \mathcal{F})$ is called a **measurable space,** and elements of $\mathcal{F}$ is henceforth called $\mathcal{F}-$**measurable sets** instead of events.*

Definition 1.3.2. *A real valued function $x : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}-$**measurable** if*

$$\Big\{\omega : x(\omega) \le a\Big\} \text{ for all } a \in \mathbb{R}.$$

*The function x is also called a **real−valued ($\mathcal{F}-$measurable) random variable.***

Definition 1.3.3. *A **probability measure** $\mathcal{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathcal{P} : \mathcal{F} \to [0, 1]$ such that*

(i) $\mathcal{P}(\Omega) = 1$;

(ii) for any disjoint sequence $\{\alpha_i\}_{i\ge1} \subset \mathcal{F}$(that is $\alpha_i \bigcap \alpha_j = \emptyset$ if $i \ne j$)

$$\mathcal{P}\Big(\bigcup_{i=1}^{\infty} \alpha_i\Big) = \sum_{i=1}^{\infty} \mathcal{P}(\alpha_i).$$

*The triple $(\Omega, \mathcal{F}, \mathcal{P})$ is called a **probability space.***

Definition 1.3.4. *If $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, we set*

$$\overline{\mathcal{F}} = \Big\{\alpha \subset \Omega : \exists \beta, \gamma \in \mathcal{F} \text{ such that } \beta \subset \alpha \subset \gamma, \quad \mathcal{P}(\beta) = \mathcal{P}(\gamma)\Big\}.$$

*Then $\overline{\mathcal{F}}$ is a σ-algebra and is called the **completion** of $\mathcal{F}$. If $\mathcal{F} = \overline{\mathcal{F}}$, the probability space $(\Omega, \mathcal{F}, P)$ is said to be **complete.***

Definition 1.3.5. ***(Stochastic processes)*** *Families of random variables which are functions of time, are known as stochastic processes (or random processes or random functions).*

*Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A **filtration** is a family $\Big\{\mathcal{F}_t\Big\}_{t\ge0}$ of increasing sub σ-algebras of $\mathcal{F}$(that is $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \le t < s < \infty$). The filtration is said to be **right continuous** if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \ge 0$. When the probability space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}-$null sets.*

From now on, unless otherwise specified, we shall always work on a given complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\Big\{\mathcal{F}_t\Big\}_{t\ge0}$ satisfying the usual conditions.

*A family $\{x(t)\}_{t\in I}$ of $\mathbb{R}^d-$valued random variables is called a **stochastic process*** *with parameter set (or index set) I and state space $\mathbb{R}^d$. The parameter set I is usually the halfline $\mathbb{R}_+ = [0, \infty)$, but it may also be an interval $[a, b]$, the non negative integers or even subsets of $\mathbb{R}^d$. Note that for each fixed $t \in I$, we have a random variable*

$$\omega \to x(t, \omega) \in \mathbb{R}^d \text{ for every } \omega \in \Omega.$$

On the other hand, for each fixed $\omega \in \Omega$, we have a function

$$t \to x(t, \omega) \in \mathbb{R}^d \text{ for every } t \in I$$

which is called a sample path of the process.

*Let $\{x(t)\}_{t \geq 0}$ be an $\mathbb{R}^d-$valued stochastic process. It is said to be $\{\mathcal{F}_t\}-$**adapted** (or simply, **adapted**) if for every t, $x(t)$ is $\mathcal{F}_t-$measurable.*

Definition 1.3.6. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. A (standard) one$-$dimensional Brownian motion is a real$-$valued continuous $\{\mathcal{F}_t\}-$adapted process $\{W(t)\}_{t \geq 0}$ with the following properties:*

(i) $W(0) = 0$ *almost surly,*

(ii) *for $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$,*

(iii) *for $0 \leq s < t < \infty$, the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$.*

Definition 1.3.7. *The solution $x = 0$ of the system described by the equation*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{1.3.1}$$

*where x and x_0 are elements of $\mathbb{R}^n$ is said to be **stable** if for every $\epsilon > 0$, there exist a $\delta(t_0, \epsilon) > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$, for every $t \geq t_0$.*

Definition 1.3.8. *The solution $x = 0$ of the system (1.3.1) is said to be **asymptotically stable** if it is stable and if there exists a $\delta(t_0) > 0$ such that $\|x(t_0)\| < \delta$ implies $\lim_{t \to \infty} x(t) = 0$.*

Definition 1.3.9. *The solution $x(0) = x_0$ of the system* (1.3.1) *is said to be* ***exponential stability*** *if it is stable and if there exists* $N > 0,\ \alpha > 0$ *such that* $\|x(t)\| \leq \|x_0\|\ Ne^{-\alpha t},\ \ \forall\ t \geq 0.$

The positive number α is the convergence rate.

Definition 1.3.10. *(i) The trivial solution of equation*

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t) \quad on \ \ t \geq t_0 \tag{1.3.2}$$

is said to be ***stochastically stable*** *or stable in probability if for every pair of* $\epsilon \in (0,1)$ *and* $r > 0$, *there exists a* $\delta = \delta(\epsilon, r, t_0) > 0$ *such that*

$$P\Big\{|x(t;t_0,x_0)| < r \text{ for all } t \geq t_0\Big\} \geq 1 - \epsilon$$

whenever $|x_0| < \delta$.

(ii) The trivial solution is said to be ***stochastically asymptotically stable*** *if it is stochastically stable and moreover, for every* $\epsilon \in (0,1)$, *there exists a* $\delta_0 = \delta_0(\epsilon, t_0) > 0$ *such that*

$$P\Big\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\Big\} \geq 1 - \epsilon$$

whenever $|x_0| < \delta$.

Lemma 1.3.1. *(Schur Complement) Given constant matrices* Ω_1, Ω_2 *and* Ω_3 *with appropriate dimensions, where* $\Omega_1^T = \Omega_1$ *and* $\Omega_2^T = \Omega_2 > 0$, *then*

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \\ * & -\Omega_2 \end{bmatrix} < 0, \quad or \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \\ * & \Omega_1 \end{bmatrix} < 0.$$

Lemma 1.3.2. *(Jensen's inequality) For any $n \times n$ constant matrix $M > 0$, any scalars a and b with $a < b$ and a vector function $x(t) : [a, b] \longrightarrow \mathbb{R}^n$ such that integrations concerned are well defined, then the following inequality holds*

$$\Big[\int_a^b x(s)ds\Big]^T M \Big[\int_a^b x(s)ds\Big] \leq (b-a)\Big[\int_a^b x^T(s)Mx(s)ds\Big].$$

Definition 1.3.11. *(Itô's formula) Let $x(t)$ be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(t)dt + g(t)dB(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t), t)$ is again an Itô process with the stochastic differential given by

$$dV(x(t), t) \quad = \quad \mathcal{L}V(x(t), t)dt + V_x(x(t), t)g(t)dB(t)$$

where

$$\mathcal{L}V(x(t), t) \quad = \quad V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}trace\Big(g^T(t)V_{xx}(x(t), t)g(t)\Big)$$

$$V_t(x(t), t) = \frac{\partial V(x(t), t)}{\partial t}, V_x(x(t), t) = \Big(\frac{\partial V(x(t), t)}{\partial x_1}, \frac{\partial V(x(t), t)}{\partial x_2}, ..., \frac{\partial V(x(t), t)}{\partial x_d}\Big),$$

and

$$V_{xx}(x(t), t) = \Big(\frac{\partial^2 V(x(t), t)}{\partial x_i \partial x_j}\Big)_{d\times d}.$$

1.4 Thesis Organization

1.5 Thesis Outline and Contribution Overview

This thesis is dedicated to investigate the stability analysis of neural networks with different time-delays. Different kinds of neural networks like HNNs, SNNs and neutral type neural networks are considered. The time-varying delays are assumed to be bounded by a positive scalar. The main objectives of this thesis are to propose less conservative conditions to the global exponential stability problems and to develop efficient algorithms to deal with the global stability problems of DNNs. The feasible matrices can be efficiently accomplished by resorting standard numerical softwares and the optimal scalar values can be obtained by solving corresponding convex optimization problems.

In **Chapter 1**, the author has discussed briefly about the notations, basic definitions, lemmas and preliminary facts. Review of literature, thesis outline, objectives of the study, contribution overview and methodologies used has also been discussed.

In **Chapter 2**, the author deals with stability analysis problem for uncertain stochastic neural networks with discrete interval and distributed time-varying delays. The parameter uncertainties are assumed to be norm bounded and the delay is assumed to be time-varying and belong to a given interval, which means that the lower and upper bounds of interval time-varying delays are available. Based on the new Lyapunov-Krasovskii functional and stochastic stability theory, delay-interval dependent stability criteria are obtained in terms of linear matrix inequalities. Finally, two numerical examples and comparisons are given to show the effectiveness of the proposed LMI conditions.

In **Chapter 3**, the sufficient conditions are derived for the global exponential stability of stochastic Cohen-Grossberg neural networks with multiple time-varying delays by using linear matrix inequality (LMI) approach. In addition, an example is provided to illustrate the applicability of the result using LMI toolbox in MATLAB.

In **Chapter 4**, the author investigates the delay-probability-distribution-dependent robust stability problem for a class of uncertain Markovian jump stochastic neural networks with time-varying delays. The information of probability distribution of the time delay is considered and transformed into parameter matrices of the transferred stochastic neutral networks model. Based on the Lyapunov-Krasovskii functional and stochastic analysis approach, a novel delay-probability-distribution-dependent sufficient condition is obtained in the linear matrix inequality (LMI) form such that delayed Markovian jump stochastic neural networks are robustly globally asymptotically stable in the mean square for all admissible uncertainties. An important feature of the result is that the stability conditions are dependent on the probability distribution of delays and upper bound of the derivative is allowed to be greater than or equal to 1. Numerical examples are given for the comparison to illustrate the effectiveness of the new results derived by the author.

In **Chapter 5** the author deals with the passivity analysis of Markovian jumping neural networks with leakage time-varying delays. Based on a Lyapunov functional that accounts for the mixed time-delays, a leakage delay-dependent passivity conditions are derived in terms of linear matrix inequalities (LMIs). The mixed delays includes leakage time-varying delays, discrete time-varying delays and distributed time-varying delays. By employing a novel Lyapunov–Krasovskii functional having triple integral terms, new passivity leakage delay-dependent criteria are established to guarantee the passivity performance. This performance not only depends on the upper bound of the time-varying leakage delay $\sigma(t)$ but also it depends on the upper bound of the derivative of the time-varying leakage delay σ_μ. While estimating the upper bound of derivative of the Lyapunov–Krasovskii functional, the discrete and distributed delays should be treated so as to appropriately develop less conservative results. Two numerical examples are given to show the validity and potential of the developed criteria.

In **Chapter 6** the author investigates the delay-dependent passivity problem of Markovian jumping neural networks of neutral type with time delays in the leakage term and mode-dependent delays. Delay-dependent passivity conditions are derived by taking the inherent characteristic of such kinds of neural networks into account. An improved Lyapunov–Krasovskii functional (LKF) with the triple integral terms and quadruple integrals is constructed and the results are derived in terms of LMI s. The information of the mode-dependent of all delays have been taken into account in the constructed Lyapunov–Krasovskii functional and novel stability criterion is derived.The advantage is that as many as possible of the Lyapunov matrices are chosen to be mode-dependent. Finally numerical example is provided to demonstrate the effectiveness and less conservativeness of the proposed theoretical results.

1.6 Objective of the Study

The objective is to study the stability analysis of

- ✠ Stochastic Cohen- Grossberg neural networks with multiple time - varying delays.
- ✠ Uncertain stochastic neural networks with discrete interval and distributed time-varying delays.
- ✠ Markovian jumping stochastic neural networks with time - varying delays.
- ✠ Markovian jumping neural networks with leakage time - varying delays.
- ✠ Markovian jumping neural networks with time delays in leakage term and mode-dependent delays.

1.7 Methodology

The stability analysis problem for uncertain stochastic neural networks , Stochastic Cohen- Grossberg neural networks , Markovian jumping neural networks and Markovian jumping stochastic neural networks with mixed delays are considered. New set of sufficient conditions are derived to guarantee the stability of the equilibrium point, which are different from the existing ones. The methodologies used are described as follows.

Lyapunov method is a powerful method for determining the stability or instability of fixed points of nonlinear autonomous systems. This method has more theoretical importance than practical value and can be used to derive and prove other stability results. Its final statement for linear time invariant system is elegant and easily tested using MATLAB. However, it is computationally more involved than the other methods for examining the stability of linear systems. Its importance lies in its generality since it can be applied to all nonlinear and linear systems without taking into account whether or not these systems are time invariant or time varying.

Lyapunov functions are functions which can be used to prove the stability of a certain fixed point in a dynamical system or autonomous differential equation. An orbit is called Lyapunov stable if the forward orbit of any point in a small enough neighborhood of it stays in a small (but perhaps, larger) neighborhood.

Various criteria have been developed to prove stability or instability of an orbit. Under favourable circumstances, the problem of stability may be reduced to a well-studied problem involving eigenvalues of matrices. A more general method involves Lyapunov functions where as Lyapunov stability theorems give only sufficient condition.

Linear Matrix Inequalities and its techniques have emerged as powerful design tools in areas ranging from control Engineering to system identification and structural design. Three factors make LMI techniques appealing:

- ✠ A variety of design specifications and constraints can be expressed as LMIs.
- ✠ Once formulated in terms of LMIs, a problem can be solved exactly by efficient convex optimization algorithms (the "LMI solvers").
- ✠ While most problems with multiple constraints or objectives lack analytical solutions in terms of matrix equations, they often remain tractable in the LMI framework. This makes LMI-based design a valuable alternative to classical "analytical" methods.

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